THE $\{1,s\}$-WEIGHTED DAVENPORT CONSTANT IN $C_n^k$

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Abstract
Let $G$ be a finite abelian group and let $\emptyset \neq A \subset \mathbb{Z}$. The $A$-weighted Davenport constant of $G$ is the smallest positive integer $D_A(G)$ such that every sequence $x_1 \cdot x_2 \cdot \ldots \cdot x_{D_A(G)}$ over $G$ has a non-empty subsequence $(x_{j_1}, \ldots, x_{j_l})$ such that $\epsilon_1 x_{j_1} + \epsilon_2 x_{j_2} + \ldots + \epsilon_l x_{j_l} = 0$ for some $\epsilon_1, \epsilon_2, \ldots, \epsilon_l \in A$. In this paper, we obtain both upper and lower bounds for $D_{\{1,s\}}(C_n^k)$, where $C_n$ denotes the cyclic group of order $n$, $s^2 \equiv 1 \pmod{n}$ and $s \not\equiv \pm 1 \pmod{n}$. These bounds become sharp in some “small” cases.

1. Introduction

Given a finite abelian group $G$ written additively, the zero-sum problems study conditions to ensure that a given sequence over $G$ has a non-empty subsequence with prescribed properties (for instance length, repetitions, and weights) such that the sum of their terms equals 0, the identity of $G$. This kind of problem dates back to the 1960s, with the works of Erdős, Ginzburg and Ziv [7] and Olson [13, 14]. It has been extensively studied for abelian groups, but in recent times several results over non-abelian groups have emerged. This paper deals with a weighted problem over $C_n^k$, where $C_n$ denotes the cyclic group of order $n$. First of all, some definitions and notations are required.

1.1. Definitions and Notations

By a sequence $S$ over a finite group $G$ we mean a finite and unordered element of the free abelian monoid $F(G)$ equipped with the sequence concatenation product

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denoted by \( \cdot \). A sequence \( S \in \mathcal{F}(G) \) has the form

\[
S = \prod_{i=1}^{k} g_i = g_1 \cdot \ldots \cdot g_k \in \mathcal{F}(G),
\]

where \( g_1, \ldots, g_k \in G \) are the terms of \( S \) and \( k = |S| \geq 0 \) is the length of \( S \). Since the sequences are unordered, \( S = \prod_{i=1}^{k} g_{\tau(i)} \in \mathcal{F}(G) \) for any permutation \( \tau : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\} \). For a given \( S \in \mathcal{F}(G) \) and for \( t \geq 0 \), we abbreviate \( S[t] = S \cdot \ldots \cdot S \) \( t \) times. For \( g \in G \), the multiplicity of the term \( g \) in \( S \) is denoted by \( v_g(S) = \#\{i \in \{1, 2, \ldots, k\} : g_i = g\} \), and therefore our sequence \( S \) may also be written as \( S = \prod_{g \in G} g^{v_g(S)} \). A sequence \( T \) is a subsequence of \( S \) if \( v_g(T) \leq v_g(S) \) for all \( g \in G \); for this we use the notation \( T \mid S \). In this case, we write \( S \cdot T[-1] = \prod_{g \in G} g^{v_g(S) - v_g(T)} \). We also define:

\[
\sigma(S) = g_1 + \ldots + g_k \in G, \quad \text{the sum of } S;
\]

\[
\Sigma(S) = \bigcup_{T \mid S \mid |T| \geq 1} \{\sigma(T)\} \subset G, \quad \text{the set of subsums of } S;
\]

\[
S \cap K = \prod_{g \in G} g^{v_g(S)}, \quad \text{the subsequence of } S \text{ that lie in a subset } K \text{ of } G.
\]

The sequence \( S \) is called zero-sum free if \( 0 \notin \Sigma(S) \), and \( S \) is zero-sum sequence if \( \sigma(S) = 0 \).

### 1.2. The Davenport Constant

One of the most important types of zero-sum problems involves the small Davenport constant \( d(G) \) of a finite abelian group \( G \). Namely, let \( d(G) \) be the maximal integer such that there exists a sequence over \( G \) (repetition allowed) of length \( d(G) \) which is zero-sum free, that is,

\[
d(G) = \sup\{|S| > 0 : S \in \mathcal{F}(G) \text{ is zero-sum free}\}.
\]

It is well-known that \( d(C_n) = n - 1 \). The inequality

\[
d(C_{n_1} \oplus \ldots \oplus C_{n_k}) \geq \sum_{i=1}^{k} (n_i - 1)
\]

holds true, considering the zero-sum free sequence formed by the concatenation of \( n_i - 1 \) copies of a generator of \( C_{n_i} \) for each \( 1 \leq i \leq k \). Olson proved that the equality holds true for abelian \( p \)-groups [13] and for \( k \leq 2 \) [14].

Many variations and generalizations of zero-sum problems have been considered over the years. In this paper, we consider the following weighted problem. Let
$A \subset \mathbb{Z}$ be a set of weights. In order to avoid trivial cases, we assume that $A \neq \emptyset$ and $A$ does not contain any multiple of $\exp(G)$, the exponent of $G$. Let

$$\sigma_A(S) = \{a_1g_1 + \ldots + a_kg_k \in G : a_i \in A\} \text{ the set of } A\text{-weighted sums of } S,$$

$$\Sigma_A(S) = \bigcup_{|T| \geq 1} \sigma_A(T) \subset G \text{ the set of } A\text{-weighted subsequence sums of } S.$$ 

Moreover, $S$ is $A$-zero-sum free if $0 \not\in \Sigma_A(S)$, and an $A$-zero-sum sequence if $0 \in \sigma_A(S)$.

The $A$-weighted Davenport constant of an abelian group $G$ is defined by

$$D_A(G) = \inf \{k > 0 : \text{ every } S \in \mathcal{F}(G) \text{ with } |S| \geq k \text{ satisfies } 0 \in \Sigma_A(S)\}.$$ 

For instance, every abelian group $G$ yields $D_{\{1\}}(G) = d(G) + 1$, where $d(G)$ is the (unweighted) small Davenport constant of $G$. Using the Pigeonhole Principle, Adhikari et al. [1] proved that $D_{\{\pm 1\}}(C_n) = \lceil \log_2(n) \rceil + 1$. More generally, Adhikari, Grynkiewicz and Sun [3] proved that if $n_1 \mid n_2 \mid \ldots \mid n_k$ and $G = C_{n_1} \oplus \ldots \oplus C_{n_k}$, then

$$\sum_{i=1}^{k} \lfloor \log_2(n_i) \rfloor + 1 \leq D_{\{\pm 1\}}(G) \leq \lceil \log_2(|G|) \rceil + 1, \quad (1.2)$$

and Marchan, Ordaz and Schmid [12] removed the hypothesis $n_1 \mid n_2 \mid \ldots \mid n_k$.

In addition, Adhikari, David and Urroz [2] showed that if $A = \{1, 2, 3, \ldots, r\}$, then $D_A(C_n) = \lceil \frac{r}{2} \rceil$ (it was previously proved for $n$ prime by Adhikari and Rath [4]). Also in [2], it is proved that if $A$ is the set of all quadratic residues modulo a squarefree number $n$ and $\omega(n)$ is the number of distinct prime factors of $n$, then $D_A(C_n) = 2\omega(n) + 1$.

Although very little is known about the $A$-weighted Davenport constant for a general set $A$, Halter-Koch [11] provided an arithmetical interpretation of certain types of weighted Davenport constants in terms of algebraic integers and of binary quadratic forms.

In this paper, we fix an integer $k \geq 1$, and establish some upper and lower bounds for $D_{\{1, s\}}(C_n^k)$, where

$$s^2 \equiv 1 \pmod{n}, \quad \text{but} \quad s \not\equiv \pm 1 \pmod{n}. \quad (1.3)$$

In fact, for $s \equiv 1 \pmod{n}$ we have $D_{\{1, s\}}(C_n^k) = d(C_n^k) + 1$, and for $s \equiv -1 \pmod{n}$ we have $D_{\{1, s\}}(C_n^k) = D_{\{\pm 1\}}(C_n^k)$.

The main motivation for considering this set of weights was the inverse zero-sum problem related to the small Davenport constant over the non-abelian group $C_n \rtimes_s C_2$. Indeed, in [5] the authors considered the inverse problem over $D_{2n}$, the dihedral group of order $2n$. The main argument used the $\{\pm 1\}$-weighted Davenport constant over $C_n$ in order to obtain the structure of the product-one free sequences
of maximum length. The idea for \( C_n \times_s C_2 \) would be similar, using the \( \{1, s\}\)-weighted Davenport constant over \( C_n \). However, the upper bounds provided in this paper were only able to solve the inverse problem for values of \( n \) with sufficiently large factors, namely, factors \( n_1 \) and \( n_2 \) as in Lemma 1 (in particular, the proof would not work for \( n = 2^t, t \geq 3 \)). Nevertheless, in [6] the authors completely solved the inverse problem over \( C_n \times_s C_2 \), using the set of weights but without using directly the bounds on \( D_{\{1,s\}}(C_k^n) \) presented here.

This paper is organized as follows. In Section 2, we present a useful factorization of \( n \) in order to set a crucial projection, yielding an isomorphism in some cases. In Section 3, we prove lower bounds for \( D_{\{1,s\}}(C_k^n) \). In Sections 4 and 5, we prove upper bounds for \( D_{\{1,s\}}(C_k^n) \) when the factorization of \( n \) generates an isomorphism and a projection, respectively; in particular, we prove a relation among these two cases. In Section 6, we discuss the tightness of these bounds, and conclude the exact value in two families of “small” cases.

2. The Natural Projection/Isomorphism

We note that conditions (1.3) guarantee that \( n \) can be neither an odd prime power nor twice an odd prime power, for otherwise we would have \( s \equiv \pm 1 \pmod{n} \). The following lemma, which is also [6, Lemma 2.2], and which we reproduce its proof here for convenience, ensures that these conditions suffice to factor \( n \) nicely.

**Lemma 1 ([6, Lemma 2.2]).** Let \( n \geq 8 \) and \( s \) be positive integers satisfying the conditions (1.3).

(i) If both \( n \neq p^t \) and \( n \neq 2p^t \) for every prime \( p \) and every integer \( t \geq 1 \), then there exist coprime integers \( n_1, n_2 \geq 3 \) such that \( s \equiv -1 \pmod{n_1} \), \( s \equiv 1 \pmod{n_2} \), and either (A) \( n = n_1n_2 \) or (B) \( n = 2n_1n_2 \).

(ii) If \( n = 2^t \) for some \( t \geq 3 \), then (B) \( n = 2n_1n_2 \), where either \( (n_1, n_2) = (1, 2^{t-1}) \) satisfies \( s \equiv 1 \pmod{n_2} \) or \( (n_1, n_2) = (2^{t-1}, 1) \) satisfies \( s \equiv -1 \pmod{n_1} \).

**Proof.** Let \( n = 2^t m \), where \( m \) is odd and \( t \geq 0 \) is an integer. Since \( m \) divides \( s^2 - 1 \) and \( \gcd(s - 1, s + 1) \in \{1, 2\} \) (depending on whether \( s \) is even or odd), each prime power factor of \( m \) divides either \( s + 1 \) or \( s - 1 \). Let \( m_1 = \gcd(m, s + 1) \) and \( m_2 = \gcd(m, s - 1) \), so that \( m = m_1m_2 \). In addition, \( s^2 \equiv 1 \pmod{2^t} \) implies that either \( s \equiv \pm 1 \pmod{2^t} \) or \( s \equiv 2^{t-1} \pm 1 \pmod{2^t} \) (depending on whether \( t \geq 3 \)). We consider some cases:

(i) **CASE** \( n \neq 2^t \), that is, \( m \neq 1 \).

(i.1) **SUBCASE** \( t = 0 \). In this case, \( n = m = m_1m_2 \). We set \( n_1 = m_1 \) and \( n_2 = m_2 \), hence \( n = n_1n_2 \) is the desired factorization, as in (A). This is the only case where \( s \) can be even; in the following, \( s \) must be odd.
(i.2) **SUBCASE** \( t = 1 \). It is possible to set either \( n_1 = 2m_1 \) and \( n_2 = m_2 \) or \( n_1 = m_1 \) and \( n_2 = 2m_2 \), hence \( n = n_1n_2 \) is the desired factorization, as in (A).

(i.3) **SUBCASE** \( t \geq 2 \) and \( m \geq 3 \). If \( s \equiv -1 \pmod{2^t} \), then we set \( n_1 = 2^tm_1 \) and \( n_2 = m_2 \). If \( s \equiv 1 \pmod{2^t} \), then we set \( n_1 = m_1 \) and \( n_2 = 2^tm_2 \). Therefore, \( n = n_1n_2 \) is the desired factorization, as in (A). In the case that \( t \geq 3 \), it is possible that \( s \equiv 2^{t-1} \pm 1 \pmod{2^t} \). If \( s \equiv 2^{t-1} - 1 \pmod{2^t} \), then we set \( n_1 = 2^{t-1}m_1 \) and \( n_2 = m_2 \). If \( s \equiv 2^{t-1} + 1 \pmod{2^t} \), then we set \( n_1 = m_1 \) and \( n_2 = 2^{t-1}m_2 \). Hence, \( n = 2n_1n_2 \) is the desired factorization, as in (B).

(ii) **CASE** \( n = 2^t \) with \( t \geq 3 \), that is, \( m = 1 \). In this case, \( s \equiv 2^{t-1} \pm 1 \pmod{2^t} \), which implies that \( s \equiv \pm 1 \pmod{2^{t-1}} \). For the negative sign, it follows that \((n_1,n_2) = \langle 2^{t-1},1 \rangle\), and for the positive sign, it follows that \((n_1,n_2) = \langle 1,2^{t-1} \rangle\). Therefore, \( n = 2n_1n_2 \) is the factorization, as in (B).

\[ \square \]

By the previous lemma and Chinese Remainder Theorem, there exists a natural projection

\[ \Psi_0 : C_n \to C_{n_1} \oplus C_{n_2} \]

satisfying

\[ \Psi_0(e) = (e_1,e_2) \quad \text{and} \quad \Psi_0(s \cdot e) = (-e_1,e_2), \]

where \( C_{n_1} = \langle e_1 \rangle \), \( C_{n_2} = \langle e_2 \rangle \) and \( C_n = \langle e \rangle \). In Case (A) of previous lemma, \( \Psi_0 \) is an isomorphism.

Let \( \bar{0} \) be the identity of \( C^k_\nu, \bar{a} = (a_1, \ldots, a_k) \in C^k_\nu \), and write \( t \cdot \bar{a} = (ta_1, \ldots, ta_k) \). Denote by

\[ \Psi : C^k_n \to C^k_{n_1} \oplus C^k_{n_2} \quad \text{(2.1)} \]

the similar projection satisfying

\[ \Psi(\bar{a}) = (\bar{a},\bar{a}) \quad \text{and} \quad \Psi(s \cdot \bar{a}) = (-\bar{a},\bar{a}), \]

where the first coordinate is the restriction to \( C^k_{n_1} \), and the second coordinate is the restriction to \( C^k_{n_2} \). In Case (A) of Lemma 1, \( \Psi \) is an isomorphism. Moreover, for \( g = (u,v) \in C^k_{n_1} \oplus C^k_{n_2} \), we write \( sg = (-u,v) \).

When we look at the sequences over \( C^k_{n_1} \oplus C^k_{n_2} \), since \( s \equiv -1 \pmod{n_1} \) and \( s \equiv 1 \pmod{n_2} \), in the second coordinate we have an unweighted problem, while in the first coordinate we have a \( \{\pm 1\} \)-weighted problem. In the case that \( n = n_1n_2 \), finding zero-sum sequences over \( C^k_n \) is equivalent to finding zero-sum sequences over \( C^k_{n_1} \oplus C^k_{n_2} \). In the case that \( n = 2n_1n_2 \), zero-sums over \( C^k_n \) imply zero-sums over \( C^k_{n_1} \oplus C^k_{n_2} \), and \( k + 1 \) disjoint zero-sums over \( C^k_{n_1} \oplus C^k_{n_2} \) generate a zero-sum over \( C^k_n \), since \( C^k_n/(C^k_{n_1} \oplus C^k_{n_2}) \simeq C^k_2 \) and \( d(C^k_2) = k \).
It is worth mentioning that the bounds presented here refer to the parities of \( n_1 \) and \( n_2 \) (see [8, Conjecture 1.10] for a conjecture regarding the Erdős-Ginzburg-Ziv constant over \( C_n^3 \) that is concerned with the parity of \( n \)). For the upper bound, our method consists basically of two steps: using the set of weights \{±1\} in \( C_{n_1}^k \) to produce more than \( d(C_{n_2}^k) \) disjoint sums in \( \{\overline{0}_{n_1}\} \oplus C_{n_2}^k \), hence obtaining a zero-sum over \( C_n^k \). However, the method can be improved, provided \( n_2 \) is even in the following way: in the first step and with a little more effort, it is possible to obtain sums in \( \{\overline{0}_{n_1}\} \oplus C_{n_2}^k \), where \( C_{n_2}^k = \{2g : g \in C_{n_2}^k\} \). This effort is rewarded, since now it is only required \( d(C_{n_2}^k) + 1 \) sums for the second step. This method will be further applied for other zero-sum problems in future papers.

3. The Lower Bounds

In this section, we provide good although not always tight lower bounds. The first one deals with the case \( n = n_1n_2 \).

**Theorem 1.** Let \( n \) and \( s \) be as in (1.3), and write \( n = n_1n_2 \) as in Lemma 1. Then:

(i) for any \( n_1 \) and \( n_2 \), we have
\[
D_{\{1,s\}}(C_n^k) \geq D_{\{\pm1\}}(C_{n_1}^k) + d(C_{n_2}^k) \geq k(\lfloor \log_2(n_1) \rfloor + n_2 - 1) + 1;
\]

(ii) if \( n_2 \) is odd and either \( n_1 > n_2 \) or \( n_1 \) is even, then
\[
D_{\{1,s\}}(C_n^k) \geq k(2n_2 - 1) + 1.
\]

**Proof.** First of all, the projection defined in (2.1) ensures that we may consider \( S \) over \( C_{n_1}^k \oplus C_{n_2}^k \) instead of over \( C_n^k \). For each case, we exhibit \( \{1,s\}\)-zero-sum free sequences \( S \).

(i) Let \( S'_1 \) be a \( \{\pm1\}\)-zero-sum free sequence over \( C_{n_1}^k \) with \( |S'_1| = D_{\{\pm1\}}(C_{n_1}^k) - 1 \) and \( S'_2 \) be a zero-sum free sequence over \( C_{n_2}^k \) with \( |S'_2| = d(C_{n_2}^k) \). Set
\[
S_1 = \prod_{g \in S'_1} (g, \overline{0}_{n_2}) \in F(C_{n_1}^k \oplus C_{n_2}^k),
\]
\[
S_2 = \prod_{h \in S'_2} (\overline{0}_{n_1}, h) \in F(C_{n_1}^k \oplus C_{n_2}^k),
\]
and \( S = S_1 \cdot S_2 \). Then \( S \in F(C_{n_1}^k \oplus C_{n_2}^k) \) is \( \{1,s\}\)-zero-sum free and \( |S| = |S_1| + |S_2| = D_{\{\pm1\}}(C_{n_1}^k) + d(C_{n_2}^k) - 1 \), which proves (i).
Let \( C_{n_1} = \langle e_1 \rangle \) and \( C_{n_2} = \langle e_2 \rangle \). Set
\[
g_i = \begin{cases} 0, \ldots, 0, e_1, 0, \ldots, 0 \quad & \text{if } i \text{ is odd,} \\ 0, \ldots, 0, e_2, 0, \ldots, 0 \quad & \text{if } i \text{ is even,} \end{cases}
\]
where \( e_1 \) is in the \( i \)th position and \( e_2 \) is in the \( (k+i) \)th position for \( 1 \leq i \leq k \). Moreover, for \( 1 \leq i \leq k \),
\[
S = \prod_{1 \leq i \leq k} g_i^{[2n_2 - 1]} \in \mathcal{F}(C_{n_1}^k \oplus C_{n_2}^k).
\]

Then \(|S| = k(2n_2 - 1)\) and we claim that \( S \) is \( \{1, s\} \)-zero-sum free. Suppose otherwise, then there exists a non-empty subsequence \( T \upharpoonright S \) such that \( 0 \in \sigma_{\{1,s\}}(T) \). Assume that \( g_i \mid T \). By the \((k+i)\)th position, we have \( v_{g_i}(T) = n_2 \). These \( g_i \)'s are the only possible terms of \( T \) that modify the \( i \)th position. If \( n_1 \) is even then the \( i \)th position of the \( \{1, s\} \)-weighted sum is odd, hence it can not be \( 0 \). If \( n_1 \) is odd and \( n_1 > n_2 \), then the \( \{1, s\} \)-weighted sum in the \( i \)th position is an odd element of \([-n_2, n_2]\), therefore it can not be \( 0 \) modulo \( n_1 \).

The second result deals with the case \( n = 2n_1n_2 \).

**Theorem 2.** Let \( n \) and \( s \) be as in (1.3), and write \( n = 2n_1n_2 \) as in Lemma 1. Then
\[
D_{\{1,s\}}(C_n^s) \geq k \cdot D_{\{1,s\}}(C_{n_1n_2}^k) + 1.
\]

**Proof.** Let \( T_1 \in \mathcal{F}(C_{n_1n_2}^k) \) be a \( \{1, s\} \)-zero-sum free sequence of length \(|T_1| = D_{\{1,s\}}(C_{n_1n_2}) - 1\). The sequence \( T_2 = T_1 \oplus (n_1n_2)e \), where \( C_{2n_1n_2} = \langle e \rangle \), is \( \{1, s\} \)-zero-sum free over \( C_{2n_1n_2} \), and \(|T_2| = D_{\{1,s\}}(C_{2n_1n_2}) \). Let \( S \in \mathcal{F}(C_{2n_1n_2}^k) \) be the sequence formed by concatenation of the sequences \( T_2 \) in each coordinate, that is,
\[
S = \prod_{1 \leq j \leq k} \prod_{g \in \text{the } j \text{th coordinate}} (0, \ldots, 0, g, 0, \ldots, 0).
\]

Hence, \(|S| = k \cdot |T_2| = k \cdot D_{\{1,s\}}(C_{n_1n_2}) \) and \( S \) is \( \{1, s\} \)-zero-sum free. \( \square \)

4. **The Upper Bounds for** \( n = n_1n_2 \)

The projection \( \Psi : C_n^k \to C_{n_1}^k \oplus C_{n_2}^k \) is actually an isomorphism in this case. Hence, finding zero-sum sequences over \( C_n^k \) is equivalent to finding zero-sum sequences over \( C_{n_1}^k \oplus C_{n_2}^k \). The goal in most cases in this section is to find many subsequences with a \( \{\pm 1\} \)-zero-sum in the first coordinate, in order to obtain a zero-sum in the second coordinate.

Before claiming the main results of this section, we prove two simpler upper bounds.
Proposition 1. Let $n$ and $s$ be as in (1.3), and write $n = n_1 n_2$ as in Lemma 1. The following hold.

(i) For every $n_2$,

$$D_{\{s\}}(C_n^k) \leq (d(C_{n_2}^k) + 1) D_{\{s\}}(C_{n_1}^k) \leq (d(C_{n_2}^k) + 1) \left( \lfloor k \log_2(n_1) \rfloor + 1 \right).$$

(ii) For $n_2$ even,

$$D_{\{s\}}(C_n^k) \leq (d(C_{n_2}^k) + 1) \left( \lfloor k \log_2(n_1) \rfloor + k + 1 \right).$$

Proof. As in the proof of Theorem 1, we consider $S \in \mathcal{F}(C_{n_1} \oplus C_{n_2})$ instead of $S \in \mathcal{F}(C_n)$.

(i) Suppose that $|S| = (d(C_{n_2}^k) + 1) D_{\{s\}}(C_{n_1}^k)$. Choose a minimal subsequence $A_1 \mid S$ with $|A_1| \leq D_{\{s\}}(C_{n_1}^k)$ such that $(\bar{0}_{n_1}, b_1) \in \sigma_{\{1,s\}}(A_1)$, where $b_1 \in C_{n_2}^k$. Inductively, we may construct $A_{j+1} \mid S \cdot (A_1 \cdot \ldots \cdot A_j)^{-1}$ for $j = 1, \ldots, d(C_{n_2}^k)$ such that $|A_{j+1}| \leq D_{\{1,s\}}(C_{n_1}^k)$ and $(\bar{0}_{n_1}, b_{j+1}) \in \sigma_{\{1,s\}}(A_{j+1})$, where $b_{j+1} \in C_{n_2}^k$. On the other hand, it follows that we have $d(C_{n_2}^k) + 1$ disjoint $\{1,s\}$-weighted sums in $\bar{0}_{n_1} \oplus C_{n_2}^k$, therefore we are done.

(ii) The proof of this inequality is similar to the previous one. The only difference is that each $\lfloor k \log_2(n_1) \rfloor + k + 1$ terms yield a $\{1,s\}$-weighted sum into $\bar{0}_{n_1} \oplus C_{n_2}^k$. In fact, let $T \mid S$ with $|T| = \lfloor k \log_2(n_1) \rfloor + k + 1$. We may construct $2^{|T|} > 2^k n_1^k$ sums using the terms of $T$, therefore the Pigeonhole Principle ensures that there exist distinct subsequences (but not necessarily pairwise disjoint) $T_1, T_2, \ldots, T_{2^k} \mid T$, such that the first coordinates of $\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_{2^k})$ are all the same in $C_{n_1}^k$. Again by the Pigeonhole Principle, there exist two of them, say $T_1$ and $T_2$, such that $\sigma(T_1)$ and $\sigma(T_2)$ have the same parity in all their last $k$ coordinates. We may transform $T_1$ and $T_2$ into disjoint subsequences. Indeed, if $T_1$ and $T_2$ share the term $(a,b)$, then the first coordinate of $\sigma(T_1)$ equals the first coordinate of $\sigma(T_2)$ if and only if the first coordinate of $\sigma(T_1 \cdot (a,b)^{[-1]})$ equals the first coordinate of $\sigma(T_2 \cdot (a,b)^{[-1]})$, as well as all the last $k$ coordinates of $\sigma(T_1)$ have the same parity as those of $\sigma(T_2)$ if and only if all the last $k$ coordinates of $\sigma(T_1 \cdot (a,b)^{[-1]})$ have the same parity as those of $\sigma(T_2 \cdot (a,b)^{[-1]})$, where “last $k$ coordinates” denotes the restriction of the elements to $C_{n_2}^k$. Therefore, assuming that $T_1$ and $T_2$ are disjoint, we obtain

$$\sum_{g \in T_1} sg + \sum_{h \in T_2} h \in \bar{0}_{n_1} \oplus C_{n_2}^k.$$  

It is possible to construct $d(C_{n_2}^k) + 1$ disjoint $\{1,s\}$-weighted sums as the previous one, therefore we are done.

□
Our goal can be achieved faster once the subsequences with an \( \{1,s\}\)-zero-sum in the first coordinate have "small" length. For example, if we have two terms \((a_1, b_1) \cdot (a_2, b_2) \mid S\) with \(a_1 = \pm a_2\), then these two terms generate a \( \{1,s\}\)-zero-sum in the first coordinate. For three or four terms, we need the following:

Lemma 2. Let \( m \geq 2 \) be an integer, and let \( S \in \mathcal{F}(C^k_m) \) with \( |S| \geq \sqrt{2m^k} \). Suppose that every term of \( S \) has multiplicity one, and the equation \( g_1 \cdot g_2 = 0 \) has no solution with \( g_1 \cdot g_2 \mid S \). If \( \bar{n}_m \mid S \), then there exists \( T \mid S \) with \( 3 \leq |T| \leq 4 \) such that \( \sum_{g \in T} e_g \cdot g = \bar{n}_m \) with \( e_g \in \{ \pm 1 \} \).

Proof. Since \( |S| \geq \sqrt{2m^k} > \sqrt{2m^k + 1/4} - 1/2 \), we have that \( \binom{|S|}{1} + \binom{|S|}{2} > m^k \). Therefore, the Pigeonhole Principle implies that \( S \) contains distinct subsequences \( T_1 \) and \( T_2 \) of lengths one or two such that \( \sigma(T_1) = \sigma(T_2) \). Notice that \( T_1 \) and \( T_2 \) have no common terms, otherwise either \( T_1 = T_2 \) or \( \bar{n}_m \mid S \). Therefore, \( \sigma(T_1) - \sigma(T_2) = \bar{n}_m \) and the latter has three or four terms.

Theorem 3. Let \( n \) and \( s \) be as in (1.3), and write \( n = n_1 n_2 \) as in Lemma 1. The following hold.

(i) For \( n \) odd,
\[
\mathcal{D}_{\{1,s\}}(C^k_n) \leq 2d(C^k_{n_2}) + \frac{n_1^k - 1}{4} + \frac{\sqrt{2n_1^k}}{2}.
\]

(ii) For \( n_1 \) even,
\[
\mathcal{D}_{\{1,s\}}(C^k_n) \leq 2d(C^k_{n_2}) + \frac{n_1^k + 2k - 2}{4} + \frac{\sqrt{2n_1^k}}{2}.
\]

(iii) For \( n_2 \) even,
\[
\mathcal{D}_{\{1,s\}}(C^k_n) \leq 2d(C^k_{n_2}) + \frac{n_1^k (k \cdot 2^{k+1} + 1) + 2 \sqrt{2n_1^k} + k(2^{k+1} + 4) + 7}{4(k + 1)}.
\]

Proof. As in the proof of Theorem 1, we consider \( S \in \mathcal{F}(C^k_{n_1} \oplus C^k_{n_2}) \) instead of \( S \in \mathcal{F}(C^k_n) \).

(i) In this case, \( n_1 \) and \( n_2 \) are both odd. Suppose that \( |S| \geq 2d(C^k_{n_2}) + \frac{n_1^k - 1}{4} + \frac{\sqrt{2n_1^k}}{2} \).

For \( 1 \leq i \leq u \), let \( A_i = (\bar{a}_i, b_i) \mid S \) be the terms of \( S \) with \( \bar{n}_{n_1} \) in the first coordinate. Furthermore, for \( 1 \leq i \leq v \), let \( A_{u+i} = (a_{u+i}, b_{u+i}) \cdot (a'_{u+i}, b'_{u+i}) \mid S \cdot \left( \prod_{1 \leq i \leq u} A_i \right)^{-1} \) denote the subsequences of \( S \) formed by two terms such that their first coordinates are distinct than \( \bar{n}_{n_1} \) and are equals or inverses in \( C^k_{n_1} \).

From \( S \), if we remove the terms of each \( A_i \) with \( 1 \leq i \leq u + v \), there will
remain at most \( \frac{n^k - 1}{2} \) terms (otherwise there would exist two equal or inverse terms in the first coordinate, that could be included in the previous \( A_{u+i} \), \( 1 \leq i \leq v \)).

Let \( T = S \cdot \left( \prod_{1 \leq i \leq u+v} A_i \right)^{-1} \), so that

\[
T = \sum_{1 \leq i \leq u+v} A_i \tilde{A}_i
\]

\[
|T| = |S| - (u + 2v) \leq \frac{n^k - 1}{2}.
\]

While \( |T| \geq \sqrt{2n^k} \), Lemma 2 ensures that there exists \( T_1 \mid T \) with \( |T_1| \in \{3, 4\} \) such that \( \tilde{0}_{n_2} \in \sigma_{\{1,s\}}(T_1) \). Inductively, whenever possible, we construct \( T_2, \ldots, T_w \) satisfying the same conditions than \( T_1 \), hence

\[
|T| - 4w = |S| - (u + 2v + 4w) < \sqrt{2n^k}.
\]

It follows that there exist \( u + v + w \) disjoint subsequences of \( S \) whose \( \{1, s\} \)-weighted sum belong to \( \{\tilde{0}_{n_2}\} \bordermath{\oplus} C^k_{n_2} \). Since

\[
u + v + w = \frac{u + 2v}{4} + \frac{u + 2v + 4w}{4} + \frac{u}{2} > \frac{|S| - \frac{n^k - 1}{4}}{4} + \frac{|S| - \sqrt{2n^k}}{4} \geq d(C^k_{n_2}),
\]

we are done.

(ii) This case is almost completely similar to the previous one. The only difference is the following: after defining \( A_i \) for \( 1 \leq i \leq u + v \), the subsequence \( T = S \cdot \left( \prod_{1 \leq i \leq u+v} A_i \right)^{-1} \) satisfies

\[
|T| = |S| - (u + 2v) \leq \frac{n^k + 2^k - 2}{2},
\]

since there are \( 2^k - 1 \) non-zero elements of \( C^k_{n_1} \) that are their own inverses. The remainder follows exactly the same steps as Case (i).

(iii) In this case, \( n_1 \) is odd and \( n_2 \) is even. This proof is also similar to Case (i). Suppose that \( S \in \mathcal{F}(C^k_{n_1} \oplus C^k_{n_2}) \) with \( |S| \geq 2d(C^k_{n_2}) + \frac{n_2^k(k2^{k+1}+1)+2\sqrt{2n^k+2k^{k+1}+3}}{4(k+1)} \).

For \( 1 \leq i \leq u \), let \( A_i \mid S \) such that \( |A_i| = 1 \), and its only term lies in \( \{\tilde{0}_{n_2}\} \oplus C^k_{n_2} \).

For \( 1 \leq i \leq v \), let \( A_{u+i} \mid S \cdot \left( \prod_{1 \leq i \leq u} A_i \right)^{-1} \) such that \( |A_{u+i}| = 2 \) and \( \sigma_{\{1,s\}}(A_{u+i}) \cap \{\tilde{0}_{n_1}\} \bordermath{\oplus} C^k_{n_2} \neq \emptyset \). Let \( T_1 = S \cdot \left( \prod_{1 \leq i \leq u+v} A_i \right)^{-1} \), so that

\[
|T_1| = |S| - (u + 2v) \leq 2^k \left( \frac{n^k - 1}{2} + (2^k - 1) \right) = 2^{k-1}(n^k + 1) - 1.
\]
For $1 \leq i \leq w$, let $A_{u+v+i} \mid T_1$ such that $|A_{u+v+i}| = 2$ and $\sigma_{\{1,s\}}(A_{u+v+i}) \cap \left[\{\overrightarrow{0}_{n_1}\} \oplus C_{n_2}^k\right] \neq \emptyset$. Let $T_2 = T_1 \cdot \left(\prod_{1 \leq i \leq u+v+w} A_i\right)^{-1}$, so that $|T_2| = |S| - (u + 2v + 2w) \leq \frac{n_1^k - 1}{2}$.

For $1 \leq i \leq t$, let $A_{u+v+w+i} \mid T_2$ be as in Lemma 2 in such way that $\sigma_{\{1,s\}}(A_{u+v+w+i}) \cap \left[\{\overrightarrow{0}_{n_1}\} \oplus C_{n_2}^k\right] \neq \emptyset$. We have $|S| - (u + 2v + 2w + 4t) < \sqrt{2n_1^k}$.

Since $d(C_n^k) = k$, the disjoint subsequences $A_i$ for $1 \leq i \leq u + v + w + t$ produce at least $u + v + \left\lceil \frac{w + t}{k+1} \right\rceil$ terms into $\{\overrightarrow{0}_{n_1}\} \oplus C_{n_2}^k$. Since

$$u + v + \frac{w + t}{k + 1} = \frac{u + 2v + 2w + 4t}{4(k + 1)} + \frac{u + 2v + 2w}{4(k + 1)} + \frac{k(u + 2v)}{2(k + 1)} + \frac{u}{2}$$

$$> \frac{|S| - \sqrt{2n_1^k}}{4(k + 1)} + \frac{|S| - \frac{n_1^k - 1}{2}}{4(k + 1)} + \frac{k}{2} \left(\frac{|S| - \frac{n_1^k - 1}{2}}{4(k + 1)} + \frac{k}{2}\right) \left(\frac{|S| - \frac{n_1^k - 1}{2}}{4(k + 1)} + \frac{k}{2}\right)$$

$$\geq d(C_{n_2}^k) + 1,$$

we are done.

\[\square\]

**Remark 1.** Notice that the previous theorem only used subsequences of length at most four in order to obtain a zero-sum in the first coordinate. It unfolds that it is possible to go further with a similar of Lemma 2, considering subsequences of length at most six, eight, etc. However, the main terms involving $d(C_{n_2}^k)$ and $n_1^k$ keep the same, the only change is that we get a larger denominator involving $\sqrt{2n_1^k}$, and we get other smaller terms of the form $\sqrt{2n_1^k}$, $\sqrt{2n_1^k}$, etc. For instance, if $n$ is odd and we consider subsequences up to six terms then we obtain the following inequality:

$$D_{\{1,s\}}(C_n^k) \leq 2d(C_{n_2}^k) + \frac{n_1^k - 1}{4} + \frac{\sqrt{2n_1^k}}{6} + \frac{\sqrt[3]{6n_1^k}}{3}.$$

Alternatively, we could use the remainder of at most $\sqrt{2n_1^k}$ terms to form subsequences of length much larger whose $\{1,s\}$-weighted sum is zero in the first coordinate and a better term (in some sense) in the second coordinate, but it also does not change the main terms. For these reasons, we decide to make it explicit until the first “error term”.
5. The Upper Bounds for \( n = 2n_1n_2 \)

Due to the projection \( \Psi \), we have \( C^k_n/(C^k_{n_1} \oplus C^k_{n_2}) \simeq C^k_2 \). Therefore, one can simply multiply each bound of Theorem 3 by \( k+1 \), obtaining upper bounds for \( D_{\{1,s\}}(C^k_n) \).

Indeed, it is possible to find \( k+1 \) disjoint subsequences whose \( \{1,s\} \)-weighted sums are \( \overline{0}_{n_1} \oplus \overline{0}_{n_2} \), that is, belong to \( C^k_2 \). Since \( d(C^k_2) = k \), there exists a \( \{1,s\} \)-zero-sum subsequence. This proves the following result.

**Proposition 2.** Let \( n \) and \( s \) be as in (1.3), and write \( n = 2n_1n_2 \) as in Lemma 1. Then

\[
D_{\{1,s\}}(C^k_n) \leq (k+1)D_{\{1,s\}}(C^k_{n_1n_2}).
\]

In the case that \( k = 1 \), \( n_2 \) is even and \( s \equiv n_2 + 1 \pmod{2n_2} \), we are able to get a precise bound. For this, we need the following lemmas:

**Lemma 3** (See [10, Theorem 11.1]). Let \( m \geq 3 \) be an integer and let \( S \in \mathcal{F}(C_m) \) be a zero-sum free sequence of length \( |S| > m/2 \). Then, there exists \( g \mid S \) such that

\[
v_g(S) \geq \max \left\{ m - 2|S| + 1, |S| - \left\lceil \frac{m-1}{3} \right\rceil \right\}.
\]

In addition, for a residue class \( r \pmod{m} \), denote by \( \varpi \) the integer such that \( 0 \leq \varpi \leq m - 1 \) and \( r \equiv \varpi \pmod{m} \). Then, there exists an integer \( t \) with \( \gcd(t,m) = 1 \) and \( \sum_{g \mid S} gt < m \). Moreover, for every \( 1 \leq k \leq \sum_{g \mid S} gt \), there exists \( T_k \mid S \) such that \( \sum_{g \mid T_k} gt = k \).

We also need the value of \( D_{\{1,m+1\}}(C_{2m}) \), where \( m \) is even (so that \((m+1)^2 \equiv 1 \pmod{2m})\).

**Theorem 4.** Let \( m \geq 4 \) be an even integer. Then

\[
D_{\{1,m+1\}}(C_{2m}) = m + 1.
\]

Moreover, each sequence of length \( m+1 \) has a proper subsequence whose \( \{1,m+1\} \)-weighted sum is 0 \( \in C_{2m} \).

**Proof.** The lower bound \( D_{\{1,m+1\}}(C_{2m}) \geq m + 1 \) follows the same steps than Theorem 2, hence it is only required to prove the upper bound. Let \( C_{2m} = \langle e \rangle \), so that \( C_m = \langle 2e \rangle \). Let

\[
S = \prod_{1 \leq i \leq m+1} (t_i e) \in \mathcal{F}(C_{2m})
\]

of length \( |S| = m + 1 \). We observe that \( D_{\{1,m+1\}}(C_m) = m = d(C_m) + 1 \), and that

\[
\begin{cases}
    (m+1)a \equiv a \pmod{2m} & \text{if } a \text{ is even,} \\
    (m+1)a \equiv m + a \pmod{2m} & \text{if } a \text{ is odd.}
\end{cases}
\]
Since \(|S| > m\), the projection \(\Psi_1 : C_{2m} \to C_m\) that maps \([(t \mod 2m)e]\) into \([(t \mod m)e]\) implies that there exists \(S_1 \mid S\) with \(|S_1| \leq m\) such that either \(0 \in \sigma_{1,m+1}(S_1)\) or \(m \in \sigma_{1,m+1}(S_1)\). If \(0 \in \sigma_{1,m+1}(S_1)\), then we are done. Therefore, we assume that \(m \in \sigma_{1,m+1}(S_1)\).

Notice that if \(m \mid S\), then \(|S \cdot (me)^{-1}| = m\), thus there exists \(S' \mid S \cdot (me)^{-1}\) with \(|S'| \leq m\) such that either \(0 \in \sigma_{1,m+1}(S')\) or \(m \in \sigma_{1,m+1}(S')\). If \(0 \in \sigma_{1,m+1}(S')\), then we are done. Otherwise, \(m \in \sigma_{1,m+1}(S')\), which implies that \(0 \in \sigma_{1,m+1}(S' \cdot me)\). If \(|S'| < m\), then \(S\) has a proper \(\{1, m + 1\}\)-weighted zero-sum subsequence over \(C_{2m}\). Thus, we may assume that \(|S'| = m\) and \(S'\) has no proper \(\{1, m + 1\}\)-weighted zero-sum subsequence. It follows from Lemma 3 that \(S' = (te)^{[m]}\) for some integer \(1 \leq t \leq m\) such that \(\gcd(t, m) = 1\) (in particular, \(t\) is odd). In this case, we have \(0 = (m + 1)te + te + \cdots + te \in \sigma_{1,m+1}(S')\), and again we are done.

Hence, we assume that \(m \nmid S\). Furthermore, suppose that \(S\) is zero-sum free over \(C_{2m}\) (otherwise, we are done). Since \(|S| > |C_{2m}|/2\), after a possible change of generators (as in Lemma 3), we assume that \(t_1, \ldots, t_{m+1}\) are positive integers in \([1, 2m - 1]\) such that \(\sum_{i=1}^{m+1} t_i \leq 2m - 1\).

In this way, \(S_1\) can be taken as an unweighted sum, that is, \(\sigma(S_1) = me\). If \(S_1\) has some odd term, say \(ae \mid S_1\), then

\[(m + 1)a + \sum_{t_i \in (S_1 \cup (ae)^{-1})} t_i \equiv 0 \pmod{2m},\]

hence, we are done. Otherwise, we assume that every term of any sequence \(S_1\) with \(\sigma(S_1) = me\) is even.

**CLAIM:** \((2e) \mid S_1\) and \(e^{[2]} \mid S \cdot S_1^{-1}\).

If this is the case, we just consider \(S_2 = (S_1 \cdot e^{[2]}) \cdot (2e)^{-1}\), therefore, we are done.

**Proof of the claim.** If \(e^{[2]} \mid S\), then

\[t_1 + \cdots + t_{m+1} \geq 1 + 2(m - 1) = 1 + 2m > 2m,\]

a contradiction. If \(2e \nmid S_1\) then \(t_i e \mid S_1\) implies \(t_i \geq 4\) is even. We have that \(4|S_1| \leq \sum_{t_i \mid S_1} t_i = m\), hence \(|S_1| \leq \frac{m}{4}\). Therefore

\[|S \cdot S_1^{-1}| = |S| - |S_1| \geq m + 1 - \frac{m}{4} = \frac{3m}{4} + 1,\]

which implies

\[\sum_{t_i \mid S \cdot S_1^{-1}} t_i \geq \frac{3m}{4} + 1.\]
Let \( v_1 = \pi_e(S) = \pi_e(S \cdot S_1^{-1}) \). Since \( \sum_{t \in S \cdot S_1^{-1}} t_i \leq m - 1 \) and \( |S_1| \leq \frac{n}{3} \), the average term of the subsequence \( S \cdot (S_1 \cdot e^{[v_1]}[1]) \) is at least 2. Therefore

\[
\frac{m - 1 - v_1}{\frac{3m}{4} + 1 - v_1} \geq \frac{\sum_{t \in S \cdot S_1^{-1}} t_i - v_1}{|S \cdot S_1^{-1}| - v_1} \geq 2,
\]

which implies \( v_1 \geq \frac{m}{2} + 3 \). If \( |S_1| = 1 \), then \( S_1 = m \ | \ S \), a contradiction. Therefore, \( |S_1| \geq 2 \). The Pigeonhole Principle ensures that there exists \( t_i \in S_1 \) such that \( 1 \leq t_i \leq \frac{n}{3} \). In this case, we replace the term \( t_i e \) by \( t_i \) copies of \( e \), which is possible since \( v_1 > \frac{m}{2} \geq t_i \). This leads us to a contradiction, hence, we are done.

\[ \square \]

**Corollary 1.** Let \( n = 2n_1n_2 \) and \( s \) be as in Lemma 1 such that \( n_2 \) is even and \( s \equiv n_2 + 1 \quad (\text{mod} \ 2n_2) \). Then

\[ D_{\{1,s\}}(C_n) = n_1n_2 + 1. \]

**Proof.** In view of Theorem 2, it is only required to prove the upper bound. Let \( S \in F(C_n) \) such that \( |S| = D_{\{1,s\}}(C_n) + 1 = n_1n_2 + 1 \). Since \( |S| \geq n_2 + 1 \), Theorem 4 ensures that there exists a subsequence \( T_1 \ | \ S \) with \( |T_1| \leq n_2 \) whose \( \{1,s\} \)-weighted sum belongs to the subgroup \( C_{n_1} \). Now construct \( T_2 \ | \ S \cdot T_1^{-1} \) with the same property. Inductively, for \( j \leq n_1 \), we construct \( T_j \ | \ S \cdot (T_1 \cdots T_{j-1})^{-1} \) with \( |T_j| \leq n_2 \) such that an \( \{1,s\} \)-weighted sum belongs to \( C_{n_1} \). Since \( d(C_{n_1}) = n_1 - 1 \), this corollary follows.

\[ \square \]

6. The Tightness of the Bounds

For \( n = n_1n_2 \), if \( n_2 \) is odd and either \( n_1 > n_2 \) or \( n_1 \) is even, then the second bound of Theorem 1 is useful only when \( n_2 \geq \lfloor \log_2(n_1) \rfloor \). Otherwise, the first bound is more useful. The worst bounds are those where \( n_2 \) is odd and \( n_1 < n_2 \), in which case we were unable to find a good lower bound. On the other hand, Proposition 1 is more useful than Theorem 3 only when \( n_1 \) is very small compared to \( n_2 \); otherwise, Theorem 3 turns out to be of better use.

For \( n = 2n_1n_2 \), we provided a good lower bound in terms of the previous case. In the particular case \( k = 1 \), the problem splits into two subcases: \( n_2 \) even with \( s \equiv n_2 + 1 \quad (\text{mod} \ 2n_2) \) (whose exact value has been established in Corollary 1) and \( n_1 \) even with \( s \equiv n_1 - 1 \quad (\text{mod} \ 2n_1) \). In both cases, the values of \( D_{\{1,s\}}(C_n) \) depend on \( D_{\{1,s\}}(C_{n_1}) \) and \( D_{\{1,s\}}(C_{n_2}) \). On the other hand, the upper bound is asymptotically sharp for large values of \( k \).

**Notation.** Let \( f, g \) be positive functions. As usual, we use the asymptotic Bachmann-Landau notations:
\begin{itemize}
  \item $f(n) \sim g(n)$ means that $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$,
  \item $f(n) = O(g(n))$ means that there exists $c > 0$ such that $f(n) \leq c \cdot g(n)$ for every positive integer $n$, and
  \item $f(n) = o(g(n))$ means that $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.
\end{itemize}

**Conjecture 1.** Let $n$ and $s$ be as in (1.3), and write $n = n_1 n_2$ as in Lemma 1. We expect that

\[ D_{\{1,s\}}(C_n^k) = \begin{cases} 2kn_2 + O(k \cdot \log_2(n_1)) & \text{if } n_2 \text{ is odd,} \\ kn_2 + O(k \cdot \log_2(n_1)) & \text{if } n_2 \text{ is even,} \end{cases} \]

\[ D_{\{1,s\}}(C_n) = \begin{cases} 2n_2 + \lfloor \log_2(n_1) \rfloor & \text{if } n_2 \text{ is odd,} \\ n_2 + \lfloor \log_2(n_1) \rfloor & \text{if } n_2 \text{ is even.} \end{cases} \]

In the case that $n = 2n_1 n_2$, we expect that

\[ D_{\{1,s\}}(C_n^k) = k \cdot D_{\{1,s\}}(C_{n_1 n_2}^k) + 1. \]

Remember that $d(C_n) = n - 1$ and $d(C_n^2) = 2n - 2$. It is conjectured that if $k \geq 3$ then equality in (1.1) holds for $C_n^k$, that is, $d(C_n^k) = k(n - 1)$. Girard [9] showed that $d(C_n^k) \sim kn$ as $n \to \infty$. As a consequence of the upper bounds and Girard’s result, we have the following.

**Corollary 2.** Let $n$ and $s$ be as in (1.3), and write $n = n_1 n_2$ as in Lemma 1.

(i) With $n_1$ fixed, it follows that

\[ D_{\{1,s\}}(C_n^k) = \begin{cases} 2kn_2(1 + o(1)) & \text{if } n_2 \text{ is odd,} \\ kn_2(1 + o(1)) & \text{if } n_2 \text{ is even.} \end{cases} \]

(ii) With $n_2$ fixed, it follows that

\[ D_{\{1,s\}}(C_n^k) = O(k \cdot \log_2(n_1)). \]

The next proposition and corollary yield the exact values of $D_{\{1,s\}}(C_n)$ when $n_1 \in \{3, 5\}$ and $n_2$ is even for $n = n_1 n_2$ and for $n = 2n_1 n_2$, respectively, proving particular cases of Conjecture 1. It is worth mentioning that Theorems 1(i) and 3(iii) yield

\[ n_2 + \lfloor \log_2(n_1) \rfloor \leq D_{\{1,s\}}(C_n) \leq n_2 + \lfloor \log_2(n_1) \rfloor + 1 \]

provided $n = n_1 n_2$, $n_2$ is even and $n_1 \in \{3, 5\}$, and similar intervals for the cases where $n = 2n_1 n_2$. 

Proposition 3. Let $n$ and $s$ be as in (1.3), and write $n = n_1 n_2$ as in Lemma 1, where $n_1 \in \{3, 5\}$ and $n_2$ even. Then

$$D_{\{1,n\}}(C_n) = n_2 + \lfloor \log_2(n_1) \rfloor.$$ 

Proof. The lower bound $D_{\{1,n\}}(C_n) \geq n_2 + \lfloor \log_2(n_1) \rfloor$ is given by item (i) of Theorem 1. Therefore we just need to prove that if $S \in \mathcal{F}(C_{n_1} \oplus C_{n_2})$ and $|S| = n_2 + \lfloor \log_2(n_1) \rfloor$, then $S$ is not $\{1, s\}$-zero-sum free. Let $C_{n_1} = \langle e_1 \rangle$, $C_{n_2} = \langle e_2 \rangle$, $C_{\mathbb{Z}/2} = \langle 2e_2 \rangle$, and $e_2 + C_{\mathbb{Z}/2} = \{ te_2 \in C_{n_2} : t \text{ is odd} \}$.

Let $k \in \mathbb{Z}$ such that $|S \cap (\{0\} \oplus C_{n_2})| = n_2 - k$. Since $d(C_{n_2}) = n_2 - 1$, if $k \leq 0$, then, we are done. Hence, assume $k \geq 1$.

(i) CASE $n_1 = 3$, $|S| = n_2 + 1$: It holds $|S \cap (\{e_1, 2e_1\} \oplus C_{n_2})| = k + 1 \geq 2$.

From these, it is possible to obtain a zero-sum in the first coordinate using any subsequence formed by two terms $(g_1, h_1) \cdot (g_2, h_2) \mid S \cap (\{e_1, 2e_1\} \oplus C_{n_2})$.

In fact, if $g_1 = g_2$, then $(g_1 - g_2, h_1 + h_2) = (0, h_1 + h_2)$, and if $g_1 = e_1$ and $g_2 = 2e_1$ then $(g_1 + g_2, h_1 + h_2) = (0, h_1 + h_2)$.

Let $S = T_1 \cdot T_2 \cdot T_3$, where

$$T_1 \mid S \cap (\{0\} \oplus C_{n_2}),$$
$$T_2 \mid S \cap (\{e_1, 2e_1\} \oplus C_{\mathbb{Z}/2}),$$
$$T_3 \mid S \cap (\{e_1, 2e_1\} \oplus (e_2 + C_{\mathbb{Z}/2})).$$

Then it is possible to obtain at least

$$\left\lfloor \frac{|T_1|}{2} \right\rfloor + \left\lfloor \frac{|T_2|}{2} \right\rfloor + \left\lfloor \frac{|T_3|}{2} \right\rfloor = \left\lfloor \frac{n_2 - k}{2} \right\rfloor + \left\lfloor \frac{|T_2|}{2} \right\rfloor + \left\lfloor \frac{k + 1 - |T_2|}{2} \right\rfloor \geq \frac{n_2 - k - 1}{2} + \frac{|T_2| - 1}{2} + \frac{k + 1 - |T_2| - 1}{2} \geq \frac{n_2}{2} - 1$$

elements in $\{0\} \oplus C_{\mathbb{Z}/2}$. Since $d(C_{\mathbb{Z}/2}) = \frac{n_2}{2} - 1$, if the latter inequality is strict, then we are done. Otherwise, $n_2 - k$, $|T_2|$ and $k + 1 - |T_2|$ are all odd, hence $k$ is odd. Therefore, after removing the sums of those pairs, it remains one term from each subsequence $T_1$, $T_2$ and $T_3$. Let $(0, t_1 e_2) \mid T_1$, $(ue_1, 2t_2 e_2) \mid T_2$ and $(ve_1, (2t_3 - 1)e_2) \mid T_3$ these remainder terms. If $t_1$ is even, then we have one more term in $\{0\} \oplus C_{\mathbb{Z}/2}$, and we are done. Hence, we suppose that $t_1$ is odd. In this case,

$$(0, (t_1 + 2t_2 + 2t_3 - 1)e_2) \in \sigma_{\{1,n\}}((0, t_1 e_2) \cdot (ue_1, 2t_2 e_2) \cdot (ve_1, (2t_3 - 1)e_2)),$$

and again we have one more element in $\{0\} \oplus C_{\mathbb{Z}/2}$. Therefore, we are done.
CASE \( n_1 = 5, |S| = n_2 + 2 \): As in the previous case, we have
\[
|S \cap (\{e_1, 2e_1, 3e_1, 4e_1\} \oplus C_{n_2})| = k + 2 \geq 3.
\]

Define \( S = T_1 \cdot T_2 \cdot T_3 \cdot T_4 \cdot T_5 \), where
\[
T_1 \mid S \cap (\{0\} \oplus C_{n_2}),
T_2 \mid S \cap (\{e_1, 4e_1\} \oplus C_{n_2 + n_2}),
T_3 \mid S \cap (\{e_1, 4e_1\} \oplus C_{2n_2}),
T_4 \mid S \cap (\{2e_1, 3e_1\} \oplus C_{n_2 + n_2}),
T_5 \mid S \cap (\{2e_1, 3e_1\} \oplus C_{2n_2}).
\]

Notice that the terms in each \( T_i \) can be grouped in pairs in order to obtain products in \( \{0\} \oplus C_{2n_2} \). Since \( n_2 \) and \( |S| \) are even, \( |T_2| + |T_3| + |T_4| + |T_5| = k + 2 \) and
\[
\sum_{i=1}^{5} \left\lfloor \frac{|T_i|}{2} \right\rfloor \geq \frac{n_2 - k - 1}{2} + \sum_{i=2}^{5} \left\lfloor \frac{|T_i| - 1}{2} \right\rfloor + \frac{1}{2} = \frac{n_2}{2} - 1,
\]
it is possible to obtain at least \( \frac{n_2}{2} - 1 \) disjoint subsums in \( \{0\} \oplus C_{2n_2} \). If the previous inequality is strict, then we are done. Otherwise, it remains at least one term in four of the subsequences \( T_i \) for \( 1 \leq i \leq 5 \). We observe that one term from each \( T_2 \) and \( T_3 \) generates a subsum in \( \{0\} \oplus C_{n_2} \), as well as one term from each \( T_4 \) and \( T_5 \) generates a subsum in \( \{0\} \oplus C_{n_2} \). If \( |T_1| \) is even, then it remains the terms from \( T_2, T_3, T_4 \) and \( T_5 \), and we obtain two subsums in \( \{0\} \oplus C_{n_2} \), which produce one more subsum in \( \{0\} \oplus C_{2n_2} \). Otherwise, either \( T_2 \) and \( T_3 \) generate one more subsum in \( \{0\} \oplus C_{n_2} \) or \( T_4 \) and \( T_5 \) generate one more subsum in \( \{0\} \oplus C_{n_2} \), and adding the remainder term from \( T_1 \) led us to one more subsum in \( \{0\} \oplus C_{2n_2} \). Hence, we are done.

\( \square \)

**Remark 2.** The previous argument can be adapted to higher values of \( n_1 \), but the inequality obtained will be weaker. Thereby, it will be required to produce more subsums in \( \{0\} \oplus C_{2n_2} \) using at most one remainder term from each subsequence \( T_1, \ldots, T_{n_1} \). This creates several cases as \( n_1 \) grows.

From the previous proposition and Corollary 1, it follows that:

**Corollary 3.** Let \( n \) and \( s \) be as in (1.3), and write \( n = 2n_1n_2 \) as in Lemma 1, where \( n_1 \in \{3, 5\} \) and \( n_2 \) even. Then
\[
D_{(1,s)}(C_n) = n_2 + \lceil \log_2(n_1) \rceil + 1.
\]
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