TWO-DIMENSIONAL, THREE-DIMENSIONAL AND N-DIMENSIONAL RECURRENCE RELATIONS OF JACOBSTHAL

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Abstract

The Jacobsthal model is a second order recursive sequence. This work presents a discussion on the two-dimensional, three-dimensional and n-dimensional recurrence relations from the one-dimensional recursive model $J_{n+1} = J_n + 2J_{n-1}$, for all n in $\mathbb{N}$, with $J_0 = 0$ and $J_1 = 1$ being their initial terms. In view of the complexification process of the Jacobsthal sequence, the mathematical properties of the Jacobsthal numbers are considered as $J(n, m)$, $J(n, m, p)$ and $J(n_1, n_2, n_3, \cdots, n_n)$, allowing us to explore derived identities inherent in this model.

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1. Introduction

The Jacobsthal sequence is a linear recurrence sequence of second order, discussed in some work such as [1, 2, 3, 4, 6, 7, 8, 9, 12, 13], whose numbers are defined by the recurrence $J_{n+1} = J_n + 2J_{n-1}$, for all n in $\mathbb{N}$, with $J_0 = 0$ and $J_1 = 1$ being their initial terms. Thus, we have the following tabulation for the Jacobsthal numbers:

J_0	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9	J_{10}
0	1	1	3	5	11	21	43	85	171	341

Table 1: Jacobsthal numbers.

The characteristic equation of this sequence is described by:

$$x^2 - x - 2 = 0.$$

This characteristic equation has as roots $x_1 = 2$ and $x_2 = -1$, with $x_1 + x_2 = 1$, $x_1x_2 = -2$ and $x_1 - x_2 = 3$.

The process of complexifying these numbers is related to the addition of the imaginary unit i, increasing the dimensionality and its corresponding algebraic representation, as we can see in [5]. Thus, in this work, we will discuss aspects concatenated to recurrence relations and two-dimensional and three-dimensional identities are presented from the Jacobsthal one-dimensional recurrence relation.

From this, two-dimensional identities will be explored, with two variables m and n for the numbers $J(n, m)$, derived from the recurrence relations $J(n + 1, m) = J(n, m) + 2J(n-1, m)$ and $J(n, m+1) = J(n, m) + 2J(n, m-1)$, with the following defined initial values: $J(0, 0) = 0$; $J(1, 0) = 1$; $J(0, 1) = i$; and $J(1, 1) = 1 + i$. We also explore identities for numbers $J(n, m, p)$ based on three-dimensional recurrence relations $J(n+1, m, p) = J(n, m, p)+2J(n-1, m, p)$ and $J(n, m+1, p) = J(n, m, p)+ 2J(n, m-1, p)$, with the initial conditions: $J(0, 0, 0) = 0$; $J(1, 0, 0) = 1$; $J(0, 1, 0) = i$; $J(0, 0, 1) = q$; $J(1, 1, 0) = 1 + i$; $J(1, 0, 1) = 1 + q$; $J(1, 1, 1) = 1 + i + q$ and $J(0, 1, 1) = i + q$, with i and q being imaginary units. However, this work presents only two-dimensional and three-dimensional numbers and their extension from the integers.

2. Jacobsthal Two-Dimensional Recurrence Relations

In this section, we expand the dimension of this sequence and insert the imaginary unit i, presenting the two-dimensional relationship of the Jacobsthal sequence based on the works of [10, 11].

Definition 1. The numbers in the form $J(n,m)$ are defined as the *Jacobsthal two-dimensional sequence*, satisfying the following recurrence relations:

$$\begin{cases} J(n+1,m) &= J(n,m)+2J(n-1,m) \\ J(n,m+1) &= J(n,m)+2J(n,m-1) \end{cases}$$

with the initial conditions: $J(0,0)=0$; $J(1,0)=1$; $J(0,1)=i$; and $J(1,1)=1+i$ where $i^2=-1$.

Based on this statement, we present some theorems supported by Definition 1.

Theorem 1. *For any integer $n \geq 0$, we have $J(n,0)=J_n$.*

Proof. By the recurrence $J(n+1,m)=J(n,m)+2J(n-1,m)$ and with the initial values $J(0,0)=0=J_0$, $J(1,0)=1=J_1$, the second finite induction principle can be applied to the value of $m=0$, varying the value of n, obtaining:

$$\begin{aligned} J(2,0) &= J(1,0)+2J(0,0)=1=J_2 \\ J(3,0) &= J(2,0)+2J(1,0)=3=J_3 \\ J(4,0) &= J(3,0)+2J(2,0)=5=J_4 \\ \vdots &= \vdots \\ J(n-1,0) &= J_{n-1} \\ J(n,0) &= J_n \\ J(n+1,0) &= J(n,0)+2J(n-1,0)=J_n+2J_{n-1}=J_{n+1}. \end{aligned}$$

□

Theorem 2. *For any integer $m \geq 0$, we have $J(0,m)=J_m i$.*

Proof. The proof of this theorem is performed in an analogous way to Theorem 1. □

Theorem 3. *For any integer $n \geq 0$, we have $J(n,1)=J_n+(J_n+J_{n-1})i$.*

Proof. The proof of this theorem is performed in an analogous way to Theorem 1. □

Theorem 4. *For any integer $m \geq 0$, we have $J(1,m)=J_{m+1}+J_m i$.*

Proof. The proof of this theorem is performed in an analogous way to Theorem 1. □

Theorem 5. *For n, m in $\mathbb{N}$, the Jacobsthal two-dimensional numbers can be described by:*

$$J(n,m)=J_nJ_{m+1}+J_{n+1}J_m i.$$

Proof. By defining the value of the natural number n, we can perform a proof by means of induction on m. For $n = 0$, this proof is supported by Theorem 2. Theorem 4, on the other hand, refers to $n = 1$. Therefore, when we define the value of $n = 2$ and vary m, we obtain:

$$\begin{array}{rcl} J(2,0) & = & 1 \\ J(2,1) & = & 1+2i \\ J(2,2) & = & J(2,1)+2J(2,0) = 3+2i = J_3+2J_2 i \\ J(2,3) & = & J(2,2)+2J(2,1) = 5+6i = J_4+2J_3 i \\ (2,4) & = & J(2,3)+2J(2,2) = 11+10i = J_5+2J_4 i \\ \vdots & = & \vdots \\ J(2,m-1) & = & J_m + 2J_{m-1} i \\ J(2,m) & = & J_{m+1} + 2J_m i \\ J(2,m+1) & = & J(2,m) + 2J(2,m-1) \\ & = & J_{m+1} + 2J_m i + 2[J_m + 2J_{m-1} i] \\ & = & J_{m+2} + 2J_{m+1} i. \end{array}$$

Finally, when we define $n = 3$ and vary m, we have:

$$\begin{array}{rcl} J(3,0) & = & 3 \\ J(3,1) & = & 3+4i \\ J(3,2) & = & J(3,1)+2J(3,0) = 9+4i = 3J_3+4J_2 i \\ J(3,3) & = & J(3,2)+2J(3,1) = 15+12i = 3J_4+4J_3 i \\ J(3,4) & = & J(3,3)+2J(3,2) = 33+20i = 3J_5+4J_4 i \\ \vdots & = & \vdots \\ J(3,m-1) & = & 3J_m + 4J_{m-1} i \\ J(3,m) & = & 3J_{m+1} + 4J_m i \\ J(3,m+1) & = & J(3,m) + 2J(3,m-1) \\ & = & 3J_{m+1} + 4J_m i + 2[3J_m + 4J_{m-1} i] \\ & = & 3J_{m+2} + 4J_{m+1} i. \end{array}$$

Therefore, we can observe that to generalize the two-dimensional Jacobsthal relation, we have:

$$\begin{aligned} J(1,m) &= J_1 J_{m+1} + J_2 J_m i \\ J(2,m) &= J_2 J_{m+1} + J_3 J_m i \\ J(3,m) &= J_3 J_{m+1} + J_4 J_m i \\ \vdots &= \vdots \\ J(n,m) &= J_n J_{m+1} + J_{n+1} J_m i. \end{aligned}$$

□

3. Two-Dimensional Jacobsthal Identities for Integers

In this section, we explore some two-dimensional identities, noting properties related to the numbers present in the Jacobsthal sequence.

Theorem 6. *The sum of the first n numbers $J(n,m)$ with index n greater than zero is described by:*

$$\sum_{l=1}^{n} 2J(l,m) = J_{m+1}(J_{n+1} - 1) + (J_n + J_{n+1} - 1)J_m i.$$

Proof. This theorem can be proved by induction, analogously to the procedures presented in Section 2. □

Theorem 7. *For the numbers $J(n,m)$ with any integer indices we can write:*

$$J(-n,-m) = \frac{(-1)^{n+m+3}}{2^{n+m+1}} J(n,m).$$

Proof. Using the relation $J(n,m) = J_n J_{m+1} + (J_{n-1} + J_n)J_m i$ and performing its extension to integer indices, we have that:

$$\begin{aligned} J(-n,-m) &= J_{-n} J_{-(m+1)} + J_{-(n-1)} J_{-m} i \\ &= \frac{(-1)^{n+1}}{2^n} J_n . \frac{(-1)^{m+2}}{2^{m+1}} J_{m+1} + \frac{(-1)^{n+2}}{2^{n+1}} J_{n+1} . \frac{(-1)^{m+1}}{2^m} J_m i \\ &= \frac{(-1)^{n+m+3}}{2^{n+m+1}} J_n J_{m+1} + \frac{(-1)^{n+m+3}}{2^{n+m+1}} J_{n+1} J_m i \\ &= \frac{(-1)^{n+m+3}}{2^{n+m+1}} (J_n J_{m+1} + J_{n+1} J_m i) \\ &= \frac{(-1)^{n+m+3}}{2^{n+m+1}} J(n,m). \end{aligned}$$

□

Theorem 8. *The sum of the first n numbers $J(n,m)$ with index n less than zero can be described by:*

$$\sum_{l=1}^{n} 2J(-l,-m) \quad = \quad \frac{(-1)^{m+3}}{2^{m+1}} J_m i + \frac{(-1)^{n+m+4}}{2^{n+m+2}} (J_{m+1}J_{n+1} + J_{n+2}J_m i).$$

Proof. Starting from the relation $J(-n+1,-m) = J(-n,-m) + 2J(-n-1,-m)$, there is the possibility of rewriting it as:

$$\begin{aligned}
2J(-n-1,-m) &= J(-n+1,-m) - J(-n,-m)\\
2J(-2,-m) &= J(0,-m) - J(-1,-m)\\
2J(-3,-m) &= J(-1,-m) - J(-2,-m)\\
2J(-4,-m) &= J(-2,-m) - J(-3,-m)\\
2J(-5,-m) &= J(-3,-m) - J(-4,-m)\\
\vdots &= \vdots\\
2J(-n-3,-m) &= J(-n-1,-m) - J(-n-2,-m)\\
2J(-n-2,-m) &= J(-n,-m) - J(-n-1,-m)\\
2J(-n-1,-m) &= J(-n+1,-m) - J(-n,-m).
\end{aligned}$$

Using the equations $J(0,-m) = \frac{(-1)^{m+3}}{2^{m+1}} J_m i$ and $J(-n,-m) = \frac{(-1)^{n+m+3}}{2^{n+m+1}} J(n,m)$, we have:

$$\begin{aligned}
\sum_{l=1}^{n} 2J(-l,-m) &= 2J(-2,-m) + 2J(-3,-m) + 2J(-4,-m) + 2J(-5,-m)\\
&\quad + \cdots + 2J(-n-3,-m) + 2J(-n-2,-m) + 2J(-n-1,-m)\\
&= J(0,-m) + J(-n+1,-m)\\
&= \frac{(-1)^{m+3}}{2^{m+1}} J_m i + \frac{(-1)^{n+m+4}}{2^{n+m+2}} (J_{m+1}J_{n+1} + J_{n+2}J_m i)\\
&= \frac{(-1)^{m+3}}{2^{m+1}} J_m i + (J_{m+1}J_{n+1} + (1 + J_{n+2})J_m i).
\end{aligned}$$

□

Theorem 9. *For any integer indexes, we have:*

$J(n+1,m+1) = J(n,m) + 2J(n-1,m) + 2[J(n,m-1) + 2J(n-1,m-1)].$

Proof. This theorem can be proved by induction, analogously to the procedures presented in Section 2. □

4. Jacobsthal Three-Dimensional Recurrence Relations

Definition 2. The numbers in the form $J(n, m, p)$ are defined as the *three-dimensional Jacobsthal sequence* and satisfy the following recurrence relations:

$$\begin{cases} J(n+1, m, p) &= J(n, m, p) + 2J(n-1, m, p) \\ J(n, m+1, p) &= J(n, m, p) + 2J(n, m-1, p) \\ J(n, m, p+1) &= J(n, m, p) + 2J(n, m, p-1) \end{cases}$$

with the initial conditions: $J(0, 0, 0) = 0$; $J(1, 0, 0) = 1$; $J(0, 1, 0) = i$; $J(0, 0, 1) = q$; $J(1, 1, 0) = 1 + i$; $J(1, 0, 1) = 1 + q$; $J(1, 1, 1) = 1 + i + q$ and $J(0, 1, 1) = i + q$ where $i^2 = -1$.

Starting from this point, we point out some theorems related to Definition 2.

Theorem 10. *For any integer $n \geq 0$, we have: $J(n, 0, 0) = J_n$.*

Proof. By the recurrence relation $J(n+1, m, p) = J(n, m, p) + 2J(n-1, m, p)$ and given the initial values $J(0, 0, 0) = 0 = J_0$, $J(1, 0, 0) = 1 = J_1$, according to the principle of finite induction, in which the values of $m = 0$ and $p = 0$ are defined, varying the value of n, we obtain:

$$\begin{aligned} J(2, 0, 0) &= J(1, 0, 0) + 2J(0, 0, 0) = 1 = J_2 \\ J(3, 0, 0) &= J(2, 0, 0) + 2J(1, 0, 0) = 3 = J_3 \\ J(4, 0, 0) &= J(3, 0, 0) + 2J(2, 0, 0) = 5 = J_4 \\ \vdots &= \vdots \\ J(n-1, 0, 0) &= J_{n-1} \\ J(n, 0, 0) &= J_n \\ J(n+1, 0, 0) &= J(n, 0, 0) + 2J(n-1, 0, 0) = J_n + 2J_{n-1} = J_{n+1}. \end{aligned}$$

□

Theorem 11. *For any integer $m \geq 0$, we have $J(0, m, 0) = J_m i$.*

Theorem 12. *For any integer $p \geq 0$, we have $J(0, 0, p) = J_p q$.*

Theorem 13. *For any integer $n \geq 0$, we have $J(n, 1, 0) = J_n + J_{n+1} i$.*

Theorem 14. *For any integer $n \geq 0$, we have $J(n, 0, 1) = J_n + J_{n+1} q$.*

Theorem 15. *For any integer $n \geq 0$, we have $J(n, 1, 1) = J_n + J_{n+1} i + 2J_{n-1} q$.*

Theorem 16. *For any integer $m \geq 0$, we have $J(1, m, 0) = J_{m+1} + J_m i$.*

Theorem 17. *For any integer $m \geq 0$, we have $J(0, m, 1) = J_m i + J_{m+1} q$.*

Theorem 18. *For any integer* $m \geq 0$*, we have* $J(1,m,1) = J_{m+1} + J_m i + J_{m+1} q$.

Theorem 19. *For any integer* $p \geq 0$*, we have* $J(1,0,p) = J_{p+1} + J_p q$.

Theorem 20. *For any integer* $p \geq 0$*, we have* $J(0,1,p) = J_{p+1} i + J_p q$.

The proofs of Theorems 11 - 20 theorems are omitted as they are very similar to that of Theorem 10.

Theorem 21. *For any integer* $p \geq 0$*, we have* $J(1,1,p) = J_{p+1} + J_{p+1} i + J_p q$.

Proof. By the recurrence relation $J(n,m,p+1) = J(n,m,p) + 2J(n,m,p-1)$ and previously defining the initial values as $J(1,1,1) = 1 + i + j$, $J(1,1,0) = 1 + i$, we can apply the second principle of finite induction, setting the value $n = 1$, $m = 1$ and varying $p = 0, 1, 2, ...$, in the following way:

$$\begin{aligned}
J(1,1,2) &= J(1,1,1) + 2J(1,1,0) = 3 + 3i + q = J_3 + J_3 i + J_2 q \\
J(1,1,3) &= J(1,1,2) + 2J(1,1,1) = 5 + 5i + 3q = J_4 + J_4 i + J_3 q \\
J(1,1,4) &= J(1,1,3) + 2J(1,1,2) = 11 + 11i + 5q = J_5 + J_5 i + J_4 q \\
\vdots &= \vdots \\
J(1,1,p-1) &= J_p + J_p i + J_{p-1} q \\
J(1,1,p) &= J_{p+1} + J_{p+1} i + J_p q \\
J(1,1,p+1) &= J(1,1,p) + 2J(1,1,p-1) \\
&= J_{p+1} + J_{p+1} i + J_p q + 2(J_p + J_p i + J_{p-1} q) \\
&= J_{p+2} + J_{p+2} i + J_{p+1} q.
\end{aligned}$$

□

Theorem 22. *For* n, m, p *in* $\mathbb{N}$*, the three-dimensional Jacobsthal numbers can be described by:* $J(n,m,p) = J_n J_{m+1} J_{p+1} + J_{n+1} J_m J_{p+1} i + J_{n+1} J_{m+1} J_p q$.

Proof. Setting $n = 0$, $p = 2$ and varying m, we obtain:

$$\begin{aligned}
J(0,2,2) &= J(0,1,2) + 2J(0,0,2) = 3i + 3q = 3J_2 i + J_3 q \\
J(0,3,2) &= J(0,2,2) + 2J(0,1,2) = 9i + 5q = 3J_3 i + J_4 q \\
J(0,4,2) &= J(0,3,2) + 2J(0,2,2) = 15i + 11q = 3J_4 i + J_5 q \\
\vdots &= \vdots \\
J(0,m-1,2) &= 3J_{m-1} i + J_m q \\
J(0,m,2) &= 3J_m i + J_{m+1} q \\
J(0,m+1,2) &= J(0,m,2) + 2J(0,m-1,2) \\
&= 3J_m i + J_{m+1} q + 2(3J_{m-1} i + J_m q) \\
&= 3J_{m+1} i + J_{m+2} q.
\end{aligned}$$

Setting $n = 1$, $p = 2$ and varying m, we obtain:

$$\begin{aligned}
J(1,2,2) &= J(1,1,2) + 2J(1,0,2) = 9 + 3i + 3q = 3J_3 + 3J_2 i + J_3 q \\
J(1,3,2) &= J(1,2,2) + 2J(1,1,2) = 15 + 9i + 5q = 3J_4 + 3J_3 i + J_4 q \\
J(1,4,2) &= J(1,3,2) + 2J(1,2,2) = 33 + 15i + 11q = 3J_5 + 3J_4 i + J_5 q \\
\vdots &= \vdots \\
J(1,m-1,2) &= 3J_m + 3J_{m-1} i + J_m q \\
J(1,m,2) &= 3J_{m+1} + 3J_m i + J_{m+1} q \\
J(1,m+1,2) &= J(1,m,2) + 2J(1,m-1,2) \\
&= 3J_{m+1} + 3J_m i + J_{m+1} q + 2(3J_m + 3J_{m-1} i + J_m q) \\
&= 3J_{m+2} + 3J_{m+1} i + J_{m+2} q.
\end{aligned}$$

Setting $n = 2$, $p = 2$ and varying m, we obtain:

$$\begin{aligned}
J(2,2,2) &= J(2,1,2) + 2J(2,0,2) = 9 + 9i + 9q = 3J_3 + 9J_2 + 2J_3 \\
J(2,3,2) &= J(2,2,2) + 2J(2,1,2) = 15 + 27i + 15q = 3J_4 + 9J_3 + 3J_4 \\
J(2,4,2) &= J(2,3,2) + 2J(2,2,2) = 33 + 45i + 33q = 3J_5 + 9J_4 + 3J_5 \\
\vdots &= \vdots \\
J(2,m-1,2) &= 3J_m + 9J_{m-1} i + 3J_m q \\
J(2,m,2) &= 3J_{m+1} + 9J_m i + 3J_{m+1} q \\
J(1,m+1,2) &= J(2,m,2) + 2J(2,m-1,2) \\
&= 3J_{m+1} + 9J_m i + 3J_{m+1} q + 2(3J_m + 9J_{m-1} i + 3J_m q) \\
&= 3J_{m+2} + 9J_{m+1} i + 3J_{m+2} q.
\end{aligned}$$

Therefore, we can observe that:

$$\begin{aligned}
J(0,m,2) &= 3J_m i + J_{m+1} q = J_0 J_{m+1} J_3 + J_1 J_m J_3 i + J_2 J_{m+1} J_2 q \\
J(1,m,2) &= 3J_{m+1} + 3J_m i + J_{m+1} q = J_1 J_{m+1} J_3 + J_2 J_m J_3 i + J_3 J_{m+1} J_2 q \\
J(2,m,2) &= 3J_{m+1} + 9J_m i + 3J_{m+1} q = J_2 J_{m+1} J_3 + J_3 J_m J_3 i + J_3 J_{m+1} J_2 q \\
\vdots &= \vdots \\
J(n,m,2) &= J_n J_{m+1} J_3 + J_{n+1} J_m J_3 i + J_{n+1} J_{m+1} J_2 q.
\end{aligned}$$

Thus, we can infer that the generalization of the Jacobsthal three-dimensional recurrence relation can be given by:

$$\begin{aligned} J(n,m,1) &= J_nJ_{m+1}J_2 + J_{n+1}J_mJ_2i + J_{n+1}J_{m+1}J_1q \\ J(n,m,2) &= J_nJ_{m+1}J_3 + J_{n+1}J_mJ_3i + J_{n+1}J_{m+1}J_2q \\ J(n,m,3) &= J_nJ_{m+1}J_4 + J_{n+1}J_mJ_4i + J_{n+1}J_{m+1}J_3q \\ \vdots &= \vdots \\ J(n,m,p) &= J_nJ_{m+1}J_{p+1} + J_{n+1}J_mJ_{p+1}i + J_{n+1}J_{m+1}J_pq. \end{aligned}$$

□

5. Three-Dimensional Jacobsthal Identities for Integers

In this section we explore some three-dimensional identities, noting the properties inherent to the numbers that are part of the Jacobsthal sequence, as well as their extension from the integers.

Theorem 23. *The sum of the first n numbers $J(n,m,p)$ with indexes greater than zero can be described by:*

$$\begin{aligned} \sum_{l=1}^{n} 2J(l,m,p) &= (J_{n+1}-1)J_{m+1}J_{p+1} + (J_{n+2}-1)J_mJ_{p+1}i \\ &\quad +(J_{n+2}-1)J_{m+1}J_pq. \end{aligned}$$

Proof. Given the recurrence relation $J(n+1,m,p) = J(n,m,p) + 2J(n-1,m,p)$, we can rewrite it as:

$$\begin{aligned} 2J(n-1,m,p) &= J(n+1,m,p) - J(n,m,p) \\ 2J(0,m,p) &= J(2,m,p) - J(1,m,p) \\ 2J(1,m,p) &= J(3,m,p) - J(2,m,p) \\ 2J(2,m,p) &= J(4,m,p) - J(3,m,p) \\ 2J(3,m,p) &= J(5,m,p) - J(4,m,p) \\ 2J(4,m,p) &= J(6,m,p) - J(5,m,p) \\ 2J(5,m,p) &= J(7,m,p) - J(6,m,p) \\ \vdots &= \vdots \\ 2J(n-4,m,p) &= J(n-2,m,p) - J(n-3,m,p) \\ 2J(n-3,m,p) &= J(n-1,m,p) - J(n-2,m,p) \\ 2J(n-2,m,p) &= J(n,m,p) - J(n-1,m,p) \\ 2J(n-1,m,p) &= J(n+1,m,p) - J(n,m,p). \end{aligned}$$

Using $J(1,m,p) = J_{m+1}J_{p+1} + J_mJ_{p+1}i + J_{m+1}J_pq$, $J(n,m,p) = J_nJ_{m+1}J_{p+1} + J_{n+1}J_mJ_{p+1}i + J_{n+1}J_{m+1}J_pq$, we have:

$$\begin{aligned}
\sum_{l=1}^{n} 2J(l,m,p) &= 2J(0,m,p) + 2J(1,m,p) + 2J(2,m,p) + 2J(3,m,p) \\
&\quad +2J(4,m,p) + 2J(5,m,p) + \cdots + 2J(n-4,m,p) \\
&\quad +2J(n-3,m,p) + 2J(n-2,m,p) + 2J(n-1,m,p) \\
&= -J(1,m,p) + J(n+1,m,p) \\
&= -J_{m+1}J_{p+1} - J_mJ_{p+1}i - J_{m+1}J_pq + J_{n+1}J_{m+1}J_{p+1} \\
&\quad +J_{n+2}J_mJ_{p+1}i + J_{n+2}J_{m+1}J_pq \\
&= (J_{n+1}-1)J_{m+1}J_{p+1} + (J_{n+2}-1)J_mJ_{p+1}i \\
&\quad +(J_{n+2}-1)J_{m+1}J_pq.
\end{aligned}$$

□

Theorem 24. *For numbers $J(n,m,p)$ with any integer indexes we have that:*

$$J(-n,-m,-p) = \frac{(-1)^{n+m+p+5}}{2^{n+m+p+2}}J(n,m,p).$$

Proof. Using the relation $J(n,m,p) = J_nJ_{m+1}J_{p+1} + J_{n+1}J_mJ_{p+1}i + J_{n+1}J_{m+1}J_pj$ and extending their indexes to the set of all integers, we have:

$$\begin{aligned}
J(-n,-m,-p) &= J_{-n}J_{-(m+1)}J_{-(p+1)} + J_{-(n+1)}J_{-m}J_{-(p+1)}i \\
&\quad +J_{-(n+1)}J_{-(m+1)}J_{-p}q \\
&= \frac{(-1)^{n+1}}{2^n}J_n.\frac{(-1)^{m+2}}{2^{m+1}}J_{m+1}.\frac{(-1)^{p+2}}{2^{p+1}}J_{p+1} \\
&\quad +\frac{(-1)^{n+2}}{2^{n+1}}J_{n+1}.\frac{(-1)^{m+1}}{2^m}J_m.\frac{(-1)^{p+2}}{2^{p+1}}J_{p+1}i \\
&\quad +\frac{(-1)^{n+2}}{2^{n+1}}J_{n+1}.\frac{(-1)^{m+2}}{2^{m+1}}J_{m+1}.\frac{(-1)^{p+1}}{2^p}J_pq \\
&= \frac{(-1)^{n+m+p+5}}{2^{n+m+p+2}}J_nJ_{m+1}J_{p+1} + \frac{(-1)^{n+m+p+5}}{2^{n+m+p+2}}J_{n+1}J_mJ_{p+1}i \\
&\quad +\frac{(-1)^{n+m+p+5}}{2^{n+m+p+2}}J_{n+1}J_{m+1}J_pq \\
&= \frac{(-1)^{n+m+p+5}}{2^{n+m+p+2}}(J_nJ_{m+1}J_{p+1} + J_{n+1}J_mJ_{p+1}i + J_{n+1}J_{m+1}J_pq) \\
&= \frac{(-1)^{n+m+p+5}}{2^{n+m+p+2}}J(n,m,p).
\end{aligned}$$

□

Theorem 25. *The sum of the first n numbers $J(n,m,p)$ with indexes less than zero can be described by:*

$$\begin{aligned}\sum_{l=1}^{n} 2J(-l,-m,-p) &= \frac{(-1)^{m+p+5}}{2^{m+p+2}}(J_mJ_{p+1}i + J_{m+1}J_pj)\\ &+\frac{(-1)^{n+m+p+5}}{2^{n+m+p+2}}(J_{n+1}J_{m+1}J_{p+1} + J_{n+2}J_mJ_{p+1}i\\ &+J_{n+2}J_{m+1}J_pj).\end{aligned}$$

Proof. Given the relation $J(-n+1,-m,-p) = J(-n,-m,-p)+2J(-n-1,-m,-p)$, we can rewrite it as:

$$\begin{aligned}2J(-n-1,-m,-p) &= J(-n+1,-m,-p) - J(-n,-m,-p)\\ 2J(-2,-m,-p) &= J(0,-m,-p) - J(-1,-m,-p)\\ 2J(-3,-m,-p) &= J(-1,-m,-p) - J(-2,-m,-p)\\ 2J(-4,-m,-p) &= J(-2,-m,-p) - J(-3,-m,-p)\\ 2J(-5,-m,-p) &= J(-3,-m,-p) - J(-4,-m,-p)\\ \vdots &= \vdots\\ 2J(-n-3,-m,-p) &= J(-n-1,-m,-p) - J(-n-2,-m,-p)\\ 2J(-n-2,-m,-p) &= J(-n,-m,-p) - J(-n-1,-m,-p)\\ 2J(-n-1,-m,-p) &= J(-n+1,-m,-p) - J(-n,-m,-p)\end{aligned}$$

Using $J(0,-m,-p) = \frac{(-1)^{m+p+5}}{2^{m+p+2}}(J_mJ_{p+1}i + J_{m+1}J_pj)$ and $J(-n,-m,-p) = \frac{(-1)^{n+m+p+5}}{2^{n+m+p+2}}J(n,m,p)$ we have:

$$\begin{aligned}\sum_{l=1}^{n} 2J(-l,-m,-p) &= 2J(-2,-m,-p) + 2J(-3,-m,-p) + 2J(-4,-m,-p)\\ &\quad +2J(-5,-m,-p) + \cdots + 2J(-n-3,-m,-p)\\ &\quad +2J(-n-2,-m,-p) + 2J(-n-1,-m,-p)\\ &= J(0,-m,-p) + J(-n+1,-m,-p)\\ &= \frac{(-1)^{m+p+5}}{2^{m+p+2}}(J_mJ_{p+1}i + J_{m+1}J_pj)\\ &\quad +\frac{(-1)^{n+m+p+5}}{2^{n+m+p+2}}(J_{n+1}J_{m+1}J_{p+1} + J_{n+2}J_mJ_{p+1}i\\ &\quad +J_{n+2}J_{m+1}J_pj).\end{aligned}$$

□

6. n-Dimensional Recurrence Relations

The notation used to establish the n-dimensional relation of Jacobsthal's sequence is presented in this section based on [10, 11].

Definition 3. The numbers of the form $J(n_1, n_2, n_3, \cdots, n_n)$ with n variable, for $n \geq 2$, and the following set of imaginary units $(\mu_1 = q, \mu_2 = k, \cdots, \mu_{n-1})$ are defined as the *Jacobsthal's n-Dimensional Sequence* and satisfy the recurrence relations:

$$\begin{cases} J(n_1+1, n_2, \cdots, n_n) & = & J(n_1, n_2, \cdots, n_n) + 2J(n_1-1, n_2, \cdots, n_n) \\ J(n_1, n_2+1, \cdots, n_n) & = & J(n_1, n_2, \cdots, n_n) + 2J(n_1, n_2-1, \cdots, n_n) \\ J(n_1, n_2, n_3+1, \cdots, n_n) & = & J(n_1, n_2, \cdots, n_n) + 2J(n_1, n_2, n_3-1, \cdots, n_n) \\ \vdots & = & \vdots \\ J(n_1, n_2, n_3, \cdots, n_n+1) & = & J(n_1, n_2, \cdots, n_n) + 2J(n_1, n_2, \cdots, n_n-1) \end{cases}$$

with the initial conditions:

$$\begin{aligned} J(0,0,0,\cdots,0) &= 0 \\ J(1,0,0,\cdots,0) &= 1 \\ J(0,1,0,\cdots,0) &= i \\ J(0,0,1,\cdots,0) &= \mu_1 \\ J(0,0,0,1,\cdots,0) &= \mu_2 \\ \vdots &= \vdots \\ J(0,0,0,\cdots,0,1) &= \mu_{n-1} \\ J(1,1,1,\cdots,1) &= 1+i+\mu_1+\cdots+\mu_{n-1} \\ J(0,1,1,\cdots,1) &= i+\mu_1+\cdots+\mu_{n-1} \\ J(1,0,1,\cdots,1) &= 1+\mu_1+\cdots+\mu_{n-1} \\ J(1,1,0,1\cdots,1) &= 1+i+\mu_2+\cdots+\mu_{n-1} \\ \vdots &= \vdots \\ J(1,1,1,\cdots,1,0) &= 1+i+\mu_1+\cdots+\mu_{n-2}. \end{aligned}$$

Thus, the numbers of the form $J(n_1, n_2, n_3, \cdots, n_n)$ must meet the presented n-dimensional recurrence conditions.

Theorem 26. *The numbers of the form $J(n_1, n_2, n_3, \cdots, n_n)$, so that, with n_1, n_2, n_3, …, n_n in $\mathbb{N}$, are given by:*

$$\begin{aligned} J(n_1, n_2, n_3, \cdots, n_n) = & \ (J_{n_1} J_{n_2+1} J_{n_3+1}. \cdots .J_{n_n+1}) \\ & + (J_{n_1+1} J_{n_2} J_{n_3+1}. \cdots .J_{n_n+1}) i \\ & + (J_{n_1+1} J_{n_2+1} J_{n_3}. \cdots .J_{n_n+1}) \mu_1 + \cdots \\ & + (J_{n_1+1} J_{n_2+1} J_{n_3+1}. \cdots .J_{n_n}) \mu_{n-1}. \end{aligned}$$

Proof. In this case, starting from the Jacobsthal recurrence relation, $J_{n+1} = J_n + 2J_{n-1}$, and the relations $J(n,m) = J_nJ_{m+1}+J_{n+1}J_mi$ and $J(n,m,p) = J_nJ_{m+1}J_{p+1} +J_{n+1}J_mJ_{p+1}i+J_{n+1}J_{m+1}J_pq$ are valid. Through an inductive process, we can verify that:

$$\begin{aligned} J(n,m) &= J_nJ_{m+1} + J_{n+1}J_mi \\ J(n,m,p) &= J_nJ_{m+1}J_{p+1} + J_{n+1}J_mJ_{p+1}i + J_{n+1}J_{m+1}J_pq \\ J(n,m,p,q) &= J_nJ_{m+1}J_{p+1}J_{q+1} + J_{n+1}J_mJ_{p+1}J_{q+1}i \\ &\quad +J_{n+1}J_{m+1}J_pJ_{q+1}q + J_{n+1}J_{m+1}J_{p+1}J_qk \\ \vdots &= \vdots \\ J(n_1,n_2,n_3,\cdots,n_n) &= (J_{n_1}J_{n_2+1}J_{n_3+1}.\cdots.J_{n_n+1}) \\ &\quad +(J_{n_1+1}J_{n_2}J_{n_3+1}.\cdots.J_{n_n+1})i \\ &\quad +(J_{n_1+1}J_{n_2+1}J_{n_3}.\cdots.J_{n_n+1})\mu_1 + \cdots \\ &\quad +(J_{n_1+1}J_{n_2+1}J_{n_3+1}.\cdots.J_{n_n})\mu_{n-1}. \end{aligned}$$

□

7. Conclusion

Recurrence relations and linear sequences have been studied by many mathematicians over the years, especially the Fibonacci sequence, which is one of the best known and most widespread to date. In the case of this work, we bring a study of the Jacobsthal's sequence, in which we seek to contribute specifically to the study of the second order Jacobsthal's sequence, from the introduction of the imaginary unit, aiming at the possibility of expanding its dimension. Thus, from the one-dimensional case, it was possible to investigate the behavior of two-dimensional, three-dimensional and n-dimensional order relations, based on the analysis of some identities presented. As a future perspective for this study, we intend to continue the development of this type of sequence, from the exploration of its applications in the field of science. An example of this would be an investigation of a new type of sequence in quaternion algebra using these numbers, as well as their combinatorial properties.

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