PRIME FLOOR AND PRIME CEILING FUNCTIONS

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Abstract

The prime floor function gives the greatest prime less than or equal to the input while prime ceiling function gives the least prime greater than or equal to the input. This paper conducts a thorough study of these functions and enlists several properties. The connection of these functions with the gap between consecutive primes is unraveled. Consequently, its product, sum, and harmonic sum are studied and asymptotic expressions for each are derived. Numerical calculations are performed which show close agreement between the asymptotic and actual value.

– To my father Dr. Ashutosh for his constant support and motivation

1. Introduction

The integer floor function $\lfloor . \rfloor$ and ceiling function $\lceil . \rceil$ take input and round off to the lowest and highest integer respectively. Mills [14], in 1947, showed that there exists a real number A such that $\lfloor A^{3^n} \rfloor$ is a prime number for $n \in \mathbb{N}$. This result was generalized by Kuipers [10] who showed that for each integer $c \geq 3$, there is a real number A depending only on c, such that $\lfloor A^{c^n} \rfloor$ is a prime number. Recently, many works have appeared which explore new results in this direction (see [5, 7, 13, 25, 27] and the references therein).

Focus of this work is the study of prime floor and ceiling functions defined below.

Definition 1. The *prime floor function* $\lfloor x \rfloor_{\mathbb{P}}$ gives the greatest prime less than or equal to x for $x \in \mathbb{R}^+$. For example: $\lfloor 16 \rfloor_{\mathbb{P}} = 13$, $\lfloor 26.75 \rfloor_{\mathbb{P}} = 23$, etc.

Definition 2. The *prime ceiling function* $\lceil x \rceil_{\mathbb{P}}$ gives the least prime greater than or equal to x for $x \in \mathbb{R}^+$. For example: $\lceil 16 \rceil_{\mathbb{P}} = 17$, $\lceil 26.75 \rceil_{\mathbb{P}} = 29$, etc.

These two functions were introduced in [11]. A function somewhat similar to $\lfloor . \rfloor_{\mathbb{P}}$ had been defined and studied in 1974 by Kahan [9]. Xiaoxia [28], Su [23], and Huang

[8] studied the asymptotic behavior of mean values of the functions attributed to Smarandache [22] that are identical to the ones defined here. Flórez [6] studied the primality of $|T - p|$ where T is the product of triangular numbers, and p is either the smallest prime greater than T or the greatest prime less than T. In 2015, Qarawani [16] studied the "greatest prime function" which is identical to the prime floor function, where he proved inequalities based on Firoozbakht's conjecture (see [17, p. 185]) and arithmetic-geometric mean inequality.

The utility of prime floor and ceiling functions is pretty apparent by now. Also, one can notice intriguing connections with the properties of gaps between primes.

2. Properties and Bounds of $\lfloor x \rfloor_{\mathbb{P}}$ and $\lceil x \rceil_{\mathbb{P}}$

Figures 1 and 2 are the plots of $\lfloor x \rfloor_{\mathbb{P}}$ and $\lceil x \rceil_{\mathbb{P}}$ respectively. While plotting, it was assumed that $\lfloor x \rfloor_{\mathbb{P}} = 0$ and $\lceil x \rceil_{\mathbb{P}} = 2$ for $x < 2$; this reasonable assumption helps to generalize the prime floor and prime ceiling functions over the entire domain of real numbers. The plots clearly show that $\lfloor x \rfloor_{\mathbb{P}} < x < \lceil x \rceil_{\mathbb{P}}$ and both functions increase. Other properties of the two functions are listed below.

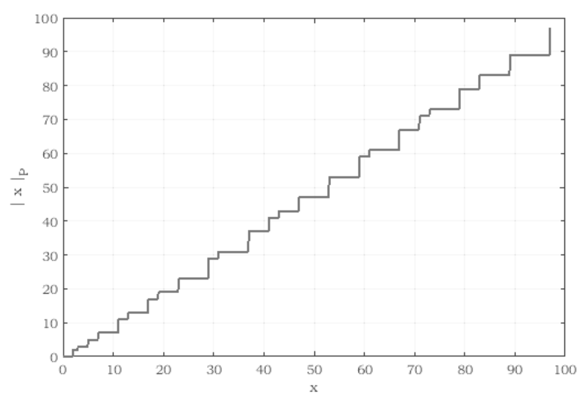

Figure 1: Plot of $\lfloor x \rfloor_{\mathbb{P}}$ for $0 \leq x \leq 97$

Proposition 1. *The prime floor and prime ceiling functions are equal for a prime input, that is* $\lfloor q \rfloor_{\mathbb{P}} = \lceil q \rceil_{\mathbb{P}} = q$*, where* q *is a prime number.*

Proposition 2. *Only the innermost function is relevant when a composition of several prime floor and/or ceiling functions are applied on* x*.*

Remark 1. For instance, $\lceil \lfloor \lceil \lfloor x \rfloor_{\mathbb{P}} \rceil_{\mathbb{P}} \rfloor_{\mathbb{P}} \rceil_{\mathbb{P}} = \lfloor x \rfloor_{\mathbb{P}}$.

Proposition 3. *We have* $\lfloor x \rfloor_{\mathbb{P}} \leq x \leq \lceil x \rceil_{\mathbb{P}}$*, where equality is achieved for prime* x*.*

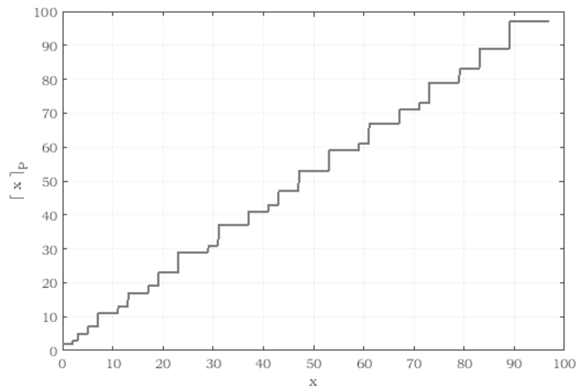

Figure 2: Plot of $\lceil x \rceil_{\mathbb{P}}$ for $0 \leq x \leq 97$

Proposition 4. *We have that* $\lfloor x \rfloor_{\mathbb{P}}^n \leq \lceil x^n \rceil_{\mathbb{P}}$ *and* $\lfloor x^n \rfloor_{\mathbb{P}} \leq \lceil x \rceil_{\mathbb{P}}^n$.

Proposition 5. *For* $x, k, \ell \in \mathbb{R}^+$, $\dfrac{k\lfloor x \rfloor_{\mathbb{P}} + \ell \lceil x \rceil_{\mathbb{P}}}{k+\ell}$ *lies between* $\lfloor x \rfloor_{\mathbb{P}}$ *and* $\lceil x \rceil_{\mathbb{P}}$*; therefore, we have*

$$\lfloor x \rfloor_{\mathbb{P}} = \left\lfloor \frac{k\lfloor x \rfloor_{\mathbb{P}} + \ell \lceil x \rceil_{\mathbb{P}}}{k+\ell} \right\rfloor_{\mathbb{P}} \quad \textit{and} \quad \lceil x \rceil_{\mathbb{P}} = \left\lceil \frac{k\lfloor x \rfloor_{\mathbb{P}} + \ell \lceil x \rceil_{\mathbb{P}}}{k+\ell} \right\rceil_{\mathbb{P}}.$$

Proposition 6. *For* $x \geq 17$*, we have*

$$\lfloor x \rfloor_{\mathbb{P}} > \frac{x}{\log x} \log\left(\frac{e^{-1}x}{\log x} \log\left(\frac{x}{\log x} \right) \right).$$

Proof. Let $\pi(x)$ denote the number of primes less than or equal to x; then $p_{\pi(x)}$ is the $\pi(x)$th prime, so $\lfloor x \rfloor_{\mathbb{P}} = p_{\pi(x)}$. Using the lower bound on p_n from [3] gives

$$\lfloor x \rfloor_{\mathbb{P}} = p_{\pi(x)} > \pi(x)\Big(\log(\pi(x) \log \pi(x)) - 1 \Big).$$

Substituting $\pi(x) > \frac{x}{\log x}$ from [18], the required result is achieved. □

Proposition 7. *For* $x \geq 60184$*, we have*

$$\lceil x \rceil_{\mathbb{P}} < \left(\frac{x}{\log x - 1.1} + 1 \right) \log\left(\left(\frac{x}{\log x - 1.1} + 1 \right) \log\left(\frac{x}{\log x - 1.1} + 1 \right) \right).$$

Proof. If x is a prime then $\lceil x \rceil_{\mathbb{P}} = p_{\pi(x)}$, otherwise $\lceil x \rceil_{\mathbb{P}} = p_{\pi(x)+1}$. In either case, $\lceil x \rceil_{\mathbb{P}} \leq p_{\pi(x)+1}$ is always true; thus using the upper bound to p_n from [19] gives

$$\lceil x \rceil_{\mathbb{P}} < (\pi(x)+1) \log\Big((\pi(x)+1) \log(\pi(x)+1) \Big),$$

which gives the desired result by substituting $\pi(x) < \frac{x}{\log x - 1.1}$ from [4]. □

It is apparent from Figures 1 and 2 that $\lfloor x \rfloor_{\mathbb{P}} \sim \lceil x \rceil_{\mathbb{P}} \sim x$ which can be rigorously proven. Using Propositions 3 and 6, we have the following corollary.

Corollary 1. *We have*

$$1 < \lim_{x\to\infty} \frac{\lfloor x \rfloor_{\mathbb{P}}}{x} \leq 1,$$

which gives $\lfloor x \rfloor_{\mathbb{P}} \sim x$ for large x.

The squeeze theorem of limits proves the final asymptotic in Corollary 1. Using it along with Propositions 3 and 7 gives the following corollary.

Corollary 2. *We have*

$$1 \leq \lim_{x\to\infty} \frac{\lceil x \rceil_{\mathbb{P}}}{x} < 1$$

which gives $\lceil x \rceil_{\mathbb{P}} \sim x$ for large x.

3. Relation with Prime Gaps

The prime gap function is given by $g_n := p_n - p_{n-1}$ where $g_1 = 2$. One hopes to use $\lfloor x \rfloor_{\mathbb{P}}$ and $\lceil x \rceil_{\mathbb{P}}$ to gain more information about prime gaps. In view of the definition of g_n, one might furthermore define $g(x) := \lceil x \rceil_{\mathbb{P}} - \lfloor x \rfloor_{\mathbb{P}}$, which acts as a continuous analogue of g_n. Figure 3 is the area plot of $g(x)$ versus x.

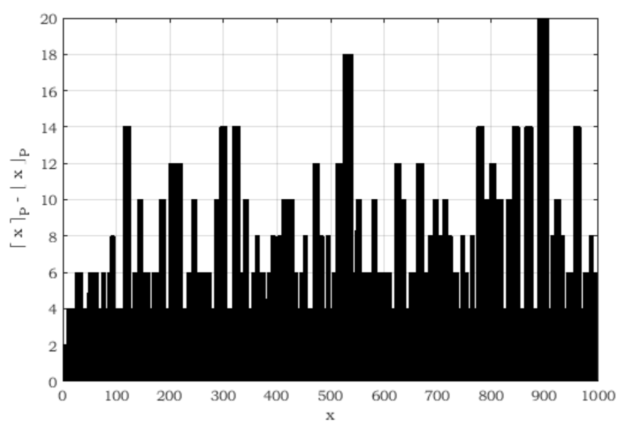

Figure 3: Plot of $\lceil x \rceil_{\mathbb{P}} - \lfloor x \rfloor_{\mathbb{P}}$ for $0 \leq x \leq 1000$

Theorem 1. *We have*

$$\int_0^x g(x)\ \mathrm{d}x = \sum_{k=1}^{\pi(x)} g_k^2 + (x - \lfloor x \rfloor_{\mathbb{P}})(\lceil x \rceil_{\mathbb{P}} - \lfloor x \rfloor_{\mathbb{P}}). \tag{1}$$

Proof. Let $I(x) = \int_{0}^{x} g(x)\ \mathrm{d}x$; the integral can be separated at prime endpoints to give

$$I(x) = \int_{0}^{p_1} \lceil x \rceil_{\mathbb{P}} - \lfloor x \rfloor_{\mathbb{P}}\ \mathrm{d}x + \int_{p_1}^{p_2} \lceil x \rceil_{\mathbb{P}} - \lfloor x \rfloor_{\mathbb{P}}\ \mathrm{d}x + \int_{p_2}^{p_3} \lceil x \rceil_{\mathbb{P}} - \lfloor x \rfloor_{\mathbb{P}}\ \mathrm{d}x$$
$$+ \ldots + \int_{\lfloor x \rfloor_{\mathbb{P}}}^{x} \lceil x \rceil_{\mathbb{P}} - \lfloor x \rfloor_{\mathbb{P}}\ \mathrm{d}x$$
$$= p_1^2 + (p_2 - p_1)^2 + (p_3 - p_2)^2 + \ldots + (\lceil x \rceil_{\mathbb{P}} - \lfloor x \rfloor_{\mathbb{P}})(x - \lfloor x \rfloor_{\mathbb{P}})$$
$$= 2^2 + \sum_{k=2}^{\pi(x)} (p_k - p_{k-1})^2 + (\lceil x \rceil_{\mathbb{P}} - \lfloor x \rfloor_{\mathbb{P}})(x - \lfloor x \rfloor_{\mathbb{P}})$$
$$= \sum_{k=1}^{\pi(x)} g_k^2 + (x - \lfloor x \rfloor_{\mathbb{P}})(\lceil x \rceil_{\mathbb{P}} - \lfloor x \rfloor_{\mathbb{P}}).$$

Hence, the connection between the prime gap function and its continuous analogue is established. □

Lemma 1. *For a continuous and differentiable function $h(x)$, we have*

$$\sum_{1 \le m \le x} h(\lceil m \rceil_{\mathbb{P}}) \sim x h(\lceil x \rceil_{\mathbb{P}}) + \lfloor x \rfloor_{\mathbb{P}} \Big(h(\lfloor x \rfloor_{\mathbb{P}}) - h(\lceil x \rceil_{\mathbb{P}}) \Big) - 2h(2) - \int_{2}^{\lfloor x \rfloor_{\mathbb{P}}} h'(t) t\ \mathrm{d}t. \quad (2)$$

Proof. Splitting the sum $\sum_{1 \le m \le x} h(\lceil m \rceil_{\mathbb{P}})$ at prime endpoints gives

$$\sum_{1 \le m \le x} h(\lceil m \rceil_{\mathbb{P}}) = \sum_{k=1}^{\pi(x)} g_k h(p_k) + (x - \lfloor x \rfloor_{\mathbb{P}}) h(\lceil x \rceil_{\mathbb{P}}). \quad (3)$$

Applying Abel's summation formula (see [1, 2]) to the sum on the right side, we have

$$\sum_{k=1}^{\pi(x)} g_k h(p_k) = \left(\sum_{k=1}^{\pi(x)} g_k \right) h(p_{\pi(x)}) - \left(\sum_{k=1}^{1} g_k \right) h(p_{\pi(1)}) - \int_{1}^{\pi(x)} \frac{\mathrm{d}h(p_u)}{\mathrm{d}u} \left(\sum_{k=1}^{u} g_k \right)\ \mathrm{d}u$$
$$\sim p_{\pi(x)} h(p_{\pi(x)}) - 2h(2) - \int_{1}^{\pi(x)} \frac{\mathrm{d}h(p_u)}{\mathrm{d}u} \times p_u\ \mathrm{d}u$$
$$= \lfloor x \rfloor_{\mathbb{P}} h(\lfloor x \rfloor_{\mathbb{P}}) - 2h(2) - \int_{1}^{\pi(x)} h'(p_u) \times \frac{\mathrm{d}p_u}{\mathrm{d}u} \times p_u\ \mathrm{d}u.$$

Substitute the above expression in Equation (3) to get

$$\begin{aligned}\sum_{1\le m\le x} h(\lceil m\rceil_{\mathbb{P}}) &\sim xh(\lceil x\rceil_{\mathbb{P}}) + \lfloor x\rfloor_{\mathbb{P}}(h(\lfloor x\rfloor_{\mathbb{P}}) - h(\lceil x\rceil_{\mathbb{P}})) - 2h(2) - \int\limits_{p_1}^{p_{\pi(x)}} h'(t)t \,\mathrm{d}t \\ &= xh(\lceil x\rceil_{\mathbb{P}}) + \lfloor x\rfloor_{\mathbb{P}}(h(\lfloor x\rfloor_{\mathbb{P}}) - h(\lceil x\rceil_{\mathbb{P}})) - 2h(2) - \int\limits_{2}^{\lfloor x\rfloor_{\mathbb{P}}} h'(t)t \,\mathrm{d}t,\end{aligned}$$

which is the desired result. □

4. Product and Sum Expressions

This section analyzes the product and sum over the prime ceiling function. Asymptotic expressions for each are given and then discussed. Similar results can be shown for the prime floor function, omitted here as they are essentially identical.

Theorem 2. *We have*

$$\prod_{m=1}^{n} \lceil m\rceil_{\mathbb{P}} \sim \left(\frac{e}{2}\right)^2 \frac{\lceil n\rceil_{\mathbb{P}}^n}{e^{\lfloor n\rfloor_{\mathbb{P}}}} \left(\frac{\lfloor n\rfloor_{\mathbb{P}}}{\lceil n\rceil_{\mathbb{P}}}\right)^{\lfloor n\rfloor_{\mathbb{P}}}. \tag{4}$$

Proof. Taking the natural logarithm of $\prod_{m=1}^{n} \lceil m\rceil_{\mathbb{P}}$ gives $\sum_{m=1}^{n} \log\lceil m\rceil_{\mathbb{P}}$ which is a special case arising with $h(x) = \log x$. Using Lemma 1, we get

$$\begin{aligned}\sum_{m=1}^{n} \log\lceil m\rceil_{\mathbb{P}} &\sim n\log\lceil n\rceil_{\mathbb{P}} + \lfloor n\rfloor_{\mathbb{P}} \log\left(\frac{\lfloor n\rfloor_{\mathbb{P}}}{\lceil n\rceil_{\mathbb{P}}}\right) - 2\log 2 - \int\limits_{2}^{\lfloor n\rfloor_{\mathbb{P}}} \frac{1}{t} \times t \,\mathrm{d}t \\ &= n\log\lceil n\rceil_{\mathbb{P}} + \lfloor n\rfloor_{\mathbb{P}} \log\left(\frac{\lfloor n\rfloor_{\mathbb{P}}}{\lceil n\rceil_{\mathbb{P}}}\right) - 2\log 2 - (\lfloor n\rfloor_{\mathbb{P}} - 2).\end{aligned}$$

Exponentiating both sides, we get

$$\prod_{m=1}^{n} \lceil m\rceil_{\mathbb{P}} \sim \left(\frac{e}{2}\right)^2 \frac{\lceil n\rceil_{\mathbb{P}}^n}{e^{\lfloor n\rfloor_{\mathbb{P}}}} \left(\frac{\lfloor n\rfloor_{\mathbb{P}}}{\lceil n\rceil_{\mathbb{P}}}\right)^{\lfloor n\rfloor_{\mathbb{P}}},$$

which is the desired result. □

The above result is numerically tested for $n \le 20$ by evaluating the actual and asymptotic values of $\log\left(\prod_{m=1}^{n} \lceil m\rceil_{\mathbb{P}}\right)$. The ratio of asymptotic to actual value is also determined, which helps to estimate the relative error.

Number n	Actual value	Asymptotic value	Ratio
2	1.386294	0	0
4	4.094344	2.518980	0.615234
6	7.649693	5.606805	0.732945
8	11.993498	9.632972	0.803183
10	16.789288	14.428762	0.859403
12	21.752133	18.555503	0.853043
14	27.150296	23.791261	0.876279
16	32.816723	29.457687	0.897642
18	38.594375	34.722771	0.899685
20	44.674308	40.693540	0.910894

Table 1: Comparison of $\log\left(\prod_{m=1}^{n} \lceil m \rceil_{\mathbb{P}}\right)$ evaluated at different n.

Table 1 suggests that the relative difference between the asymptotic and actual value reduces with increasing n. The close agreement between the two values is apparent from the ratio values, which approach unity as n increases.

Theorem 3. *We have*

$$\sum_{m=1}^{n} \lceil m \rceil_{\mathbb{P}} \sim (n - \lfloor n \rfloor_{\mathbb{P}}) \lceil n \rceil_{\mathbb{P}} + \frac{\lfloor n \rfloor_{\mathbb{P}}^2}{2} - 2. \tag{5}$$

Proof. By substituting $h(x) = x$ in Lemma 1, we get

$$\begin{aligned}
\sum_{m=1}^{n} \lceil m \rceil_{\mathbb{P}} &\sim n \lceil n \rceil_{\mathbb{P}} + \lfloor n \rfloor_{\mathbb{P}} (\lfloor n \rfloor_{\mathbb{P}} - \lceil n \rceil_{\mathbb{P}}) - 4 - \int_{2}^{\lfloor n \rfloor_{\mathbb{P}}} t \, \mathrm{d}t \\
&= n \lceil n \rceil_{\mathbb{P}} + \lfloor n \rfloor_{\mathbb{P}} (\lfloor n \rfloor_{\mathbb{P}} - \lceil n \rceil_{\mathbb{P}}) - 4 - \frac{\lfloor n \rfloor_{\mathbb{P}}^2 - 2^2}{2} \\
&= (n - \lfloor n \rfloor_{\mathbb{P}}) \lceil n \rceil_{\mathbb{P}} + \frac{\lfloor n \rfloor_{\mathbb{P}}^2}{2} - 2,
\end{aligned}$$

which simplifies to the required result. □

The above result is numerically tested for $n \leq 20$ by evaluating the actual and asymptotic values of $\sum_{m=1}^{n} \lceil m \rceil_{\mathbb{P}}$. The ratio of asymptotic to actual value is also determined, which gives an idea of the relative error.

Table 2 suggests that the relative difference between the asymptotic and actual value reduces with increasing n. The close agreement between the two values is apparent from the ratio values, which approach unity as n increases.

Number n	Actual value	Asymptotic value	Ratio
2	4	0	0
4	12	7.5	0.629167
6	24	17.5	0.729167
8	42	33.5	0.797619
10	64	55.5	0.867188
12	88	71.5	0.812500
14	118	99.5	0.843220
16	152	133.5	0.878289
18	188	161.5	0.859043
20	230	201.5	0.876087

Table 2: Comparison of $\sum_{m=1}^{n} \lceil m \rceil_{\mathbb{P}}$ values evaluated at different n.

Theorem 4. *We have*

$$\sum_{m=1}^{n} \frac{1}{\lceil m \rceil_{\mathbb{P}}} \sim \frac{n - \lfloor n \rfloor_{\mathbb{P}}}{\lceil n \rceil_{\mathbb{P}}} + \log\left(\frac{\lfloor n \rfloor_{\mathbb{P}}}{2}\right). \tag{6}$$

Proof. By substituting $h(x) = 1/x$ in Lemma 1, we get

$$\begin{aligned}\sum_{m=1}^{n} \frac{1}{\lceil m \rceil_{\mathbb{P}}} &\sim \frac{n}{\lceil n \rceil_{\mathbb{P}}} + \lfloor n \rfloor_{\mathbb{P}} \left(\frac{1}{\lfloor n \rfloor_{\mathbb{P}}} - \frac{1}{\lceil n \rceil_{\mathbb{P}}}\right) - 2 \times \frac{1}{2} + \int_{2}^{\lfloor n \rfloor_{\mathbb{P}}} \frac{t}{t^2}\, \mathrm{d}t \\ &= \frac{n - \lfloor n \rfloor_{\mathbb{P}}}{\lceil n \rceil_{\mathbb{P}}} + \log\left(\frac{\lfloor n \rfloor_{\mathbb{P}}}{2}\right),\end{aligned}$$

which simplifies to the required result. □

The harmonic sum on the left-hand side is reminiscent of the proof of Euler, where the infinitude of the set of prime numbers was shown by the divergence of the harmonic sum of prime numbers. Note, however, the harmonic sum here will have multiple repetitions of primes. This makes one wonder whether a similar weighted harmonic sum of twin primes obeys any exciting pattern. Although the harmonic sum of twin primes converges to Brun's constant [15], weighted sums might diverge. Apart from this, the reader is encouraged to wonder what connection the above result could have with the prime number theorem, if any.

Table 3 suggests that the asymptotic and actual values are in close agreement since the ratio sequence approaches unity. An additional observation by looking at all the tables is that the error decreases but at different rates for different functions. A future study might consider the error term more explicitly and offer more terms.

Number n	Actual value	Asymptotic value	Ratio
2	1	0	0
4	1.533333	0.605465	0.394868
6	1.876190	1.059148	0.564521
8	2.109957	1.343672	0.636824
10	2.291775	1.525490	0.665637
12	2.459607	1.781671	0.724372
14	2.595354	1.930626	0.743878
16	2.713001	2.048273	0.754984
18	2.824456	2.192698	0.776326
20	2.920566	2.294770	0.785728

Table 3: Comparison of $\sum_{m=1}^{n} \frac{1}{\lceil m \rceil_{\mathbb{P}}}$ evaluated at different n.

5. Conclusion and Future Scope

This paper explores the prime floor and prime ceiling functions. They are analogous in a sense to the integer floor and ceiling functions but come with implicit number-theoretic information. Asymptotic expressions for the product, sum, and sum of reciprocals of the prime ceiling function are proved. The relative closeness of the actual and asymptotic values is numerically validated. The author hopes this work proves useful for studying prime gaps for which the motivation is shared below.

Consider a sequence $(\nu_n)_{n\geq 1}$ such that $x = \nu_n$ lies between two consecutive primes p_n and p_{n+1} for all $n \geq 1$. Thus, the prime floor and prime ceiling of the sequence $(\nu_n)_{n\geq 1}$ generate all prime pairs $(\lfloor \nu_n \rfloor_{\mathbb{P}}, \lceil \nu_n \rceil_{\mathbb{P}})$. One possible example of ν_n could simply be the mean of p_n and p_{n+1}, but this would not be a closed form expression. As the definition of ν_n only poses a condition, so it would be desirable to have a ν_n with a closed form expression. This sequence would have the property of generating primes when operated upon by prime floor and prime ceiling functions. Apart from this, one could also show that

$$\sum_{n=1}^{\infty} \frac{1}{\lceil n \rceil_{\mathbb{P}}^{s}} = \sum_{k=1}^{\infty} \frac{g_k}{p_k^s}$$

where the right-hand side is a weighted prime zeta function (introduced by Glaisher [26]) which can be further explored. The prime floor and prime ceiling functions could find application in the study of prime partitions [12, 24] and maybe partition zeta function (or a variant defined for prime partitions) [20, 21].

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