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A MATRIX RELATED TO STERN POLYNOMIALS AND THE PROUHET-THUE-MORSE SEQUENCE

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Abstract

The Stern polynomials defined by $s(0;x) = 0$, $s(1;x) = 1$, and for $n \geq 1$ by $s(2n;x) = s(n;x^2)$ and $s(2n+1;x) = x\,s(n;x^2) + s(n+1;x^2)$ have only 0 and 1 as coefficients. We construct an infinite lower-triangular matrix related to the coefficients of the $s(n;x)$ and show that its inverse has only 0, 1, and -1 as entries, which we find explicitly. In particular, the sign distribution of the entries is determined by the Prouhet-Thue-Morse sequence. We also obtain other properties of this matrix and a related Pascal-type matrix that involve the Catalan, Stirling, Fibonacci, Fine, and Padovan numbers. Further results involve compositions of integers, the Sierpiński matrix, and identities connecting the Stern and Prouhet-Thue-Morse sequences.

1. Introduction

The Stern sequence, also known as Stern's diatomic sequence, is one of the most remarkable integer sequences in number theory and combinatorics. It can be defined by $s(0) = 0$, $s(1) = 1$, and for $n \geq 1$ by

$$s(2n) = s(n), \qquad s(2n+1) = s(n) + s(n+1). \tag{1}$$

The first 20 Stern numbers, starting with $n = 1$, are listed in Table 1. Numerous properties and references can be found in [5], [21, A002487], or [23]. Perhaps the most remarkable properties are the facts that the terms $s(n)$, $s(n+1)$ are always

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relatively prime, and that each positive reduced rational number occurs once and only once in the sequence $\{s(n)/s(n+1)\}_{n\geq 1}$.

In two papers published in 2007, the Stern sequence was extended to two quite different sequences of polynomials, both with interesting and useful properties: one by Klavžar, Milutinović and Petr [18], and the other by the second author and K. B. Stolarsky [11]. In this paper we only consider the sequence introduced in [11]; it is defined by $s(0;x)=0$, $s(1;x)=1$, and for $n\geq 1$ recursively by

$$s(2n;x) = s(n;x^2), \tag{2}$$
$$s(2n+1;x) = x\, s(n;x^2) + s(n+1;x^2). \tag{3}$$

The first few of these Stern polynomials are listed in Table 1; numerous properties can be found in [11] and [12]. Here we only repeat the obvious properties that $s(n;0)=1$ for $n\geq 1$, and

$$s(n;1) = s(n), \qquad s(2^n;x) = 1 \quad (n \geq 0). \tag{4}$$

To give an expression for the degree of $s(n;x)$, let $\nu(n)$ be the 2-adic valuation of n, that is, $\nu(n)$ is the greatest integer such that $2^{\nu(n)}$ divides n. Then for $n\geq 1$ we have

$$\deg s(n;x) = \frac{n-2^{\nu(n)}}{2}, \qquad \deg s(2n+1;x) = n. \tag{5}$$

Another important property of these Stern polynomials is that they are $(0,1)$-polynomials (or *Newman polynomials*), which is not the case for the Stern polynomials of Klavžar et al.

n	$s(n)$	$s(n;x)$	n	$s(n)$	$s(n;x)$
1	1	1	11	5	$1+x+x^3+x^4+x^5$
2	1	1	12	2	$1+x^4$
3	2	$1+x$	13	5	$1+x+x^2+x^5+x^6$
4	1	1	14	3	$1+x^2+x^6$
5	3	$1+x+x^2$	15	4	$1+x+x^3+x^7$
6	2	$1+x^2$	16	1	1
7	3	$1+x+x^3$	17	5	$1+x+x^2+x^4+x^8$
8	1	1	18	4	$1+x^2+x^4+x^8$
9	4	$1+x+x^2+x^4$	19	7	$1+x+x^3+x^4+x^5+x^8+x^9$
10	3	$1+x^2+x^4$	20	3	$1+x^4+x^8$

Table 1: $s(n)$ and $s(n;x)$, $1\leq n\leq 20$.

It is the main purpose of this paper to explore a matrix related to the Stern polynomials $s(n;x)$. This is best described by beginning with the infinite identity matrix; for each $n \geq 1$ we put the coefficients of $s(n;x)$ into row n, starting with the constant coefficient 1 and going left from the main diagonal, which is possible by (5). All remaining entries in this row are set to 0. We denote the resulting infinite matrix by R, and let R_N denote the submatrix consisting of the first N rows and columns; see (6) for the matrix R_{10}. Here and in all later matrices the dots stand for zeros, and R and all R_N are, by definition, lower triangular.

$$R_{10} := \begin{pmatrix} 1 \\ \cdot & 1 \\ \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 \end{pmatrix}. \tag{6}$$

We can also consider a closely related polynomial sequence $r(n;x)$ formed by taking the reciprocal polynomial of $s(n;x)$ and multiplying by an appropriate power of x to get degree n. In other words, we define

$$r(n;x) := x^n s(n; \tfrac{1}{x}). \tag{7}$$

Then row n of the lower-triangular part of the matrix R contains the coefficients of $r(n;x)$, with leading coefficient on the diagonal.

By construction, the matrices R and R_N are invertible, and obviously the inverses R^{-1} and $(R_N)^{-1}$ are again lower triangular with integer entries. However, it is rather surprising that the only entries that occur seem to be 0, 1, and -1; see (8).

$$(R_{10})^{-1} := \begin{pmatrix} 1 \\ \cdot & 1 \\ \cdot & -1 & 1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & -1 & -1 & 1 \\ \cdot & \cdot & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & -1 & \cdot & 1 \end{pmatrix}. \tag{8}$$

We also observe that all nonzero entries in the second and third lower diagonals are -1, and that in general all nonzero entries on a fixed diagonal have the same sign. The sign pattern is quite obvious in the penultimate row in (8). To make it even more obvious, row 17 of R^{-1}, going to the main diagonal, is

$$0 \quad 1 \quad -1 \quad -1 \quad 1 \quad -1 \quad 1 \quad 1 \quad -1 \quad -1 \quad 1 \quad 1 \quad -1 \quad 1 \quad -1 \quad -1 \quad 1\ .$$

When read from right to left, this is reminiscent of the Prouhet-Thue-Morse sequence (t_n) which can be defined by $t_0 = 0$ and

$$t_{2n} = t_n, \qquad t_{2n+1} = 1 - t_n. \tag{9}$$

The first 16 terms are 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0. There are several other ways of generating this sequence; see, e.g., [2, Sect. 1.6, 5.1].

We are now ready to state our main result. As usual, $(R^{-1})_{n,k}$, for $n \geq 1$ and $k \geq 1$, denotes the entry in row n and column k in the matrix R^{-1}.

Theorem 1. *The infinite lower-triangular matrix R^{-1} has only entries $0, 1, -1$. In particular, $(R^{-1})_{1,1} = 1$, and for $n \geq 2$ and $1 \leq k \leq n$ we have*

$$\left|(R^{-1})_{n,k}\right| \equiv \binom{2n-k-1}{n-k} \pmod 2, \tag{10}$$

and the sign of $(R^{-1})_{n,k}$ is $(-1)^{t_{n-k}}$, where (t_n) is the Prouhet-Thue-Morse sequence.

We prove (10) by first considering a shifted Pascal matrix P and its inverse; this will be done in Section 2. In Section 3 we complete the proof of Theorem 1, using factors of certain polynomials. We explore the row, column, and antidiagonal sums of the matrices R and R^{-1} in Section 4. The row sums of a matrix formed by a Hadamard product leads to a connection with compositions of integers; we investigate this in Section 5. Another related infinite matrix S, based on the Sierpiński gasket, and its inverse occur in Section 6, and we conclude this paper with some further remarks and results in Section 7.

2. A Pascal-Type Matrix

It has been known for some time, at least since Carlitz's paper [7] of 1960, that there is a close connection between the Stern sequence $\big(s(n)\big)$ and binomial coefficients. In fact, Carlitz showed that for a fixed $n \geq 0$, the number of odd binomial coefficients $\binom{n-k}{k}$ is given by $s(n+1)$. This indicates that it will be of interest to consider binomial coefficients (mod 2), and we use the notation

$$\binom{n}{k}^* \in \{0, 1\}, \quad \text{where} \quad \binom{n}{k}^* \equiv \binom{n}{k} \pmod 2. \tag{11}$$

Then Carlitz's result can be rewritten as

$$s(n+1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}^*; \tag{12}$$

this was also obtained in [14, p. 319]. A polynomial analogue of this identity was given in [11] in terms of the Chebyshev polynomials $U_n(x)$. A well-known explicit expansion of $U_n(x)$ (see, e.g., [7, Eqn. (2.16)]) then gives

$$s(n+1;x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}^{*} x^k, \tag{13}$$

which was first stated in this form in [10], with applications and extensions. With (7) we now have

$$r(n;x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k}^{*} x^{n-k}, \tag{14}$$

and according to the definition of $r(n;x)$ and the matrix R, the coefficient on the right of (14) is the entry in row n and column $n-k$ of R.

We now consider the "non-starred" version of the right-hand side of (14), namely the polynomials

$$\rho(n;x) := \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} x^{n-k} = \sum_{k=1}^{n} \binom{k-1}{n-k} x^k, \tag{15}$$

where, as usual, we let $\binom{n}{m} = 0$ when $0 \le n < m$. We then form an infinite lower-triangular matrix P from the coefficients of $\rho(n;x)$ and let P_N be the corresponding $N \times N$ submatrix. Then it is clear from the right-most term in (15) that column k of P is row $k-1$ of Pascal's triangle moved downward by $k-1$ places; see the matrix P_8 in (16).

$$P_8 := \begin{pmatrix} 1 & & & & & & & \\ \cdot & 1 & & & & & & \\ \cdot & 1 & 1 & & & & & \\ \cdot & \cdot & 2 & 1 & & & & \\ \cdot & \cdot & 1 & 3 & 1 & & & \\ \cdot & \cdot & \cdot & 3 & 4 & 1 & & \\ \cdot & \cdot & \cdot & 1 & 6 & 5 & 1 & \\ \cdot & \cdot & \cdot & \cdot & 4 & 10 & 6 & 1 \end{pmatrix}. \tag{16}$$

We mention in passing that other types of Pascal matrices have been studied; see, e.g., [4]. The next obvious step is to consider the inverses of the matrices P and P_N. We are going to prove the following result.

Theorem 2. *The inverse of the matrix P is the lower-triangular integer matrix given by $(P^{-1})_{1,1} = 1$, $(P^{-1})_{n,1} = 0$ for $n > 1$, and*

$$\left(P^{-1}\right)_{n,k} = (-1)^{n-k} \frac{k-1}{2n-k-1} \binom{2n-k-1}{n-k}, \quad 2 \le k \le n. \tag{17}$$

The entries in (17) are integers since the integer matrices P_N have determinant 1.

Proof of Theorem 2. We need to show that the product of the matrices P^{-1} and P is indeed the infinite identity matrix. Since the order does not matter, we consider $P^{-1}P$ and recall that, in addition to (17), by (15) we have

$$P_{j,k} = \binom{k-1}{j-k}. \tag{18}$$

To obtain the matrix product, we fix a row $n \geq 1$ and a column $k \geq 1$, and consider the sum

$$\big(P^{-1}P\big)_{n,k} = \sum_{j=k}^{n} (P^{-1})_{n,j} P_{j,k}. \tag{19}$$

First, when $n < k$, then all summands in (19) are 0. Similarly, the sum is clearly 1 when $n = k$. In the nontrivial case $k < n$, with (17) and (18), the sum in (19) becomes

$$\begin{aligned}\big(P^{-1}P\big)_{n,k} &= \sum_{j=k}^{n} (-1)^{n-j} \frac{j-1}{2n-j-1} \binom{2n-j-1}{n-j} \binom{k-1}{j-k} \\ &= \sum_{j=0}^{n-k} (-1)^{j} \frac{n-j-1}{n+j-1} \binom{n+j-1}{j} \binom{k-1}{n-j-k},\end{aligned}$$

where we have replaced j by $n-j$. Using computer algebra, for instance Maple or Mathematica, we find that the latter sum is 0, which completes the proof.

Alternatively, we can rewrite the sum in question as

$$\big(P^{-1}P\big)_{n,k} = \frac{(k-1)!(2n-2k-1)!}{(n-1)!(n-k)!} \sum_{j=0}^{n-k} (-1)^{j} \binom{n-k}{j} \binom{n+j-2}{2n-2k-1} (n-j-1),$$

and if we write $n-j-1 = (n+j-1) - 2j$, then the sum on the right becomes

$$\begin{aligned}&\sum_{j=0}^{n-k} (-1)^{j} \binom{n-k}{j} \binom{n+j-2}{2n-2k-1} (n+j-1) \\ &\qquad - \sum_{j=0}^{n-k} (-1)^{j} \binom{n-k}{j} \binom{n+j-2}{2n-2k-1} (2j) =: S_1 - S_2.\end{aligned}$$

After some further straightforward manipulations, we get

$$\begin{aligned}S_1 &= 2(n-k) \sum_{j=0}^{n-k} (-1)^{j} \binom{n-k}{j} \binom{n+j-1}{2n-2k}, \\ S_2 &= -2(n-k) \sum_{j=0}^{n-k-1} (-1)^{j} \binom{n-k-1}{j} \binom{n+j-1}{2n-2k-1}.\end{aligned}$$

Both these sums are instances of a known binomial identity, namely Equation (3.47) in [16], which gives

$$S_1 = S_2 = 2(n-k)(-1)^{n-k}\binom{n-1}{n-k}.$$

This shows, once again, that $\left(P^{-1}P\right)_{n,k} = 0$ whenever $1 \le k < n$. $\square$

As a consequence of Theorem 2 we get the first part of Theorem 1, namely (10). We can see this by reducing the matrix identity $P_N \cdot (P_N)^{-1} = I_N$ modulo 2, for all N. As an illustration, compare (20) with (8).

$$(P_8)^{-1} = \begin{pmatrix} 1 \\ \cdot & 1 \\ \cdot & -1 & 1 \\ \cdot & 2 & -2 & 1 \\ \cdot & -5 & 5 & -3 & 1 \\ \cdot & 14 & -14 & 9 & -4 & 1 \\ \cdot & -42 & 42 & -28 & 14 & -5 & 1 \\ \cdot & 132 & -132 & 90 & -48 & 20 & -6 & 1 \end{pmatrix}. \tag{20}$$

To reduce P^{-1} to R^{-1} ((17) to (10)), it is easy to see that both cases agree when $k = 1$. For $2 \le k \le n-1$ we have

$$\begin{aligned} \frac{k-1}{2n-k-1}\binom{2n-k-1}{n-k} - \binom{2n-k-1}{n-k} &= \frac{k-1-(2n-k-1)}{2n-k-1}\binom{2n-k-1}{n-k} \\ &= -2 \cdot \frac{n-k}{2n-k-1}\binom{2n-k-1}{n-k} = -2 \cdot \binom{2n-k-2}{n-k-1}, \end{aligned}$$

where in the last equation we have used a well-known identity for binomial coefficients. Hence

$$\frac{k-1}{2n-k-1}\binom{2n-k-1}{n-k} \equiv \binom{2n-k-1}{n-k} \pmod 2,$$

as desired. This last congruence is also trivially true for $k = n$.

We notice that the second and third columns of the matrix in (20) are, up to sign, identical from the third row on. In fact, we have the following.

Corollary 1. *For all $n \ge 3$,*

$$\left(P^{-1}\right)_{n,2} = -\left(P^{-1}\right)_{n,3} = (-1)^n C_{n-2}, \tag{21}$$

and for all $n \ge 2$,

$$\sum_{k=2}^{n} \left|\left(P^{-1}\right)_{n,k}\right| = C_{n-1}, \tag{22}$$

where C_n is the nth Catalan number, defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof. The identities in (21) are easy to verify by writing the binomial coefficient in (17), with $k = 2$ and $k = 3$, in terms of factorials.

To prove (22), we use the known combinatorial identities

$$\sum_{k=0}^{n} \binom{n+k}{k} = \binom{2n+1}{n+1}, \qquad \sum_{k=0}^{n} \frac{1}{n+k}\binom{n+k}{k} = \frac{1}{n}\binom{2n}{n},$$

which can be found, for instance, in [16] as special cases of (1.49) and (1.50), respectively. With these we obtain

$$\begin{aligned}\sum_{k=0}^{n} \frac{n-k}{n+k}\binom{n+k}{k} &= \sum_{k=0}^{n} \frac{2n}{n+k}\binom{n+k}{k} - \sum_{k=0}^{n}\binom{n+k}{k}\\ &= 2\cdot\binom{2n}{n} - \frac{2n+1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n} = C_n.\end{aligned}$$

Finally, with (17) we get upon replacing k by $n-k$,

$$\sum_{k=2}^{n} \left|\left(P^{-1}\right)_{n,k}\right| = \sum_{k=0}^{n-1} \frac{n-k-1}{n+k-1}\binom{n+k-1}{k} = C_{n-1}.$$

Keeping in mind that the term for $k = 1$ on the left vanishes when $n \geq 2$, this proves (22). $\square$

In concluding this section we remark that by (19) it is clear that the inverse of each finite matrix P_N is the corresponding $N \times N$ submatrix of P^{-1}. We also note that very recently Qi et al. [22] studied what amounts to the matrices P_N and also obtained their inverses. However, the approach and methods of proof in [22] are quite different from ours.

3. Proof of the Second Part of Theorem 1

Our way of proving the remainder of Theorem 1 consists of constructing polynomials from the conjectured rows of R^{-1} and the columns of R. We define

$$p_n(x) := \sum_{i=0}^{n-1} \left(R^{-1}\right)_{n,n-i} x^i, \tag{23}$$

that is, the coefficient sequence is row n of R^{-1}, with constant coefficient on the main diagonal. Second, we define

$$q_k(x) := \sum_{i=0}^{k-1} R_{k+i,k} x^i, \tag{24}$$

that is, the coefficient sequence is column k of R, with constant coefficient on the main diagonal.

The proof of Theorem 1 is complete if we can show that $R^{-1}R = I$, the infinite identity matrix. Our main tool for achieving this is to write both polynomials $p_n(x)$ and $q_k(x)$ as products. We begin with the easier case.

Lemma 1. *If the integer $k \geq 1$ is such that $k-1$ has the binary expansion $k-1 = \sum_{i\geq 0} u_i 2^i$, then*

$$q_k(x) = \prod_{i\geq 0} \left(1 + x^{2^i}\right)^{u_i}. \tag{25}$$

Proof. We prove a shifted version of the lemma. By (18), taken modulo 2, we have

$$q_{k+1}(x) = \sum_{j=0}^{k} \binom{k}{j}^* x^j \quad (k \geq 0), \tag{26}$$

with the starred binomial coefficient defined in (11). We claim: if $k \geq 0$ has the binary expansion $k = \sum_{i\geq 0} v_i 2^i$, then

$$q_{k+1}(x) = \prod_{i\geq 0} \left(1 + x^{2^i}\right)^{v_i}. \tag{27}$$

By a well-known congruence of Lucas, which holds for any prime modulus (see, e.g., [17]), we have

$$\binom{k}{j} \equiv \binom{v_d}{w_d} \cdots \binom{v_1}{w_1}\binom{v_0}{w_0} \pmod{2}, \tag{28}$$

where j has the binary representation $j = w_d 2^d + \cdots + w_1 2 + w_0$. Now, expanding the right-hand side of (27), we see that $q_{k+1}(x)$ is a $(0,1)$-polynomial. Furthermore, x^j, $0 \leq j \leq k$, has coefficient 1 exactly when the nonzero binary digits of j are a subset of the corresponding nonzero binary digits of k. But this, by (28), is equivalent to (26), which completes the proof. $\square$

n	$p_n(x)$	$q_n(x)$
1	1	1
2	1	$1+x$
3	$1-x$	$1+x^2$
4	1	$(1+x)(1+x^2)$
5	$(1-x)(1-x^2)$	$1+x^4$
6	$1-x^2$	$(1+x)(1+x^4)$
7	$1-x$	$(1+x^2)(1+x^4)$
8	1	$(1+x)(1+x^2)(1+x^4)$
9	$(1-x)(1-x^2)(1-x^4)$	$1+x^8$
10	$(1-x^2)(1-x^4)$	$(1+x)(1+x^8)$

Table 2: Factorizations of $p_n(x)$ and $q_n(x)$, $1 \leq n \leq 10$.

The polynomial factorization (25) is illustrated in Table 2, which also indicates that the polynomials $p_n(x)$ have a similar product representation.

Lemma 2. *Given an integer $n \geq 2$, let $\alpha \geq 1$ be the integer such that $2^{\alpha-1}+1 \leq n \leq 2^\alpha$, and let $2^\alpha - n = \sum_{i\geq 0} b_i 2^i$, $b_i \in \{0,1\}$. Then*

$$p_n(x) = \prod_{i\geq 0} \left(1 - x^{2^i}\right)^{b_i}. \tag{29}$$

Proof. Let $\overline{p}_n(x)$ be the polynomial obtained from $p_n(x)$ by taking the nonnegative residues (mod 2) of its coefficients. From the definition of $p_n(x)$ and from (10) we have for $n \geq 2$,

$$\overline{p}_n(x) = \sum_{j=1}^{n} \binom{2n-j-1}{n-j}^* x^{n-j} = \sum_{j=1}^{n} \binom{n+j-1}{j}^* x^j. \tag{30}$$

Now set $b := 2^\alpha - n$ and note that $0 \leq b \leq 2^{\alpha-1}-1$. We claim that for $0 \leq j \leq n-1$ we have

$$\binom{n+j-1}{j} \equiv \binom{b}{j} \pmod 2. \tag{31}$$

To prove this, we expand the left-hand binomial coefficient as

$$\begin{aligned}\binom{2^\alpha - b + j - 1}{j} &= \frac{(2^\alpha - b+j-1)(2^\alpha - b + j - 2)\cdots(2^\alpha - b)}{1\cdot 2\cdots j}\\ &= (-1)^j \frac{\left((b-j+1)-2^\alpha\right)\left((b-j+2)-2^\alpha\right)\cdots\left(b-2^\alpha\right)}{1\cdot 2\cdots j}.\end{aligned}$$

Since $j \leq n-1 \leq 2^\alpha - 1$, the terms 2^α in the numerator vanish modulo 2, and the last expression reduces to $\binom{b}{j}$ (mod 2), as claimed in (31).

Then with (30) we get

$$\overline{p}_n(x) = \sum_{j=1}^{n} \binom{b}{j}^* x^j = \sum_{j=1}^{b} \binom{b}{j}^* x^j,$$

where in the second equation we used the fact that $b < 2^{\alpha-1} \leq n$. But now the same argument as in the proof of Lemma 1, based on (28), shows that

$$p_n(x) = \prod_{i\geq 0} \left(1 \pm x^{2^i}\right)^{b_i}, \tag{32}$$

where the b_i are the binary digits of $b = 2^\alpha - n$.

To complete the proof, we note that the definition (9) of the Prouhet-Thue-Morse sequence implies

$$\prod_{i\geq 0} \left(1 - x^{2^i}\right) = \sum_{j\geq 0} (-1)^{t_j} x^j = 1 - x - x^2 + x^3 - x^4 + x^5 + \cdots; \tag{33}$$

see also, for instance, [1, Prop. 2]. By definition of $p_n(x)$, its sign pattern is the same as that in (33), namely the coefficient of x^j, if it is nonzero, is $(-1)^{t_j}$. Hence, comparing the left-hand side of (33) with (32) and using the uniqueness of binary expansions, we see that the signs in (32) must all be negative. This gives (29), and the proof is complete. □

We are now ready to finish the proof of Theorem 1.

Proof of Theorem 1, Part 2. Multiplying (23) by (24), we get

$$p_n(x)q_k(x) = \sum_{i=0}^{n+k-2} \left(\sum_{j=0}^{i} \left(R^{-1}\right)_{n,n-(i-j)} R_{k+j,k} \right) x^i. \tag{34}$$

When $i = n-k$, the inner sum becomes

$$\sum_{j=0}^{n-k} \left(R^{-1}\right)_{n,k+j} R_{k+j,k} = \sum_{j=k}^{n} \left(R^{-1}\right)_{n,j} R_{j,k},$$

and now we see that the right-hand side is the (n,k)th entry of the matrix product $R^{-1}R$. When $k = n$, we get the product of the constant coefficients of $p_n(x)$ and $q_n(x)$, namely 1. Therefore it remains to show that all other entries of $R^{-1}R$ vanish; by (34), this means:

$$\text{If } 1 \le k < n, \text{ then the coefficient of } x^{n-k} \text{ in } p_n(x)q_k(x) \text{ is } 0. \tag{35}$$

We first assume that $n-k$ is odd. By the conditions of Lemma 2 we see that $p_n(x)$ contains $1-x$ as a factor if and only if n is odd. Similarly, by Lemma 1, $q_k(x)$ is divisible by $1+x$ if and only if k is even. Then either $p_n(x)q_k(x)$ has no factor $1 \pm x$ or it has the factor $(1-x)(1+x) = 1-x^2$. In either case the coefficients of odd powers of x in $p_n(x)q_k(x)$ vanish, which proves (35) for the case $n-k$ odd.

Now we let $n-k$ be even and assume that

$$n - k \equiv 2^{\beta-1} \pmod{2^\beta}, \qquad \beta \ge 2. \tag{36}$$

This means that $n \ge 2^{\beta-1}+1$, and if α is as in the hypothesis of Lemma 2, then

$$2^\alpha - n \equiv 2^\beta - \overline{n} \pmod{2^\beta}, \tag{37}$$

where $\overline{n}$ is the smallest nonnegative residue of n modulo 2^β. We also see that the binary digits $b_0, b_1, \dots, b_{\beta-1}$ of $b = 2^\alpha - n$ depend only on $\overline{n}$. Next, by (36) we have

$$k - 1 \equiv \overline{k-1} \equiv \overline{n} - 2^{\beta-1} - 1 \pmod{2^\beta}, \tag{38}$$

where again $\overline{k-1}$ is the smallest nonnegative residue of $k-1$ modulo 2^β, and we see that the binary digits $u_0, u_1, \dots, u_{\beta-1}$ in Lemma 1 depend only on $\overline{k-1}$.

Combining (37) and (38), we get

$$(2^\alpha - n) + (k-1) \equiv (2^\beta - \overline{n}) + (\overline{n} - 2^{\beta-1} - 1) = 2^{\beta-1} - 1 \pmod{2^\beta},$$

which means that the last β binary digits of $(2^\alpha - n) + (k-1)$ are always $01\dots1$. This, in turn, implies that the polynomial $p_n(x)q_k(x)$, given that (36) holds, contains the factors

$$\big(1 \pm x^{2^{\beta-2}}\big)\big(1 \pm x^{2^{\beta-3}}\big)\cdots\big(1 \pm x^{2^0}\big),$$

and either neither of the factors $1 - x^{2^{\beta-1}}$ and $1 + x^{2^{\beta-1}}$, or both, which then gives the product $1 - x^{2^\beta}$; see Table 3 for an illustration. In particular this means that, by (36), the coefficient of x^{n-k} vanishes, which completes the proof of (35) and of the theorem. □

$\overline{n}$	$\overline{k}$	$b_2b_1b_0$	$u_2u_1u_0$	$p_n(x) \pmod{x^8}$	$q_k(x) \pmod{x^8}$
0	4	000	011	0	$(1+x^2)(1+x)$
1	5	111	100	$(1-x^4)(1-x^2)(1-x)$	$(1+x^4)$
2	6	110	101	$(1-x^4)(1-x^2)$	$(1+x^4)(1+x)$
3	7	101	110	$(1-x^4)(1-x)$	$(1+x^4)(1+x^2)$
4	0	100	111	$(1-x^4)$	$(1+x^4)(1+x^2)(1+x)$
5	1	011	000	$(1-x^2)(1-x)$	0
6	2	010	001	$(1-x^2)$	$(1+x)$
7	3	001	010	$(1-x)$	$(1+x^2)$

Table 3: Illustration of the proof of Theorem 1 for $\beta = 3$.

4. Row, Column, and Antidiagonal Sums

In this section we explore further properties of the infinite matrices P and R and their inverses. In particular, we consider the sums mentioned in the title, provided they are defined. It is clear that for all four infinite matrices in question, the diagonal sums diverge, but for R and R^{-1} we discuss the proportion of nonzero entries in the diagonals in Section 7.

Given a matrix M, we denote the sums of its nth row, kth column, and nth antidiagonal by $r_n(M)$, $c_k(M)$, and $a_n(M)$, respectively.

Corollary 2. *Let P be the infinite matrix defined in Section 2. Then*

$$r_n(P) = F_n, \qquad c_k(P) = 2^{k-1}, \qquad a_n(P) = \mathcal{P}_{n+2}, \tag{39}$$

where F_n is the nth Fibonacci number and $\mathcal{P}_n$ is the nth Padovan number. Furthermore, we have

$$r_n(P^{-1}) = (-1)^n \mathcal{F}_{n-2}, \qquad a_n(P^{-1}) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^{n-1} j}{2n-3j-2}\binom{2n-3j-2}{n-2j-1}, \tag{40}$$

where $\mathcal{F}_n$ is the nth Fine number.

The column sums $c_n(P^{-1})$ diverge; see also (20). The Fibonacci numbers, satisfying the recurrence relation $F_{n+1} = F_n + F_{n-1}$, are taken with the usual initialization $F_0 = 0$, $F_1 = 1$. The Padovan numbers are defined by a third-order recurrence relation, namely $\mathcal{P}_0 = 1$, $\mathcal{P}_1 = \mathcal{P}_2 = 0$, and $\mathcal{P}_{n+1} = \mathcal{P}_{n-1} + \mathcal{P}_{n-2}$ for $n \geq 3$. Numerous properties and references can be found in [21, A000931].

The Fine numbers have various combinatorial definitions; see [9] or [21, A000957] for properties and references. One property we will use is the sum

$$\mathcal{F}_n = \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k(k+1)\binom{2n-k}{n}; \tag{41}$$

see [9, p. 251]. Finally, the unnamed sequence on the right of (40) is listed as A132364 in [21], where some properties can also be found.

Proof of Corollary 2. We know from (15) that $P_{n,k} = \binom{k-1}{n-k}$. This immediately gives

$$r_n(P) = \sum_{k=1}^{n}\binom{k-1}{n-k} = \sum_{k=0}^{n-1}\binom{n-k-1}{k} = F_n,$$

where the right-hand equation is a well-known identity that is easy to verify with the Fibonacci recursion. Next, using the binomial theorem, we have

$$c_k(P) = \sum_{n=k}^{2k-1}\binom{k-1}{n-k} = \sum_{n=0}^{k-1}\binom{k-1}{n} = 2^{k-1}.$$

Further,

$$a_n(P) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} P_{n-j,j+1} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{j}{n-2j-1},$$

and using the sum on the right, we get

$$\begin{aligned} a_{n+1}(P) + a_n(P) &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\left(\binom{j}{n-2j} + \binom{j}{n-2j-1}\right) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\binom{j+1}{n-2j} \\ &= \sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor}\binom{j}{n+2-2j} = a_{n+3}(P), \end{aligned}$$

where in the last line we have used the fact that the summand for $j = 0$ is zero. Thus the sequence $a_n(P)$ satisfies the recurrence relation for the Padovan numbers, and it is easy to verify by direct computation that $a_n(P) = \mathcal{P}_{n+2}$ for $n = 1, 2, 3$. This proves the identities in (39).

To prove the identities in (40), we use (17). First, we get

$$\begin{aligned} r_n(P^{-1}) &= \sum_{k=2}^{n} (-1)^{n-k} \frac{k-1}{2n-k-1} \binom{2n-k-1}{n-1} \\ &= \sum_{k=2}^{n} (-1)^{n-k} \frac{k-1}{n-1} \binom{2n-k-2}{n-2}, \end{aligned}$$

where we have used a well-known combinatorial identity. Shifting the summation, we then get

$$r_n(P^{-1}) = \frac{(-1)^n}{n-1} \sum_{k=0}^{n-2} (-1)^k (k-1) \binom{2(n-2)-k}{n-2} = (-1)^n \mathcal{F}_{n-2},$$

where the second equality follows from comparison with (41). Finally, the right-hand identity in (40) follows immediately from (17) and the fact that

$$a_n(P^{-1}) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(P^{-1}\right)_{n-j,j+1}.$$

This completes the proof of the corollary. □

Corollary 3. *Let R be the infinite matrix defined in Section 1. Then*

$$r_n(R) = s(n) \qquad \text{and} \qquad c_k(R) = 2^{d(k-1)}, \tag{42}$$

where $d(k-1)$ is the sum of binary digits of $k-1$. Furthermore, $a_n(R)$ is the sequence defined recursively by $a_1(R) = 1$, $a_2(R) = 0$, and

$$a_{2n+2}(R) = a_n(R), \qquad a_{2n+1}(R) = a_n(R) + a_{n+1}(R) \qquad (n \geq 1). \tag{43}$$

Before proving these results, we note that $c_k(R)$ is also the number of odd entries in row $k-1$ of Pascal's triangle, as can be seen in the proof below. The equivalence of the two forms of $c_k(R)$ is in fact a well-known result due to Glaisher [15], based on Lucas's congruence (28). The sequence $c_{k+1}(R)$ is known as Gould's sequence; numerous properties, remarks, and references can be found in [21, A001316].

The sequence in (43) is listed in [21] as A106345, where the explicit formula

$$a_n(R) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{j}{n-1-2j}^{*} \tag{44}$$

can also be found. A proof is given below.

Proof of Corollary 3. The first identity in (42) follows from the definition of the matrix R and from (4). Next, by definition of the polynomial $q_k(x)$ in Section 3 and by (26), we have

$$c_k(R) = q_k(1) = \sum_{j=0}^{k-1} \binom{k-1}{j}^*,$$

which counts the number of odd entries in row $k-1$ of Pascal's triangle, as claimed. On the other hand, by Lemma 1 we have

$$c_k(R) = q_k(1) = \prod_{i\geq 0} 2^{u_i} = 2^{\sum_{i\geq 0} u_i} = 2^{d(k-1)},$$

which proves the second identity in (42). Next, by (14) and (15) we have $R_{n,k} = \binom{n-1}{n-k}^*$, and then (44) follows from the fact that

$$a_n(R) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} R_{n-j,j+1}.$$

Finally, to show that the sequence $a_n(R)$ satisfies (43), we note that from (28) we get for nonnegative integers r, s and $a, b \in \{0, 1\}$ (as also used by Carlitz [7, p. 18f.]),

$$\binom{2r+a}{2s+b}^* = \begin{cases} 0 & \text{when } a = 0 \text{ and } b = 1, \\ \binom{r}{s}^* & \text{otherwise.} \end{cases} \tag{45}$$

Using this and (44), we first get

$$\begin{aligned} a_{2n+2}(R) &= \sum_{j\geq 0} \binom{j}{2n+1-2j}^* = \sum_{i\geq 0} \binom{2i+1}{2n-1-4i}^* \\ &= \sum_{i\geq 0} \binom{i}{n-1-2i}^* = a_n(R), \end{aligned}$$

where we have used (45) twice. Further, splitting j into even and odd, we get

$$\begin{aligned} a_{2n+1}(R) &= \sum_{j\geq 0} \binom{j}{2n-2j}^* = \sum_{i\geq 0} \binom{2i}{2n-4i}^* + \sum_{i\geq 0} \binom{2i+1}{2n-2-4i}^* \\ &= \sum_{i\geq 0} \binom{i}{n-2i}^* + \sum_{i\geq 0} \binom{i}{n-1-2i}^* = a_{n+1}(R) + a_n(R). \end{aligned}$$

This proves the identities in (43); the initial conditions are easy to verify by direct computation. □

Corollary 4. *If R is the infinite matrix defined in Section 1, then*

$$r_n(R^{-1}) = \begin{cases} 1 & \text{when } n \text{ is a power of } 2, \\ 0 & \text{otherwise,} \end{cases} \tag{46}$$

and $a_n(R^{-1})$ is the sequence defined recursively by $a_1(R^{-1}) = 1$, $a_2(R^{-1}) = 0$, and

$$a_{2n}(R^{-1}) = -a_{n+1}(R^{-1}), \quad a_{2n+1}(R^{-1}) = a_n(R^{-1}) + a_{n+1}(R^{-1}) \quad (n \geq 1). \tag{47}$$

To supplement Corollary 4, we consider $n = 2^{\alpha-1} + 1$ for $\alpha \geq 2$. Then $2^\alpha - n = 2^{\alpha-1} - 1$, and by Lemma 2 the polynomial $p_n(x)$ has degree $2^{\alpha-1} - 1 = n - 2$, and all $n - 1$ coefficients are nonzero; see also Table 2. Hence the column sums of R^{-1} diverge for $k \geq 2$, while $c_1(R^{-1}) = 1$; see (8) as an illustration.

The sequence of antidiagonal sums is given by

$$a_n(R^{-1}) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{2n-3j-2}{n-2j-1}^* (-1)^{t_{n-2j-1}}, \tag{48}$$

where (t_n) is the Prouhet-Thue-Morse sequence defined in (9). The sequence (47) is listed in [21] as A342682; starting with $n = 1$, the first 30 terms are

$$1, 0, 1, -1, 1, 1, 0, -1, 0, -1, 2, 0, 1, 1, -1, 0, -1, 1, -1, -2, 1, 0, 2, -1, 1, -1, 2, 1, 0, 0.$$

Proof of Corollary 4. By definition of the polynomials $p_n(x)$ at the beginning of Section 3, and by Lemma 2, we have $r_n(R^{-1}) = p_n(1) = 0$ unless $2^\alpha - n = 0$, that is, n is a power of 2. This proves (46).

Next, using Theorem 1 and the fact that

$$a_n(R^{-1}) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(R^{-1}\right)_{n-j,j+1},$$

we immediately get (48). Finally, to obtain the recurrence relations in (47) we proceed as we did in the previous proof. With (48) we get

$$a_{2n}(R^{-1}) = \sum_{j \geq 0} \binom{4n-3j-2}{n-2j-1}^* (-1)^{t_{2n-2j-1}},$$

and with (45) we see that the terms with even j vanish. We therefore set $j = 2i - 1$ ($i \geq 1$) and note that by (9) we have $t_{2n-4i+1} = 1 - t_{n-2i}$, so that with (45),

$$a_{2n}(R^{-1}) = \sum_{i \geq 1} \binom{4n-6i+1}{n-4i+1}^* (-1)^{t_{2n-4i+1}} = \sum_{i \geq 1} \binom{2n-3i}{n-2i}^* (-1)^{1-t_{n-2i}}.$$

This last identity can actually be taken over $i \geq 0$ since the term for $i = 0$ is

$$\binom{2n}{n} = \frac{2n}{n}\binom{2n-1}{n-1} \equiv 0 \pmod 2$$

and thus, again with (48), we have $a_{2n}(R^{-1}) = -a_{n+1}(R^{-1})$, as desired.
Next we have

$$a_{2n+1}(R^{-1}) = \sum_{j\geq 0}\binom{4n-3j}{2n-2j}^{*}(-1)^{t_{2n-2j}}.$$

This time we apply the first identity in (9), giving $t_{2n-2j} = t_{n-j}$. Then, splitting j into even and odd integers and using (45), we get

$$\begin{aligned}
a_{2n+1}(R^{-1}) &= \sum_{i\geq 0}\binom{4n-6i}{n-4i}^{*}(-1)^{t_{2n-4i}} + \sum_{i\geq 0}\binom{4n-6i-3}{2n-4i-2}^{*}(-1)^{t_{2n-4i-2}} \\
&= \sum_{i\geq 0}\binom{2n-3i}{n-2i}^{*}(-1)^{t_{n-2i}} + \sum_{i\geq 0}\binom{2n-3i-2}{n-2i-1}^{*}(-1)^{t_{n-2i-1}} \\
&= a_{n+1}(R^{-1}) + a_n(R^{-1}),
\end{aligned}$$

having again used (48). This proves the second identity in (47); the initial conditions are again easy to verify. □

5. A Connection with Compositions of Integers

Another property of the matrix R and its inverse, again related to a sequence of row sums, reveals an intriguing connection with compositions of integers. Given an integer $m \geq 1$, a *composition* of m is an ordered set (or a finite sequence) of positive integers whose elements sum to m. In contrast to partitions, sequences that differ in the order of their terms define different compositions. Let $\mathcal{C}(m)$ be the set of compositions of m. Thus, for example, the compositions of $m = 4$ are

$$\mathcal{C}(4) = \{(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,3),\ (1,2,1),\ (1,1,2),\ (1,1,1,1)\}.$$

In general, the cardinality of $\mathcal{C}(m)$ is 2^{m-1}; see, for instance, [24, p. 15].

Suppose that $\mu = (\mu_1, \ldots, \mu_\ell) \in \mathcal{C}(m)$. We define $\gamma, \delta : \mathcal{C}(m) \to \mathcal{C}(m+1)$ by

$$\gamma(\mu) = (\mu_1, \ldots, \mu_\ell + 1), \qquad \delta(\mu) = (\mu_1, \ldots, \mu_\ell, 1).$$

Then the images of $\mathcal{C}(m)$ under γ and δ are disjoint subsets of $\mathcal{C}(m+1)$, and since they both have cardinality 2^{m-1}, their union is $\mathcal{C}(m+1)$.

We now define a sequence $C(n)$, $n = 1, 2, \ldots$, of compositions. Let $C(1) = () = \emptyset$, the empty composition, and $C(2) = (1)$, the unique composition of $m = 1$. Then for $n \geq 2$, let

$$C(2n-1) = \gamma(C(n)), \qquad C(2n) = \delta(C(n)). \tag{49}$$

We can see by induction that for any $m \geq 1$, the terms $C(2^{m-1}+1), \dots, C(2^m)$ of this sequence are exactly the 2^{m-1} compositions $\mathcal{C}(m)$; see Table 4.

Next, if $\mu = (\mu_1, \dots, \mu_\ell) \in \mathcal{C}(m)$ as before, we define the products of the parts of μ and a corresponding sequence by

$$f(\mu) := \mu_1 \cdots \mu_\ell, \qquad f(n) := f(C(n)) \qquad (n \geq 2), \tag{50}$$

with $f(1) = 1$, the usual convention for an empty product. This sequence is A124758 in [21]. See Table 4 for the first 16 values of $f(n)$.

n	$C(n)$	$f(n)$	n	$C(n)$	$f(n)$
1	()	1	9	(4)	4
2	(1)	1	10	(3, 1)	3
3	(2)	2	11	(2, 2)	4
4	(1, 1)	1	12	(2, 1, 1)	2
5	(3)	3	13	(1, 3)	3
6	(2, 1)	2	14	(1, 2, 1)	2
7	(1, 2)	2	15	(1, 1, 2)	2
8	(1, 1, 1)	1	16	(1, 1, 1, 1)	1

Table 4: $C(n)$ and $f(n)$, $1 \leq n \leq 16$.

Returning to the matrix R of Section 1, we now consider the Hadamard (or element-wise) product $R \odot |R^{-1}|$ of R and the matrix consisting of the absolute values of R^{-1}; see the upper-left 10×10 submatrix in (51).

$$\left(R \odot |R^{-1}|\right)_{10} = \begin{pmatrix} 1 &&&&&&&&& \\ \cdot & 1 &&&&&&&& \\ \cdot & 1 & 1 &&&&&&& \\ \cdot & \cdot & \cdot & 1 &&&&&& \\ \cdot & \cdot & 1 & 1 & 1 &&&&& \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 &&&& \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 &&& \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 && \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 \end{pmatrix}. \tag{51}$$

The row sums of this matrix are 1, 1, 2, 1, 3, 2, 2, 1, 4, 3. Comparing them with Table 4, we see that these values are identical with $f(1), \dots, f(10)$. The following result shows that this is not a coincidence.

Theorem 3. *For all $n \geq 1$ we have*

$$r_n\left((R \odot |R^{-1}|\right) = f(n), \tag{52}$$

where r_n is the nth row sum and $f(n)$ is the product of parts, as defined in (50). Furthermore, for all $n \geq 1$ we have the explicit expression

$$r_n\big(R \odot |R^{-1}|\big) = \sum_{k=1}^{n} \binom{k-1}{n-k}^* \binom{2n-k-1}{n-k}^*. \tag{53}$$

Proof. By (14) or (18) and by (10) we have

$$\big((R \odot |R^{-1}|\big)_{n,k} = \binom{k-1}{n-k}^* \binom{2n-k-1}{n-k}^*,$$

which immediately gives (53). We prove (52) by showing that the sequences on the right of (52) and (53) satisfy the same recurrence relations.

We begin with the sequence $f(n)$. In Table 4 we see that $f(1) = f(2) = 1$. Denote the final part in the composition $C(n)$ by $\mu_\ell(n)$; then by (49) and (50) we have

$$f(2n-1) = \left(1 + \frac{1}{\mu_\ell(n)}\right) \cdot f(n), \qquad f(2n) = f(n) \qquad (n \geq 2). \tag{54}$$

To deal with the first identity in (54), we let $\nu(n)$ be the 2-adic valuation of n, and we claim that

$$\mu_\ell(n) = \nu(n-1) + 1 \qquad (n \geq 2); \tag{55}$$

see Table 4 for the first 15 instances of this. We prove (55) by induction on n. The cases $n = 2, 3$ are clear by Table 4. Now suppose that (55) holds up to $2n-2$, for some $n \geq 2$. By the first identity in (49) and by the induction hypothesis we have

$$\begin{aligned} \mu_\ell(2n-1) &= \mu_\ell(n) + 1 = (\nu(n-1) + 1) + 1 \\ &= \nu(2(n-1)) + 1 = \nu((2n-1) - 1) + 1, \end{aligned}$$

as claimed, where we have also used an obvious property of the 2-adic valuation. Next, by the second identity in (49) we have $\mu_\ell(2n) = 1$, while $\nu(2n-1) + 1 = 1$, so (55) holds almost trivially in this case. Altogether we have thus shown that

$$f(2n-1) = \left(1 + \frac{1}{\nu(n-1) + 1}\right) \cdot f(n), \qquad f(2n) = f(n) \qquad (n \geq 2). \tag{56}$$

To deal with the right-hand side of (53), we denote this sum by $g(n)$ and verify that $g(1) = g(2) = 1$. Also,

$$g(2n) = \sum_{k=1}^{2n} \binom{k-1}{2n-k}^* \binom{4n-k-1}{2n-k}^*.$$

When k is odd, then both factors of each summand vanish by (45). Hence we set $k = 2j$ and get, also by (45),

$$g(2n) = \sum_{j=1}^{n} \binom{2j-1}{2n-2j}^* \binom{4n-2j-1}{2n-2j}^*$$
$$= \sum_{j=1}^{n} \binom{j-1}{n-j}^* \binom{2n-j-1}{n-j}^* = g(n). \tag{57}$$

For the odd case, it is more convenient to consider $g(2n+1)$ rather than $g(2n-1)$. We split the corresponding sum according to $k = 2j$ and $k = 2j+1$, obtaining

$$g(2n+1) = \sum_{k=1}^{2n+1} \binom{k-1}{2n-k+1}^* \binom{4n-k+1}{2n-k+1}^*$$
$$= \sum_{j=1}^{n} \binom{2(j-1)+1}{2n-2j+1}^* \binom{4n-2j+1}{2n-2j+1}^* + \sum_{j=0}^{n} \binom{2j}{2n-2j}^* \binom{4n-2j}{2n-2j}^*.$$

Using (45) again, we get

$$g(2n+1) = \sum_{j=1}^{n} \binom{j-1}{n-j}^* \binom{2n-j}{n-j}^* + \sum_{j=0}^{n} \binom{j}{n-j}^* \binom{2n-j}{n-j}^*. \tag{58}$$

Denoting these last two sums by $T_1(n)$ and $T_2(n)$, we claim that

$$\frac{T_2(n)}{T_1(n)} = \nu(n) + 1 \qquad (n \geq 1), \tag{59}$$

where $\nu(n)$ is again the 2-adic valuation of n. It is important to note that $T_1(n) \neq 0$ and $T_2(n) \neq 0$ for any $n \geq 1$ since the summand belonging to $j = n$ is always 1 in both cases. We assume (59) to be true and note that a shift in the index of summation gives

$$T_2(n) = \sum_{j=1}^{n+1} \binom{j-1}{n+1-j}^* \binom{2(n+1)-j-1}{n+1-j}^* = g(n+1).$$

Then with (58) we get

$$g(2n+1) = \left(1 + \frac{1}{\nu(n)+1}\right) g(n+1).$$

Replacing n by $n-1$ in this last identity and recalling (57) and the initial conditions $g(1) = g(2) = 1$, we see that the sequences $f(n)$ and $g(n)$ satisfy the same recurrence relation (56), and are therefore identical. This proves (52).

It remains to prove (59). We do so by induction on n. The cases $n = 1, 2$ can be verified by direct computation. Suppose now that (59) holds up to $2n - 1$, for some $n \geq 2$. Then, using (45) again repeatedly, we get

$$\begin{aligned} T_1(2n) &= \sum_{j=1}^{2n} \binom{j-1}{2n-j}^* \binom{4n-j}{2n-j}^* = \sum_{i=1}^{n} \binom{2(i-1)+1}{2n-2i}^* \binom{4n-2i}{2n-2i}^* \\ &= \sum_{i=1}^{n} \binom{i-1}{n-i}^* \binom{2n-i}{n-i}^* = T_1(n) \end{aligned}$$

and similarly, separating summands according to $j = 2i$, $j = 2i - 1$,

$$\begin{aligned} T_2(2n) &= \sum_{j=1}^{2n} \binom{j}{2n-j}^* \binom{4n-j}{2n-j}^* \\ &= \sum_{i=1}^{n} \binom{2i}{2n-2i}^* \binom{4n-2i}{2n-2i}^* + \sum_{i=1}^{n} \binom{2(i-1)+1}{2n-2i+1}^* \binom{4n-2i+1}{2n-2i+1}^* \\ &= \sum_{i=1}^{n} \binom{i}{n-i}^* \binom{2n-i}{n-i}^* + \sum_{i=1}^{n} \binom{i-1}{n-i}^* \binom{2n-i}{n-i}^* = T_2(n) + T_1(n). \end{aligned}$$

Hence by the induction hypothesis and the property $\nu(2n) = \nu(n) + 1$ we get

$$\frac{T_2(2n)}{T_1(2n)} = \frac{T_2(n) + T_1(n)}{T_1(n)} = \frac{T_2(n)}{T_1(n)} + 1 = \big(\nu(n) + 1\big) + 1 = \nu(2n) + 1, \tag{60}$$

which was to be shown for the even case. In the odd case we proceed similarly, obtaining

$$T_1(2n+1) = T_2(n), \qquad T_2(2n+1) = T_2(n);$$

we leave the details to the reader. These identities give

$$\frac{T_2(2n+1)}{T_1(2n+1)} = \frac{T_2(n)}{T_2(n)} = 1 = \nu(2n+1) + 1,$$

as desired. This proves (59) by induction, and the proof of Theorem 3 is complete. □

An alternative to the Hadamard product of R and $|R^{-1}|$ in Theorem 3 is as follows: Given an $n \geq 1$ as in (52), take an integer m such that $n \leq 2^m$ and reflect the submatrix R_{2^m} about its main antidiagonal to get R'_{2^m}. Now take the Hadamard product of R_{2^m} and R'_{2^m}. The nth row sum is then equal to $f(n)$ as well. An explanation is contained in Corollary 9 in Section 7.

6. The Sierpiński Triangle and Matrix

As we saw, the columns of the infinite matrix R of Section 1 correspond to the rows of the Pascal triangle modulo 2, which is also known as the Sierpiński triangle. If we left-justify this triangle, we get the infinite Sierpiński matrix S, and as before we denote by S_N the upper-left $N \times N$ submatrix, as shown in (61) for $N = 8$.

$$S_8 := \begin{pmatrix} 1 &&&&&&& \\ 1 & 1 &&&&&& \\ 1 & \cdot & 1 &&&&& \\ 1 & 1 & 1 & 1 &&&& \\ 1 & \cdot & \cdot & \cdot & 1 &&& \\ 1 & 1 & \cdot & \cdot & 1 & 1 && \\ 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{61}$$

It was shown by Callan [6] that the inverse matrix S^{-1} is again the Sierpiński matrix, but with entries 0, 1, and -1, and the sign pattern determined by the Prouhet-Thue-Morse sequence, as was the case with R^{-1}; see (62).

$$S_8^{-1} = \begin{pmatrix} 1 &&&&&&& \\ -1 & 1 &&&&&& \\ -1 & \cdot & 1 &&&&& \\ 1 & -1 & -1 & 1 &&&& \\ -1 & \cdot & \cdot & \cdot & 1 &&& \\ 1 & -1 & \cdot & \cdot & -1 & 1 && \\ 1 & \cdot & -1 & \cdot & -1 & \cdot & 1 & \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{pmatrix}. \tag{62}$$

The purpose of this brief section is to determine the row and antidiagonal sums of S and S^{-1}. More general kinds of diagonal sums are considered in [20].

Corollary 5. *If S is the Sierpiński matrix, then for $n \geq 1$ we have*

$$r_n(S) = 2^{d(n-1)}, \qquad a_n(S) = s(n), \tag{63}$$

where $d(n-1)$ is the sum of binary digits of $n-1$, and $s(n)$ is the nth Stern number. Furthermore,

$$r_n(S^{-1}) = \begin{cases} 1, & n = 1, \\ 0, & n \geq 2, \end{cases} \qquad a_n(S^{-1}) = \begin{cases} 0, & n \equiv 0 \pmod 3, \\ 1, & n \equiv 1 \pmod 3, \\ -1, & n \equiv 2 \pmod 3. \end{cases} \tag{64}$$

Proof. The (n,k)-entry of the matrix S is clearly $S_{n,k} = \binom{n-1}{k-1}^*$, and so

$$r_n(S) = \sum_{k=1}^{n} \binom{n-1}{k-1}^*.$$

The first identity in (63) then follows from the proof of Corollary 3. Next, we have

$$a_n(S) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} S_{n-j,j+1} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j}^* = s(n),$$

which follows from (12). To obtain the first identity of (64) we consider the matrix product $S^{-1}S = I$, the infinite identity matrix. Then we see that the nth row sum of S^{-1} is just the element $(S^{-1}S)_{n,1} = I_{n,1}$, as claimed.

For the second part of (64), we consider

$$a_n(S^{-1}) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(S^{-1}\right)_{n-j,j+1} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j}^* (-1)^{t_{n-2j-1}}. \tag{65}$$

In a way that is completely analogous to the proof of (47) we can use (65) to show that

$$a_{2n} = -a_n, \qquad a_{2n+1} = a_n + a_{n+1}, \tag{66}$$

with initial conditions $a_1 = 1$ and $a_2 = -1$, where for ease of notation we have suppressed the dependence on S^{-1}. We leave the details to the reader.

Next, we use induction and (66) to prove the second identity in (64). The cases $n = 1, 2, 3$ are easy to verify. We fix an $n \geq 4$ and suppose that the statement holds up to $n-1$. If $n \equiv 0 \pmod 3$, then $n = 2k + r$ with $r = 0, 1$, or 2 and $k \equiv r \pmod 3$. If $r = 0$, then $a_n = -a_k = 0$; if $r = 1$, then $a_n = a_k + a_{k+1} = 1 + (-1) = 0$; and if $r = 2$, then $a_n = -a_{k+1} = 0$, where in all three cases we have used (66) and the induction hypothesis. The cases $n \equiv 1, 2 \pmod 3$ are analogous. □

7. Further Remarks

We close with a few remarks related to the previous sections.

1. It may be of interest to determine the number of 1s in the nth row of the matrix R^{-1}, that is, consider the sequence 1, 1, 1, 1, 2, 1, 1, 1, 4, 2, ...; see (8). Since R^{-1} has entries 0, 1, -1 only, we can use Corollary 4 to get the following result.

Corollary 6. *The number of* 1*s in the* n*th row of the infinite matrix* R^{-1} *is*

$$\begin{cases} 1 & \text{when } n \text{ is a power of } 2, \\ \frac{1}{2} r_n & \text{otherwise,} \end{cases}$$

where $r_n = r_n(|R^{-1}|)$ *is given by*

$$r_n = \sum_{k=0}^{n-1} \binom{n-1+k}{k}^*, \tag{67}$$

and equivalently by the recurrence $r_1 = 1$ *and*

$$r_{2n} = r_n, \qquad r_{2n+1} = 2r_{n+1} \qquad (n \geq 1). \tag{68}$$

The sequence r_n also satisfies $r_n = 2^{z(n-1)}$, where $z(n-1)$ is the number of 0s in the binary expansion of $n-1$ ($n \geq 2$); see [21, A080100]. Furthermore, (68) shows that r_n is positive and even exactly when n is not a power of 2.

Proof of Corollary 6. By Corollary 4, the number of 1s in row n of R^{-1} is 1 when n is a power of 2, and is equal to the number of -1s otherwise. Hence it is half the number of nonzero terms in row n of R^{-1}. But by (10) this number of nonzero terms is

$$r_n(|R^{-1}|) = \sum_{k=1}^{n} \binom{2n-k-1}{n-k}^*,$$

and by reversing the order of summation we get (67). Finally, the recurrence (68) can be obtained from (67) in analogy to the proofs of Corollaries 4 and 5. □

2. The matrix R and Theorem 1 can be used to establish some identities connecting the Stern sequence $s(n)$ defined in (1) and the Prouhet-Thue-Morse sequence (t_n), which was defined in (9).

Corollary 7. *For any integer* $n \geq 2$ *we have*

$$\sum_{k=0}^{n-2} \binom{n-1+k}{k}^* (-1)^{t_k} s(n-k) = 1, \tag{69}$$

and as a consequence, for any $m \geq 1$,

$$\sum_{k=0}^{2^m} (-1)^{t_k} s(2^m + 1 - k) = 0. \tag{70}$$

Proof. Let V and $\mathbf{1}$ be the infinite column vectors consisting of the Stern sequence and the sequence with only 1s, respectively. Then by the definition of R we have $V = R \cdot \mathbf{1}$, and multiplying both sides of this identity by R^{-1} from the left, we get $R^{-1} \cdot V = \mathbf{1}$. This last matrix identity, together with Theorem 1 and the definition of the vector V, gives (69) upon reversing the order of summation.

To obtain (70), we set $n = 2^m + 1$ and note that for $0 \leq k \leq n-2 = 2^m - 1$, by (28) we have $\binom{n-1+k}{k} \equiv 1 \pmod 2$. Now (69) implies (70) if we also note that $t_k = 1$ for $k = 2^m$, which follows from (9). □

We now use the identity (70) to obtain the following consequence.

Corollary 8. *For any integer* $m \geq 2$ *we have*

$$\sum_{k=2^m}^{2^{m+1}-1} (-1)^{t_{k+1}} s(k) = -1. \tag{71}$$

It is useful to compare (71) with the identities

$$\sum_{k=2^m}^{2^{m+1}-1} s(k) = 3^m, \qquad \sum_{k=2^m}^{2^{m+1}-1} (-1)^{k+1} s(k) = 3^{m-1}. \tag{72}$$

The first of these was already known to Stern [25]; see also [11] for further references and remarks. The second sum in (72) follows from the first and the recurrence relations in (1). To illustrate the identities in (71) and (72), we take $m = 3$ and obtain from Table 1,

$$\begin{aligned} -1+4+3-5+2-5-3+4 &= -1, \\ 1+4+3+5+2+5+3+4 &= 27, \\ -1+4-3+5-2+5-3+4 &= 9. \end{aligned}$$

Proof of Corollary 8. We separate the term for $k = 0$ in (70), obtaining

$$(-1)^{t_0} s(2^m+1) + \sum_{k=1}^{2^m} (-1)^{t_k} s(2^m+1-k) = 0. \tag{73}$$

We recall that $t_0 = 0$ and note that $s(2^m + 1) = m + 1$, which follows by induction from the fact that (1) gives

$$s(2^m+1) = s(2^{m-1}) + s(2^{m-1}+1) = 1 + s(2^{m-1}+1),$$

with the initial condition $s(2^0 + 1) = s(2) = 1$. Hence the identity (73) becomes, after shifting the summation,

$$\sum_{k=0}^{2^m-1} (-1)^{t_{k+1}} s(2^m-k) = -(m+1). \tag{74}$$

Now we replace m by $m+1$ in (74) and subtract (74) from the identity thus obtained. This gives

$$\sum_{k=2^m}^{2^{m+1}-1} (-1)^{t_{k+1}} \big(s(2^{m+1}-k) - s(2^m-k)\big) = -1. \tag{75}$$

Finally, using the well-known symmetry of the Stern sequence between 2^m and 2^{m+1}, we get

$$s(2^{m+1}-k) - s(2^m-k) = s(2^m+k) - s(2^m-k) = s(k),$$

where the final equality was proved in Corollary 3.1 of [11]. This, with (75), gives the desired identity (71). $\square$

3. Consider the 8×8 submatrix R_8 (see (6)) and reflect it along the main antidiagonal. The resulting matrix turns out to be the inverse $(R_8)^{-1}$, up to signs. This is in fact true in general, as the following corollary shows.

Corollary 9. *For any integer $m \geq 1$ we have*

$$\left(R_{2^m}^{-1}\right)_{n,k} = (-1)^{t_{n-k}} \left(R_{2^m}\right)_{2^m+1-k,2^m+1-n}, \tag{76}$$

where (t_n) is the Prouhet-Thue-Morse sequence.

Proof. It is clear that the reflection of the matrix entry with index (n,k), i.e., row n and column k, $1 \leq n, k \leq 2^m$, is the entry with index $(2^m + 1 - k, 2^m + 1 - n)$, and vice versa. Now by (14) or (18) we have $R_{k,j} = \binom{j-1}{k-j}^*$, and thus

$$\left(R_{2^m}\right)_{2^m+1-k,2^m+1-n} = \binom{2^m - n}{n - k}^*.$$

Hence by Theorem 1 we are done if we can show that

$$\binom{2^m - n}{n - k} \equiv \binom{2n - k - 1}{n - k} \pmod 2, \qquad 1 \leq k \leq n \leq 2^m.$$

This can be done in the same way as in the proof of (31); we leave the details to the reader. $\square$

4. The matrix R can also be used to visualize an identity satisfied by the Stern polynomials $s(n;x)$ defined by (2) and (3). We divide the submatrix R_{16} into 4 blocks as follows.

$$\left|\begin{array}{cccccccc|cccccccc}
\hline
1 &&&&&&&&&&&&&&& \\
\cdot & 1 &&&&&&&&&&&&&& \\
\cdot & 1 & 1 &&&&&&&&&&&&& \\
\cdot & \cdot & \cdot & 1 &&&&&&&&&&&& \\
\cdot & \cdot & 1 & 1 & 1 &&&&&&&&&&& \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 &&&&&&&&&& \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 &&&&&&&&& \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 &&&&&&&& \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 &&&&&&& \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 &&&&&& \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 &&&&& \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 &&&& \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 &&& \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 && \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\hline
\end{array}\right|$$

Then, using the definition of R, as given just before (6), we can read off the identities

$$s(8+j;x) = s(j;x) + x^j \cdot s(8-j;x), \qquad 0 \le j \le 8. \tag{77}$$

If we take, for instance, $j = 5$, then $s(13;x) = s(5;x) + x^j \cdot s(3;x)$ or, by Table 1,

$$1 + x + x^2 + x^5 + x^6 = (1 + x + x^2) + x^5(1 + x);$$

recall that the rows of R must be read from right to left, beginning at the main diagonal, in order to obtain the coefficients of the Stern polynomials. The identity (77) is a special case of Lemma 2.1 in [11]: For any $m \ge 0$ and $0 \le j \le 2^m$ we have

$$s(2^m + j;x) = s(j;x) + x^j \cdot s(2^m - j;x).$$

5. As mentioned at the beginning of Section 4, the diagonal sums for the various infinite matrices diverge. However, we observe that in the case of the matrix R (see (6)), every entry in the main diagonal is 1, and so is every second entry in the first subdiagonal. Furthermore, two out of four entries in the second subdiagonal and one out of four entries in the third subdiagonal are 1; hence the sequence of ratios of entries 1 among all entries begins with 1, 2, 2, 4. As we will see, it is no coincidence that this is the beginning of Gould's sequence $G(n) := 2^{d(n)}$, where $d(n)$ is the sum of digits in the binary expansion of n; see [21, A001316]. We recall that the sequence $G(n)$ already occurred as column sums of R in (42).

Corollary 10. *The ratio of nonzero entries among all entries in the nth diagonals of the infinite matrices R, R^{-1}, S, and S^{-1} is $1/G(n)$, where $\big(G(n)\big)_{n\ge 0}$ is Gould's sequence and the main diagonal corresponds to $n = 0$.*

Proof. We use three facts that are likely known, but for the sake of completeness we indicate how to prove them. First, for any $m \ge 0$ the finite Sierpiński matrix S_{2^m} is symmetric about its main antidiagonal; see (61). This can be stated as

$$\binom{j}{n}^* = \binom{2^m - 1 - n}{2^m - 1 - j}^*, \qquad 0 \le j, n \le 2^m - 1. \tag{78}$$

The identity (78) can be proved by writing the right-hand binomial coefficient (without the asterisk) in terms of factorials, then cancel equal factors in numerator and denominator, and finally proceed as we did in the proof of (36).

Second, for $m \ge 0$ we require the identity

$$G(n) \cdot G(2^m - 1 - n) = 2^m, \qquad 0 \le n \le 2^m - 1. \tag{79}$$

By definition of $G(n)$, this identity is equivalent to $d(n) + d(2^m - 1 - n) = m$ which, in turn, follows from the fact that the binary expansion of $2^m - 1$ consists of m digits 1.

Third, it follows from Lucas's congruence (28), with j replaced by n and k replaced by j, that for a fixed $n < 2^m$, the term $\binom{j}{n}^*$ is periodic with period 2^m.

To prove the statement of the corollary, we first recall that $R_{n,k} = \binom{k-1}{n-k}^*$, so that the jth term of the nth diagonal ($n \geq 0$) is

$$R_{n+j,j} = \binom{j-1}{n}^*, \qquad j = 1, 2, 3, \ldots \tag{80}$$

Now let m be the smallest integer such that $n \leq 2^m - 1$; then the terms in (80) form column $n+1$ of the Sierpiński matrix S_{2^m}. So, by (78) we get

$$R_{n+j,j} = \binom{2^m - 1 - n}{2^m - j}^*, \tag{81}$$

and by definition of Gould's sequence there are $G(2^m - 1 - n)$ terms among the 2^m terms $j = 1, 2, \ldots, 2^m$ that are 1. By (79) this number is $2^m/G(n)$; so the proportion of 1s is $1/G(n)$, and by periodicity this holds for the entire nth diagonal.

To deal with the remaining three matrices, we note that Theorem 1 and Section 5 give, respectively,

$$\left|\left(R^{-1}\right)_{n+j,j}\right| = \binom{2n+j-1}{n}^*, \quad S_{n+j,j} = \left|\left(S^{-1}\right)_{n+j,j}\right| = \binom{n+j-1}{n}^*. \tag{82}$$

Since the right-hand sides of the identities in (82) are just shifted versions of the right-hand side of (80), the periodicity property shows that the statement of the corollary also holds for R^{-1}, S, and S^{-1}. $\square$

6. In this paper we have come across several "Stern-like" sequences. All of them are listed in the OEIS [21], with an appropriate shift in some cases. We list them in Table 5, with references to where they occur.

$a(2n) =$	$a(2n+1) =$	$a(1), a(2)$	Eqn.	OEIS	Remarks
$a(n)$	$a(n) + a(n+1)$	1, 1	(1)	A002487	Stern
$a(n)$	$1 - a(n)$	1, 1	(9)	A010060	P.-Thue-Morse
$a(n)$	$2 \cdot a(n)$	2, 2	(42)	A001316	$c_{n+1}(R)$; Gould
$a(n-1)$	$a(n) + a(n+1)$	1, 0	(43)	A106345	$a_n(R)$
$-a(n+1)$	$a(n) + a(n+1)$	1, 0	(47)	A342682	$a_n(R^{-1})$
$-a(n)$	$a(n) + a(n+1)$	1, −1	(66)	A049347	$a_n(S^{-1})$; $\overline{1, -1, 0}$
$a(n)$	$2 \cdot a(n+1)$	1, 1	(68)	A080100	$r_n(\lvert R^{-1}\rvert)$

Table 5: Stern-like sequences and their occurrences.

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