MULTIPLICATIVE FUNCTIONS *K*-ADDITIVE ON GENERALIZED PENTAGONAL NUMBERS

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Abstract

Let $k \geq 3$ be an integer. We prove that the set $\mathcal{P}$ of all nonzero generalized pentagonal numbers is a k-additive uniqueness set for the collection of multiplicative functions; if a multiplicative function f satisfies a multivariate Cauchy's functional equation $f(x_1+x_2+\cdots+x_k) = f(x_1)+f(x_2)+\cdots+f(x_k)$ for arbitrary $x_1, \ldots, x_k \in \mathcal{P}$, then f is the identity function $f(n) = n$ for all $n \in \mathbb{N}$. For the case $k = 3$, we assume the Generalized Riemann Hypothesis and for $k \geq 4$ the proof is unconditional. This extends B. Kim et al.'s work for $k = 2$.

1. Introduction

An arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is called *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. Let $\mathcal{M}$ denote the set of complex valued multiplicative functions.

A set $E \subseteq \mathbb{N}$ is called an *additive uniqueness set* of a set of arithmetic functions $\mathcal{F}$ if there is exactly one element $f \in \mathcal{F}$ which satisfies

$$f(m+n) = f(m) + f(n) \text{ for all } m, n \in E. \tag{1}$$

For example, $\mathbb{N}$ and $\{1\} \cup 2\mathbb{N}$ are trivially additive uniqueness sets of $\mathcal{M}$.

This concept was first introduced by C. A. Spiro [15] in 1992. She proved that the set of primes is an additive uniqueness set of $\mathcal{M}_0 = \{f \in \mathcal{M} | \ f(p_0) \neq 0 \text{ for some prime } p_0\}$ and asked whether other interesting sets were additive uniqueness sets for multiplicative functions. Later on, Spiro's work has been extended in many directions.

Let $k \geq 2$ be a fixed integer. If there is only one function $f \in \mathcal{F}$ satisfying $f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$ for arbitrary $x_i \in E$, $i \in \{1, 2, \ldots, k\}$, then E is called a *k-additive uniqueness set* of $\mathcal{F}$.

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In 2010, Fang [4] proved that the set of primes is a 3-additive uniqueness set of $\mathcal{M}_0$. In 2013, Dubickas and Šarka [3] generalized Fang's result to sums of arbitrary primes. Park showed that the set of shifted primes $\mathcal{B} = \{p-1| \ p \ \text{ is a prime}\}$ [13] and the set of k-almost primes [12] are additive uniqueness sets for $\mathcal{M}$.

In 1999 Chung and Phong [2] showed that the set of positive triangular numbers $T_n = \frac{n(n+1)}{2}$, $n \in \mathbb{N}$, and the set of positive tetrahedral numbers $Te_n = \frac{n(n+1)(n+2)}{6}$, $n \in \mathbb{N}$, are new additive uniqueness sets for $\mathcal{M}$ and Park [14] extended their work to sums of k triangular numbers, $k \geq 3$.

Park [11] proved that the set of nonzero squares is a k-additive uniqueness set of $\mathcal{M}$ for every $k \geq 3$, although it is not a 2-additive uniqueness set ([1]).

In 2018, B. Kim et al. [8] proved that the set of generalized pentagonal numbers $\mathcal{P} = \{\frac{n(3n-1)}{2}, \ n \in \mathbb{Z}\}$, is an additive uniqueness set for $\mathcal{M}$. Recently, they [9] have showed that the set of positive pentagonal numbers and the set of positive hexagonal numbers $\mathcal{H} = \{n(2n-1), \ n \in \mathbb{N}\}$ are new additive uniqueness sets for the collection of multiplicative functions. They also conjectured that among the sets of s-gonal numbers only the sets of triangular, pentagonal and hexagonal numbers are additive unique for $\mathcal{M}$.

Recently the author has proved that the set of practical numbers [6] and the set of odious numbers [7] are new k-additive uniqueness sets of $\mathcal{M}$ for every $k \geq 2$.

Although $\mathcal{P}$ is a 0-density subset of $\mathbb{N}$, it has a nice additive structure. In 1994, Guy [5] showed that each nonnegative integer can be written as the sum of three generalized pentagonal numbers. It is natural to ask which integers n can be represented by a sum of three nonzero generalized pentagonal numbers. D. Kim et al. [10] proved the following result.

Theorem 1. *Under the Generalized Riemann Hypothesis (GRH), any positive integer n is a sum of three nonzero generalized pentagonal numbers, except for $n = 1$ and 2.*

In this short note we prove the following theorem, which extends B. Kim et al.'s [8] work for $k = 2$.

Theorem 2. *Fix $k \geq 3$. The set $\mathcal{P}$ of nonzero generalized pentagonal numbers is a k-additive uniqueness set of $\mathcal{M}$; if a multiplicative function f satisfies*

$$f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$$

for arbitrary $x_1, \ldots, x_k \in \mathcal{P}$, then f is the identity function. For $k = 3$ we assume the GRH and for $k \geq 4$ the result holds unconditionally.

2. Proof

Proof of Theorem 2. The proof consists of four parts.

Case I. $k = 3$. Clearly $f(3) = 3$. In view of Theorem 1 it is sufficient to determine $f(2)$. The proof then follows by induction on n. Note that $f(10) = f(2)f(5) = f(2)[2f(2) + 1]$. On the other hand $f(10) = f(7 + 2 + 1) = f(7) + f(2) + 1$ and $f(7) = f(5 + 1 + 1) = f(5) + 2 = 2f(2) + 3$. Hence we get $f^2(2) - f(2) - 2 = 0$. Thus we obtain two solutions $f(2) = -1$ and $f(2) = 2$. The first solution yields $f(4) = f(2) + 2 = 1$ and $f(5) = 2f(2) + 1 = -1$. But this would lead to a contradiction:

$$\begin{aligned} f(12) &= f(5 + 5 + 2) = 2f(5) + f(2) = -3 \\ &= f(3)f(4) = 3f(4) = 3. \end{aligned}$$

Therefore we conclude that $f(2) = 2$ and this completes the proof for the case $k = 3$.

Case II. $k = 4$. Let $\mathcal{P}_k$ be the set of sums of k positive generalized pentagonal numbers. If $k \geq 4$, then $\mathcal{P}_k$ is the set of all positive integers except for $1, 2, \ldots, k - 1$. Indeed, let $n \geq 8$. Since every positive integer is a sum of three generalized pentagonal numbers (some of which possibly vanish), $n - 7$ can be written as a sum of i nonzero generalized pentagonal numbers for some i with $1 \leq i \leq 3$. Since $7 \in \mathcal{P}$ and

$$n = (n - 7) + 7 = (n - 7) + 2 + 5 = (n - 7) + 1 + 1 + 5$$

every integer > 7 can be written as a sum of four nonzero generalized pentagonal numbers. It can be easily checked that every positive integer ≤ 7 is a sum of four positive generalized pentagonal numbers except for 1, 2, 3. Hence every positive integer ≥ 4 can be written as a sum of four positive generalized pentagonal numbers.

It is clear that the sum of k positive generalized pentagonal numbers can represent k but cannot represent any number from 1 through $k-1$. Since sums of four positive generalized pentagonal numbers represent all integers ≥ 4, the sum

$$\underbrace{1 + \cdots + 1}_{k-4 \ \text{times}} + x + y + z + w, \tag{2}$$

where $x, y, z, w \in \mathcal{P}$, can represent all integers $\geq k$.

Now observe that $f(4) = 4$. $f(6) = f(2)f(3) = f(2 + 2 + 1 + 1) = 2f(2) + 2$ and $f(12) = 4f(3) = f(5 + 5 + 1 + 1) = 2f(5) + 2 = 2f(2) + 8$. For convenience, let $a = f(2)$, $b = f(3)$. Thus we get the system of equations

$$\begin{cases} ab = 2a + 2 \\ 2b = a + 4. \end{cases}$$

We obtain the two solutions $f(2) = -2$, $f(3) = 1$ and $f(2) = 2$, $f(3) = 3$. The first solution yields $f(5) = f(2+1+1+1) = 1$. But this would lead to a contradiction:

$$\begin{aligned} f(15) &= f(12+1+1+1) = 4f(3)+3 = 7 \\ &= f(3)f(5) = 1. \end{aligned}$$

Thus, we can conclude that $f(2) = 2$, $f(3) = 3$. So $f(n) = n$ for $n \leq 4$ and f must be the identity function by induction.

`Case III.` $k = 5$. Note that $f(5) = 5$. $f(6) = f(2)f(3) = f(2+1+1+1+1) = f(2)+4$, $f(12) = f(3)f(4) = f(5+2+2+2+1) = 3f(2)+6$ and $f(35) = f(5)f(7) = f(12+15+5+2+1) = f(3)f(4)+5f(3)+f(2)+6$. For convenience, let $a = f(2)$, $b = f(3)$ and $c = f(4)$. Thus we get the system of equations

$$\begin{cases} ab = a+4 \\ bc = 6+3a \\ 5(2a+3) = bc+5b+a+6. \end{cases}$$

We obtain the two solutions $f(2) = -\frac{5}{3}$, $f(3) = -\frac{7}{5}$, $f(4) = -\frac{5}{7}$ and $f(2) = 2$, $f(3) = 3$, $f(4) = 4$. The first solution yields $f(7) = f(2+2+1+1+1) = 2f(2)+3 = -\frac{1}{3}$ and $f(11) = f(5+2+2+1+1) = 2f(2)+7 = \frac{11}{3}$. But this would lead to a contradiction:

$$\begin{aligned} f(22) &= f(12+7+1+1+1) = 4+f(7) = \frac{11}{3} \\ &= f(2)f(11) = -\frac{55}{9}. \end{aligned}$$

Thus, we can conclude that $f(2) = 2$, $f(3) = 3$ and $f(4) = 4$. So $f(n) = n$ for $n \leq 5$ and f must be the identity function by induction.

`Case IV.` $k \geq 6$. For this case we follow closely Park's [13] argument. Let $k \geq 6$. Note that

$$\begin{aligned} (k-3)+21 &= (k-3)\cdot 1+7+7+7 \\ &= (k-3)\cdot 1+15+5+1, \\ (k-3)+9 &= (k-3)\cdot 1+5+2+2 \\ &= (k-3)\cdot 1+7+1+1, \\ (k-4)+19 &= (k-4)\cdot 1+7+5+5+2 \\ &= (k-4)\cdot 1+15+2+1+1 \\ (k-5)+39 &= (k-5)\cdot 1+35+1+1+1+1 \\ &= (k-5)\cdot 1+15+15+5+2+2. \end{aligned}$$

Let $x = f(2)$, $y = f(3)$, $z = f(5)$ and $w = f(7)$. The above equalities give rise to the system of equations

$$\begin{cases} 3w = zy + z + 1 \\ 2x + z = w + 2 \\ 2z + w = zy + 2 \\ zw + 4 = 2zy + 2x + z. \end{cases}$$

The solutions are

$$f(2) = f(3) = f(5) = f(7) = 1$$
$$f(2) = 2, f(3) = 3, f(5) = 5, f(7) = 7.$$

Consider the first solution set $f(2) = f(3) = f(5) = f(7) = 1$. Arrange positive generalized pentagonal numbers into an increasing sequence and let x_n denote the nth term. Then $f(x_1) = f(x_2) = f(x_3) = f(x_4) = 1$. As seen in *Case II*, every x_n with $n \geq 3$ can be written as a sum of five positive generalized pentagonal numbers. From the equality

$$\begin{aligned} &(k-6) + 1 + 1 + 1 + 5 + 7 + x_f \\ &= (k-6) + 15 + x_a + x_b + x_c + x_d + x_e \end{aligned} \tag{3}$$

we infer that $f(x_n) = 1$ for all $n \geq 5$ inductively.

But for sufficiently large n, x_n can be represented as a sum of k positive generalized pentagonal numbers by (2). So $f(x_n) = k$, which is a contradiction.

Hence, we conclude that $f(2) = 2$, $f(3) = 3$, $f(5) = 5$ and $f(7) = 7$. Moreover, (3) yields $f(x_n) = x_n$ for every $n \geq 1$.

If N is a sum of k positive generalized pentagonal numbers then $f(N) = N$. Otherwise, choose an integer $M \geq k$ such that $\gcd(M, N) = 1$. Then M and MN can be represented as sums of k positive generalized pentagonal numbers by (2). By the multiplicativity of f, $Mf(N) = f(M)f(N) = f(MN) = MN$. Therefore, $f(N) = N$ and this completes the proof. □

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