MULTIPLICATIVE FUNCTIONS \( k \)-ADDITIVE ON GENERALIZED PENTAGONAL NUMBERS

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Abstract

Let \( k \geq 3 \) be an integer. We prove that the set \( \mathcal{P} \) of all nonzero generalized pentagonal numbers is a \( k \)-additive uniqueness set for the collection of multiplicative functions; if a multiplicative function \( f \) satisfies a multivariate Cauchy’s functional equation

\[
f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)
\]

for arbitrary \( x_1, \ldots, x_k \in \mathcal{P} \), then \( f \) is the identity function \( f(n) = n \) for all \( n \in \mathbb{N} \). For the case \( k = 3 \), we assume the Generalized Riemann Hypothesis and for \( k \geq 4 \) the proof is unconditional. This extends B. Kim et al.’s work for \( k = 2 \).

1. Introduction

An arithmetic function \( f : \mathbb{N} \to \mathbb{C} \) is called multiplicative if \( f(1) = 1 \) and \( f(mn) = f(m)f(n) \) whenever \( \gcd(m,n) = 1 \). Let \( \mathcal{M} \) denote the set of complex valued multiplicative functions.

A set \( E \subseteq \mathbb{N} \) is called an additive uniqueness set of a set of arithmetic functions \( \mathcal{F} \) if there is exactly one element \( f \in \mathcal{F} \) which satisfies

\[
f(m + n) = f(m) + f(n) \quad \text{for all} \quad m, n \in E.
\]

(1)

For example, \( \mathbb{N} \) and \( \{1\} \cup 2\mathbb{N} \) are trivially additive uniqueness sets of \( \mathcal{M} \).

This concept was first introduced by C. A. Spiro [15] in 1992. She proved that the set of primes is an additive uniqueness set of \( \mathcal{M}_0 = \{ f \in \mathcal{M} | f(p_0) \neq 0 \text{ for some prime } p_0 \} \) and asked whether other interesting sets were additive uniqueness sets for multiplicative functions. Later on, Spiro’s work has been extended in many directions.

Let \( k \geq 2 \) be a fixed integer. If there is only one function \( f \in \mathcal{F} \) satisfying

\[
f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)
\]

for arbitrary \( x_i \in E, \; i \in \{1,2,\ldots,k\} \), then \( E \) is called a \( k \)-additive uniqueness set of \( \mathcal{F} \).

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In 2010, Fang [4] proved that the set of primes is a 3-additive uniqueness set of \( M_0 \). In 2013, Dubickas and Šarka [3] generalized Fang’s result to sums of arbitrary primes. Park showed that the set of shifted primes \( B = \{ p - 1 \mid p \text{ is a prime} \} \) [13] and the set of \( k \)-almost primes [12] are additive uniqueness sets for \( M \).

In 1999 Chung and Phong [2] showed that the set of positive triangular numbers \( T_n = \frac{n(n+1)}{2}, n \in \mathbb{N} \), and the set of positive tetrahedral numbers \( Te_n = \frac{n(n+1)(n+2)}{6}, n \in \mathbb{N} \), are new additive uniqueness sets for \( M \) and Park [14] extended their work to sums of \( k \) triangular numbers, \( k \geq 3 \).

Park [11] proved that the set of nonzero squares is a \( k \)-additive uniqueness set of \( M \) for every \( k \geq 3 \), although it is not a 2-additive uniqueness set ([1]).

In 2018, B. Kim et al. [8] proved that the set of generalized pentagonal numbers \( P = \{ \frac{n(3n-1)}{2}, n \in \mathbb{Z} \} \), is an additive uniqueness set for \( M \). Recently, they [9] have showed that the set of positive pentagonal numbers and the set of positive hexagonal numbers \( H = \{ n(2n - 1), n \in \mathbb{N} \} \) are new additive uniqueness sets for the collection of multiplicative functions. They also conjectured that among the sets of \( s \)-gonal numbers only the sets of triangular, pentagonal and hexagonal numbers are additive unique for \( M \).

Recently the author has proved that the set of practical numbers [6] and the set of odious numbers [7] are new \( k \)-additive uniqueness sets of \( M \) for every \( k \geq 2 \).

Although \( P \) is a 0-density subset of \( \mathbb{N} \), it has a nice additive structure. In 1994, Guy [5] showed that each nonnegative integer can be written as the sum of three generalized pentagonal numbers. It is natural to ask which integers \( n \) can be represented by a sum of three nonzero generalized pentagonal numbers. D. Kim et al. [10] proved the following result.

**Theorem 1.** Under the Generalized Riemann Hypothesis (GRH), any positive integer \( n \) is a sum of three nonzero generalized pentagonal numbers, except for \( n = 1 \) and 2.

In this short note we prove the following theorem, which extends B. Kim et al.’s [8] work for \( k = 2 \).

**Theorem 2.** Fix \( k \geq 3 \). The set \( P \) of nonzero generalized pentagonal numbers is a \( k \)-additive uniqueness set of \( M \); if a multiplicative function \( f \) satisfies

\[
f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)
\]

for arbitrary \( x_1, \ldots, x_k \in P \), then \( f \) is the identity function. For \( k = 3 \) we assume the GRH and for \( k \geq 4 \) the result holds unconditionally.
2. Proof

Proof of Theorem 2. The proof consists of four parts.

Case I. $k = 3$. Clearly $f(3) = 3$. In view of Theorem 1 it is sufficient to determine $f(2)$. The proof then follows by induction on $n$. Note that $f(10) = f(2)f(5) = f(2)(2f(2) + 1)$. On the other hand $f(10) = f(7 + 2 + 1) = f(7) + f(2) + 1$ and $f(7) = f(5 + 1 + 1) = f(5) + 2 = 2f(2) + 3$. Hence we get $f^2(2) - f(2) - 2 = 0$. Thus we obtain two solutions $f(2) = -1$ and $f(2) = 2$. The first solution yields $f(4) = f(2) + 2 = 1$ and $f(5) = 2f(2) + 1 = -1$. But this would lead to a contradiction:

$$f(12) = f(5 + 5 + 2) = 2f(5) + f(2) = -3$$
$$= f(3)f(4) = 3f(4) = 3.$$

Therefore we conclude that $f(2) = 2$ and this completes the proof for the case $k = 3$.

Case II. $k = 4$. Let $P_k$ be the set of sums of $k$ positive generalized pentagonal numbers. If $k \geq 4$, then $P_k$ is the set of all positive integers except for $1, 2, \ldots, k - 1$. Indeed, let $n \geq 8$. Since every positive integer is a sum of three generalized pentagonal numbers (some of which possibly vanish), $n - 7$ can be written as a sum of $i$ nonzero generalized pentagonal numbers for some $i$ with $1 \leq i \leq 3$. Since $7 \in P$ and

$$n = (n - 7) + 7 = (n - 7) + 2 + 5 = (n - 7) + 1 + 1 + 5$$

every integer $> 7$ can be written as a sum of four nonzero generalized pentagonal numbers. It can be easily checked that every positive integer $\leq 7$ is a sum of four positive generalized pentagonal numbers except for $1, 2, 3$. Hence every positive integer $\geq 4$ can be written as a sum of four positive generalized pentagonal numbers.

It is clear that the sum of $k$ positive generalized pentagonal numbers can represent $k$ but cannot represent any number from $1$ through $k - 1$. Since sums of four positive generalized pentagonal numbers represent all integers $\geq 4$, the sum

$$\underbrace{1 + \cdots + 1}_{k - 4 \text{ times}} + x + y + z + w,$$

(2)

where $x, y, z, w \in P$, can represent all integers $\geq k$.

Now observe that $f(4) = 4$. $f(6) = f(2)f(3) = f(2 + 2 + 1 + 1) = 2f(2) + 2$ and $f(12) = 4f(3) = f(5 + 5 + 1 + 1) = 2f(5) + 2 = 2f(2) + 8$. For convenience, let $a = f(2), b = f(3)$. Thus we get the system of equations

$$\begin{cases}
ab = 2a + 2 \\
2b = a + 4.
\end{cases}$$
We obtain the two solutions $f(2) = -2$, $f(3) = 1$ and $f(2) = 2$, $f(3) = 3$. The first solution yields $f(5) = f(2 + 1 + 1 + 1) = 1$. But this would lead to a contradiction:

$$f(15) = f(12 + 1 + 1 + 1) = 4f(3) + 3 = 7$$
$$= f(3)f(5) = 1.$$ 

Thus, we can conclude that $f(2) = 2$, $f(3) = 3$. So $f(n) = n$ for $n \leq 4$ and $f$ must be the identity function by induction.

**Case III.** $k = 5$. Note that $f(5) = 5$, $f(6) = f(2)f(3) = f(2 + 1 + 1 + 1) = f(2) + 4$, $f(12) = f(3)f(4) = f(5 + 2 + 2 + 2 + 1) = 3f(2) + 6$ and $f(35) = f(5)f(7) = f(12 + 15 + 5 + 2 + 1) = f(3)f(4) + 5f(3) + f(2) + 6$. For convenience, let $a = f(2)$, $b = f(3)$ and $c = f(4)$. Thus we get the system of equations

$$\begin{cases} 
    ab = a + 4 \\
    bc = 6 + 3a \\
    5(2a + 3) = bc + 5b + a + 6. 
\end{cases}$$

We obtain the two solutions $f(2) = -\frac{5}{3}, f(3) = -\frac{7}{3}, f(4) = -\frac{5}{7}$ and $f(2) = 2$, $f(3) = 3$, $f(4) = 4$. The first solution yields $f(7) = f(2 + 2 + 1 + 1 + 1) = 2f(2) + 3 = -\frac{1}{3}$ and $f(11) = f(5 + 2 + 2 + 1 + 1) = 2f(2) + 7 = \frac{11}{3}$. But this would lead to a contradiction:

$$f(22) = f(12 + 7 + 1 + 1 + 1) = 4 + f(7) = \frac{11}{3}$$
$$= f(2)f(11) = -\frac{55}{9}.$$ 

Thus, we can conclude that $f(2) = 2$, $f(3) = 3$ and $f(4) = 4$. So $f(n) = n$ for $n \leq 5$ and $f$ must be the identity function by induction.

**Case IV.** $k \geq 6$. For this case we follow closely Park’s [13] argument. Let $k \geq 6$. Note that

$$\begin{align*}
(k - 3) + 21 &= (k - 3) \cdot 1 + 7 + 7 + 7 \\
 &= (k - 3) \cdot 1 + 15 + 5 + 1, \\
(k - 3) + 9 &= (k - 3) \cdot 1 + 5 + 2 + 2 \\
 &= (k - 3) \cdot 1 + 7 + 1 + 1, \\
(k - 4) + 19 &= (k - 4) \cdot 1 + 7 + 5 + 5 + 2 \\
 &= (k - 4) \cdot 1 + 15 + 2 + 1 + 1, \\
(k - 5) + 39 &= (k - 5) \cdot 1 + 35 + 1 + 1 + 1 + 1 \\
 &= (k - 5) \cdot 1 + 15 + 15 + 5 + 2 + 2.
\end{align*}$$
Let \( x = f(2), y = f(3), z = f(5) \) and \( w = f(7) \). The above equalities give rise to the system of equations

\[
\begin{align*}
3w &= zy + z + 1 \\
2x + z &= w + 2 \\
2z + w &= zy + 2 \\
zw + 4 &= 2zy + 2x + z.
\end{align*}
\]

The solutions are

\[
\begin{align*}
f(2) &= f(3) = f(5) = f(7) = 1 \\
f(2) &= 2, f(3) = 3, f(5) = 5, f(7) = 7.
\end{align*}
\]

Consider the first solution set \( f(2) = f(3) = f(5) = f(7) = 1 \). Arrange positive generalized pentagonal numbers into an increasing sequence and let \( x_n \) denote the \( n \)th term. Then \( f(x_1) = f(x_2) = f(x_3) = f(x_4) = 1 \). As seen in Case II, every \( x_n \) with \( n \geq 3 \) can be written as a sum of five positive generalized pentagonal numbers. From the equality

\[
(k - 6) + 1 + 1 + 5 + 7 + x_f = (k - 6) + 15 + x_a + x_b + x_c + x_d + x_e
\]

we infer that \( f(x_n) = 1 \) for all \( n \geq 5 \) inductively.

But for sufficiently large \( n \), \( x_n \) can be represented as a sum of \( k \) positive generalized pentagonal numbers by (2). So \( f(x_n) = k \), which is a contradiction.

Hence, we conclude that \( f(2) = 2, f(3) = 3, f(5) = 5 \) and \( f(7) = 7 \). Moreover, (3) yields \( f(x_n) = x_n \) for every \( n \geq 1 \).

If \( N \) is a sum of \( k \) positive generalized pentagonal numbers then \( f(N) = N \). Otherwise, choose an integer \( M \geq n \) such that \( \gcd(M, N) = 1 \). Then \( M \) and \( MN \) can be represented as sums of \( k \) positive generalized pentagonal numbers by (2). By the multiplicativity of \( f \), \( Mf(N) = f(M)f(N) = f(MN) = MN \). Therefore, \( f(N) = N \) and this completes the proof.

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References

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