NOTE ON SETS WITHOUT GEOMETRIC PROGRESSIONS

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Abstract

For \( k \geq 3 \), we call a set \( G \subseteq (0, 1] \) of real numbers \( k \)-good if \( G \) contains no geometric progression of length \( k \) with integer ratio \( r > 1 \). A real number \( x \in (0, 1] \setminus G \) is called \( k \)-bad with respect to \( G \) if there exists an integer \( r > 1 \) such that \( G \cup \{x\} \) contains the \( k \)-term progression \((x, xr, xr^2, \cdots, xr^{k-1})\). Define \( \text{Bad}(G) = \{x \in (0, 1] \setminus G : \text{\( x \) is \( k \)-bad with respect to \( G \)}\)\).

1. Introduction

For an integer \( k \geq 3 \), we call a set \( G \subseteq (0, 1] \) of real numbers \( k \)-good if \( G \) contains no geometric progression of length \( k \) with integer ratio \( r > 1 \). A real number \( x \in (0, 1] \setminus G \) is called \( k \)-bad with respect to \( G \) if there exists an integer \( r > 1 \) such that \( G \cup \{x\} \) contains the \( k \)-term geometric progression \((x, xr, xr^2, \cdots, xr^{k-1})\). Define

\[
\text{Bad}(G) = \{x \in (0, 1] \setminus G : \text{\( x \) is \( k \)-bad with respect to \( G \)}\).
\]

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In 2015, Nathanson and O’Bryant [3] proved the following theorem.

**Theorem A ([3]).** Fix $k \geq 3$. There exists a unique strictly increasing sequence of positive integers $\{A_1^{(k)} < A_2^{(k)} < \cdots\}$ with $A_1^{(k)} = 1$ such that

$$G^{(k)} = \bigcup_{i=1}^{\infty} \left( \frac{1}{A_{2i}^{(k)}}, \frac{1}{A_{2i-1}^{(k)}} \right)$$

is a $k$-good set and

$$\text{Bad}(G^{(k)}) = \bigcup_{i=1}^{\infty} \left( \frac{1}{A_{2i+1}^{(k)}}, \frac{1}{A_{2i}^{(k)}} \right).$$

Nathanson and O’Bryant also proved in [3] that $A_2^{(k)} = 2^{k-1}, A_3^{(k)} = 2^k$ and

$$A_4^{(k)} = \begin{cases} 2 k^{3k-l-1} & \text{if there is a positive integer } l \text{ such that } 2^{k-1} < 3^l < 2^k, \\ 3^{k-1} & \text{otherwise}. \end{cases}$$

Afterwards, the second author determined the value of $A_5^{(k)}$.

**Theorem B ([1]).** Let $k \geq 3$ be an integer.

(i) If there is no integral power of 3 between $2^{k-1}$ and $2^k$, then

$$A_5^{(k)} = 2^{k-1}3^{i_0+1},$$

where $i_0$ is the largest integer $i$ such that $3^i < 3^{k-1}/2^k$.

(ii) If there is a positive integer $l$ such that $2^{k-1} < 3^l < 2^k$, then $k \geq 4$ and

$$A_5^{(k)} = \begin{cases} 200 & \text{if } k = 4, \\ 2^{k-1}3^{k-l} & \text{if } k \geq 5. \end{cases}$$

For the remainder of the paper we use the variable $l$ for the least integer such that $3^l > 2^{k-1}$. This convention allows for a simpler statement of the above theorem. The reader can check that case (i) above simplifies to the second case, and we get the following, valid for all $k \geq 3$:

$$A_5^{(k)} = \begin{cases} 200 & \text{if } k = 4, \\ 2^{k-1}3^{k-l} = 3^{k-1} \left( \frac{2^{l-1}}{3} \right) & \text{otherwise}. \end{cases}$$

Note that the value $\frac{1}{A_5^{(k)}}$ is the upper limit of an interval of values that are bad with respect to the set $\left( \frac{1}{A_4^{(k)}}, \frac{1}{A_3^{(k)}} \right) \cup \left( \frac{1}{A_2^{(k)}}, 1 \right)$ (write this set as $G_2^{(k)}$) because of progressions with ratio $r = 3$. The ratio $\left( \frac{3^{k-1}}{2^{l-1}} \right)$ is the ratio between $\frac{1}{2^{l-1}}$ and $\frac{1}{\frac{1}{2^{l-1}}} = 1/A_5^{(k)}$, the largest value that is excluded from $G_2^{(k)}$. When $x = \frac{1}{2^{l-1}3^{l-1}} = 1/A_5^{(k)}$, the term $3^{k-l}x = \frac{1}{2^{l-1}}$, and so the entire progression with ratio 3 starting at $x$ is no longer contained in $G_2^{(k)}$.

In this note, we obtain the value of $A_6^{(k)}$. 

**Theorem 1.** Let $k \geq 3$ be an integer.

(i) If there is no integral power of 3 between $2^{k-1}$ and $2^k$, then

$$A_6^{(k)} = 2k \cdot 3^{k-l},$$

where $l$ is the smallest integer such that $3^l > 2^k$.

(ii) If there is a positive integer $l$ such that $2^{k-1} < 3^l < 2^k$ and there is no integral power of 4 between $4 \cdot 3^{k-l-1}$ and $2 \cdot 3^{k-l}$, then

$$A_6^{(k)} = \begin{cases} 4^{k-1} & \text{for even } k, \\ 2^{2k-1} & \text{for odd } k. \end{cases}$$

(iii) If there is a positive integer $l$ such that $2^{k-1} < 3^l < 2^k$ and there is a positive integer $m$ such that $4 \cdot 3^{k-l-1} < 4m < 2 \cdot 3^{k-l}$, then $A_6^{(k)} = 216$ and

$$A_6^{(k)} = \begin{cases} \frac{1}{2} 4^{k-m} 3^{k-l} & \text{for even } k > 4, \\ 4^{k-m} 3^{k-l} & \text{for odd } k. \end{cases}$$

Based on the above results, we pose the following problem.

**Problem.** For each positive integer $i$, do there exist infinitely many positive integers $k$ such that $A_6^{(k)} = (i + 1)^{k-1}$?


**2. Proof of Theorem 1**

We first prove\(^2\) that $A_6^{(3)} = 24$ and $A_6^{(4)} = 216$. For $k = 3$, we know that $A_6^{(k)} = 4$, $A_3^{(k)} = 8$, $A_4^{(k)} = 9$ and $A_5^{(k)} = 12$. For $x \in (\frac{1}{16}, \frac{1}{12})$, we have $\frac{1}{8} < 2x < \frac{1}{4}$, $\frac{1}{8} < 3x \leq \frac{3}{4}$ and $r^2x > 1$ for $r \geq 4$. So the set $\left(\frac{1}{16}, \frac{1}{12}\right] \cup \left(\frac{1}{9}, \frac{1}{8}\right] \cup \left(\frac{1}{4}, 1\right]$ is 3-good. Furthermore, for $x \in (\frac{1}{27}, \frac{1}{17})$, we have $\frac{1}{8} < 2^2x = 4x \leq \frac{1}{2}$, $\frac{1}{8} < 3x < \frac{3}{4}$ and $r^2x > 1$ for $r \geq 5$, and so the set

$$\left(\frac{1}{24}, \frac{1}{12}\right] \cup \left(\frac{1}{9}, \frac{1}{8}\right] \cup \left(\frac{1}{4}, 1\right]$$

is also 3-good. However, for $x_0 = \frac{1}{21}$, we have $3x_0 = \frac{1}{8}$ and $\frac{1}{4} < 3^2x_0 < 1$. Thus $x_0$ is 3-bad with respect to the set (1), and so $A_6^{(3)} = 24$.

For $k = 4$, we know $A_2^{(k)} = 8$, $A_3^{(k)} = 16$, $A_4^{(k)} = 48$ and $A_5^{(k)} = 200$. For $x \in (\frac{1}{216}, \frac{1}{200})$, we have that $\frac{1}{200} < 3x < 2^2x = 4x \leq \frac{1}{48}$, while $\frac{1}{16} < 5^2x \leq \frac{1}{8}$ and $r^3x > 1$ for $r \geq 6$. When $x_0 = \frac{1}{216}$, we have $\frac{1}{48} < 6x_0 < \frac{1}{16}$ and $\frac{1}{8} < 6^2x_0 \leq 6^3x_0 = 1$. Thus,

$$\left(\frac{1}{216}, \frac{1}{200}\right] \cup \left(\frac{1}{48}, \frac{1}{16}\right] \cup \left(\frac{1}{8}, 1\right],$$

\(^2\)These values were included in [3], but we include a proof for completeness.
is 4-good and \( A_6^{(4)} = 216 \). In the following we can assume \( k \geq 5 \). We divide the proof into three cases.

**Case 1.** Assume there is no integral power of 3 between \( 2^{k-1} \) and \( 2^k \). In this case, \( A_4^{(k)} = 3^{k-1} \), \( A_5^{(k)} = 2^{k-1}3^{k-1} \), and \( 3^{l-1} < 2^{k-1} < 2^k < 3^l \). Let

\[
G_2^{(k)} = \left[ \frac{1}{3^{k-1}}, \frac{1}{2^k} \right] \cup \left( \frac{1}{3^{k-1}}, 1 \right].
\]

For \( \frac{1}{2^k 3^{k-1}} < x \leq \frac{1}{2^k 3^{k-1} - 1} \), and any \( r \geq 4 \), we have

\[
r^{k-1}x > 4^{k-1}x > \frac{4^{k-1}}{2^k 3^{k-1}} > \frac{4^{k-1}}{3^k} > 1.
\]

(2)

Note the last inequality follows since \( k \geq 5 \). When \( r = 3 \), we have \( \frac{1}{2^k} < 3^{k-l}x \leq \frac{1}{2^{k-1}} \), so \( 3^{k-l}x \notin G_2^{(k)} \). Finally, for \( r = 2 \), note that after dividing through (2) by \( 2^{k-1} \) we find that \( 2^{k-1}x > \frac{1}{2^{k-1}} \). Thus, there must exist an integer \( m \leq k - 1 \) such that \( \frac{1}{2^m} < 2^m x \leq \frac{1}{2^{m-1}} \), and so \( 2^m x \notin G_2^{(k)} \).

Thus \( \left( \frac{1}{2^k 3^{k-1}}, \frac{1}{2^{k-1} 3^{k-1} - 1} \right] \cup G_2^{(k)} \) is \( k \)-good. Now we prove that \( x_0 = \frac{1}{2^k 3^{k-1}} \) is \( k \)-bad with respect to \( G_2^{(k)} \). Using again that \( 3^{l-1} < 2^k < 3^l \),

\[
\frac{1}{3^{k-1}} = \frac{2^k}{2^k 3^{k-1}} < \frac{1}{2^k 3^{k-1} - 1} = 3x_0 < 3^2 x_0 < \cdots < 3^{k-l} x_0 = \frac{1}{2^k}
\]

and then

\[
\frac{1}{2^{k-1}} < 3^{k-l+1} x_0 < \cdots < 3^{k-1} x_0 = \frac{3^{k-1}}{2^k 3^{k-1}} = \frac{3^{l-1}}{2^k} < 1.
\]

Thus,

\[
\{ 3^i x_0 : i = 1, 2, \cdots, k-1 \} \subseteq G_2^{(k)}.
\]

That is, \( x_0 \) is \( k \)-bad with respect to \( G_2^{(k)} \). To sum up, in Case 1, we have shown that \( A_6^{(k)} = 2^k 3^{k-1} \).

For the remainder of the proof we can assume there is an integer \( l \) with \( 2^{k-1} < 3^l < 2^k \). Thus, \( A_4^{(k)} = 2^k 3^{k-l-1} \) and \( A_5^{(k)} = 2^k 3^{k-l} \). For the remainder we set

\[
G_2^{(k)} = \left( \frac{1}{2^k 3^{k-l-1}}, \frac{1}{2^k} \right] \cup \left[ \frac{1}{2^{k-1}}, 1 \right].
\]

We first prove the following lemma which will be used throughout to handle the ratio \( r = 3 \).

**Lemma 1.** Suppose that \( k \geq 3 \) and there exists an integer \( l \) with \( 2^{k-1} < 3^l < 2^k \). Then for every \( x \) in the interval \( \frac{1}{3^{k-1} 2^2} < x \leq \frac{1}{2^{k-1} 3^{k-1} - 1} = \frac{1}{A_6^{(k)}} \), there exists an integer \( 1 \leq n \leq k-1 \) such that either \( \frac{1}{A_5^{(k)}} < 3^n x \leq \frac{1}{A_4^{(k)}} \) or \( \frac{1}{A_5^{(k)}} < 3^n x \leq \frac{1}{A_4^{(k)}} \).
Thus, with respect to $G_0 < h < k$, we have $0 < n_1 \leq n_2 \leq k - 1$. Suppose that $3^{n_2} x \notin \left(\frac{1}{A_5^{(k)}}, \frac{1}{A_4^{(k)}}\right]$. Then

$$3^{n_2} x > \frac{1}{2^{k-1}} > \frac{1}{2^l} \geq 3^{n_2-1} x.$$  

Multiplying through by $\frac{1}{2^{k-l}}$ gives

$$3^{n_2-k+l} x > \frac{1}{2^{k-1}3^{k-l}} \geq 3^{n_2-k+l-1} x.$$  

So $n_1 = n_2 - k + l$. On the other hand, multiplying through the same inequality by $\frac{1}{3^{n_2-k+l}}$ gives

$$\frac{1}{A_4^{(k)}} = \frac{1}{2^{k-1}3^{k-l-1}} \geq 3^{n_2-k+l} x = 3^{n_1} x.$$  

Thus, $3^{n_1} x \in \left(\frac{1}{A_5^{(k)}}, \frac{1}{A_4^{(k)}}\right]$.  

We now return to the proof of Theorem 1.1.

Case 2. Assume there is a positive integer $l$ such that $2^{k-1} < 3^l < 2^k$ and there is no integral power of 4 between $4 \cdot 3^{k-l-1}$ and $2 \cdot 3^{k-l}$.

We consider the interval $\frac{1}{3^{n_2-k+l}} < x \leq \frac{1}{3^{n_2-k-l}}$. If $r \geq 4$, we have $r^{k-1} x \geq 4^{k-1} x > 1$. If $r = 3$ we apply Lemma 1. Since $\frac{1}{3^{n_2-k+l}} > \frac{1}{3^{n_2-k-l}}$, it implies that

$$3^n x \in \left(\frac{1}{A_4^{(k)}}, \frac{1}{A_4^{(k)}}\right] \cup \left(\frac{1}{A_3^{(k)}}, \frac{1}{A_2^{(k)}}\right]$$ for some $1 \leq n \leq k - 1$.

For the remaining ratio $r = 2$, note that $2^{k-1} x > \frac{1}{2^{k-1}}$, so there exists some integer $0 < h < k - 1$ such that $\frac{1}{2^h} < 2^{h} x \leq \frac{1}{2^{h+1}}$. Thus the set

$$G_3^{(k)} := \left(\frac{1}{4^{k-1}}, \frac{1}{2^{k-1}3^{k-1}}\right] \cup G_2^{(k)}$$

is $k$-good.

We now suppose further that $k$ is even and we will prove that $x_0 = \frac{1}{4^{k-1}}$ is $k$-bad with respect to $G_3^{(k)}$ by showing that for each $1 \leq i \leq k - 1$, the term $4^i x_0 = \frac{1}{4^{i-1}} \in G_3^{(k)}$. When $i = k - 1$ we have $4^{k-1} x_0 = 1$. Since $k$ is even, $k = 2j$ for some integer $j$ and $4^{k-j} x_0 = \frac{1}{4^j} = \frac{1}{2^j} = \frac{1}{A_j^{(k)}} \leq G_3^{(k)}$. Clearly there is no power of 4 strictly between $2^k$ and $2^{k-1}$, so it suffices to show that none of the terms $4^i x_0$ fall in the gap between $\frac{1}{A_5^{(k)}}$ and $\frac{1}{A_4^{(k)}}$, i.e., there is no $i$ with

$$\frac{1}{2^{k-1}3^{k-l}} < \frac{1}{2^{k-1}3^{k-l}} \leq \frac{1}{2^{k-1}3^{k-l}}.$$ (3)
If there were such an \( i \), multiplying through (3) by \( 2^{k-2} = 4^{j-1} \) we get

\[
\frac{1}{2 \cdot 3^{k-l}} < \frac{1}{4^{k-i-1}} \leq \frac{1}{2 \cdot 3^{k-l-1}},
\]

which contradicts the assumption that there is no integral power of 4 between \( 4 \cdot 3^{k-l-1} \) and \( 2 \cdot 3^{k-l} \). Thus, when \( k \) is even we have \( A^{(k)}_6 = \frac{1}{2} \).

We now consider odd \( k \). The argument above for even \( k \) implies that \( G^{(k)}_3 \) is still \( k \)-good and we will show, in this case, that the larger set \( \left( \frac{1}{2^{4/7}}, \frac{1}{2^{3/7}} \right] \cup G^{(k)}_3 \) is also \( k \)-good. For \( \frac{1}{2^{4/7}} < x \leq \frac{1}{2^{3/7}} \), if \( r \geq 5 \), then

\[
r^{k-1} x > \frac{1}{2} \cdot \left( \frac{5}{4} \right)^{k-1} > 1.
\]

If \( r = 4 \), using that \( k \) is odd and noting that \( \frac{1}{2^2} < 4^{k-1} x \leq \frac{1}{2^{3/7}} \), we know that

\[
4^{\frac{k-1}{2}} x \notin G^{(k)}_2. \quad \text{When } r = 3, \text{ Lemma 1 again shows there exists } 1 \leq n \leq k - 1 \text{ with } 3^n x \notin \left( \frac{1}{2^{4/7}}, \frac{1}{2^{3/7}} \right] \cup G^{(k)}_2. \quad \text{Finally, for } r = 2, \text{ we have } \frac{1}{2^2} < 2^{k-1} x \leq \frac{1}{2^{3/7}}, \text{ and so we know that } 2^{k-1} x \notin G^{(k)}_2. \quad \text{Thus } \left( \frac{1}{2^{4/7}}, \frac{1}{2^{3/7}} \right] \cup G^{(k)}_2 \text{ is } k \text{-good.}
\]

To finish this case we show that \( x_0 = \frac{1}{2^{4/7}} \) is \( k \)-bad with respect to this set by showing that \( 4^i x_0 \) is contained in it for each \( 1 \leq i \leq k - 1 \). When \( i = k - 1 \) we have \( 4^{k-1} x_0 = \frac{1}{2} > \frac{1}{A^{(k)}_6} \). Since \( k \) is odd, \( k = 2j + 1 \) for some integer \( j \). Now \( 4^{k-j-1} x_0 = \frac{1}{2^j} = \frac{1}{A^{(k)}_6} \) and there is no power of 4 strictly between \( 2^k \) and \( 2^{k-1} \), so it remains to show that there is no \( i \) with

\[
\frac{1}{2^{k-1} 3^{k-l}} < \frac{1}{2 \cdot 4^{k-i-1}} \leq \frac{1}{2 \cdot 3^{k-l-1}}.
\]

If there were, multiplying through this time by \( 2^{k-2} = 2 \cdot 4^{j-1} \) gives

\[
\frac{1}{2 \cdot 3^{k-l}} < \frac{1}{4^{k-i-j}} \leq \frac{1}{2 \cdot 3^{k-l-1}},
\]

contradicting the assumption there is no integral power of 4 between \( 4 \cdot 3^{k-l-1} \) and \( 2 \cdot 3^{k-l} \). To sum up, in Case 2, we have shown that

\[
A^{(k)}_6 = \begin{cases} 4^{k-1} & \text{for even } k, \\ 2^{2k-1} & \text{for odd } k. \end{cases}
\]

**Case 3.** Assume that there is a positive integer \( l \) such that \( 2^{k-1} < 3^{l} < 2^k \) and a positive integer \( m \) such that

\[
4 \cdot 3^{k-l-1} < 4^m < 2 \cdot 3^{k-l}. \quad (4)
\]

The same argument used in Case 2 already shows that \( \left( \frac{1}{2^{4/7}}, \frac{1}{2^{3/7}} \right] \cup G^{(k)}_2 \) is \( k \)-good for even \( k \) and that \( \left( \frac{1}{2^{4/7}}, \frac{1}{2^{3/7}} \right] \cup G^{(k)}_2 \) is \( k \)-good when \( k \) is odd.
First, we consider even \( k \) and the interval \( \frac{1}{2^{k-1}3^{k-1}} < x \leq \frac{1}{2k-1} \). Note, since \( 4^m > 4 \cdot 3^{k-1} \), that the lower bound \( \frac{1}{2^{k-1}3^{k-1}} > \frac{2}{3^{k-1}} \). If \( r \geq 5 \) then, since \( k \geq 3 \),

\[
 r^{k-1}x \geq 5^{k-1}x > \frac{5^{k-1}}{\frac{1}{2}4^{k-1}3^{k-1}} > \frac{8 \cdot 5^{k-1}}{3 \cdot 4^k} > 1.
\]

For the ratio \( r = 4 \) we consider \( 4 \cdot m \). Using the inequalities in (4) we deduce that

\[
 \frac{1}{2^{k-1}3^{k-1}} < 4^{\frac{k}{2}-m}x \leq \frac{1}{2k4m-1} < \frac{1}{2k3k-1-1}
\]

and so \( \frac{1}{4} < 4^{\frac{k}{2}-m}x < \frac{1}{4} \). Also, note that \( 1 < \frac{k}{2} - m < \frac{k}{2} - 1 \), so \( 4^{\frac{k}{2}-m} = 2^{k-2m} \), thus this observation handles the case of \( r = 2 \) as well. Finally, the ratio \( r = 3 \) can again be handled using Lemma 1, noting that

\[
 \frac{1}{2^{k-1}3^{k-1}} > \frac{2}{3^{k-1}4^{k-1}} > \frac{1}{2^{k-1}3^{k-1}}
\]

Thus, \( \left( \frac{1}{2^{k-1}3^{k-1}}, \frac{1}{4} \right] \bigcup G_{2}^{(k)} \) is \( k \)-good.

We now show \( x_0 = \frac{1}{2^{k-1}3^{k-1}} \) is \( k \)-bad with respect to \( \left( \frac{1}{2^{k-1}3^{k-1}}, \frac{1}{4} \right] \bigcup G_{2}^{(k)} \).

We have

\[
 \frac{1}{2^{k-1}3^{k-1}} 4^{\frac{k}{2}-m} = \frac{1}{2^{k-1}3^{k-1}} \quad \text{and} \quad \frac{1}{2^{k-1}3^{k-1}} 4^{\frac{k}{2}-m+1} = \frac{4}{2^{k-1}3^{k-1}} > \frac{1}{2^{k-1}3^{k-1}}.
\]

Furthermore, using (4) we find that

\[
 \frac{1}{2^{k-1}3^{k-1}} 4^{\frac{k}{2}-1} = \frac{4^m}{2^{k-1}3^{k-1}} < \frac{1}{2k-1}, \quad \frac{1}{2^{k-1}3^{k-1}} 4^{\frac{k}{2}} = \frac{4^m}{2^{k-1}3^{k-1}} > \frac{1}{2k-1},
\]

and \( \frac{1}{2^{k-1}3^{k-1}} 4^{k-1} = \frac{4^m}{2^{k-1}3^{k-1}} < 1 \). That is, \( 4^{i}x_0 = \left( \frac{1}{2^{k-1}3^{k-1}}, \frac{1}{4} \right] \bigcup G_{2}^{(k)} \) for \( 1 \leq i \leq k-1 \), so \( x_0 \) is \( k \)-bad with respect to that set. So \( A_{2}^{(k)} = \frac{1}{2} 4^{k-m}3^{k-1} \).

Now consider odd \( k \). For \( \frac{1}{2^{k-1}3^{k-1}} < x \leq \frac{1}{4^{k}} \), by the same discussion as in the odd-\( k \) part of Case 2, we know that \( x \) is \( k \)-good with respect to \( G_{2}^{(k)} \), so we take \( \frac{1}{4^{k-1}3^{k-1}} < x \leq \frac{1}{2^{k-1}3^{k-1}} \). If \( r \geq 5 \), then

\[
 r^{k-1}x \geq 5^{k-1}x > 5^{k-1} \frac{1}{4^{k-m}3^{k-1}} > 1.
\]

For \( r = 2, 4 \), it follows from (4) that

\[
 4^{k+1-m}x = 2^{k-2m+1}x > \frac{1}{4^{k-m}3^{k-1}} 2^{k-2m+1} = \frac{1}{2^{k-1}3^{k-1}}\cdot \frac{1}{2^{k-1}3^{k-1}} 2^{k-2m+1} = \frac{1}{2^{k-1}3^{k-1}} 2^{k-2m+1} < \frac{1}{2^{k-1}3^{k-1}}.
\]
Thus,
\[ 2^{k-2m+1}x = 4^{m+1}-m x \in \left( \frac{1}{A_6^{(k)}}, \frac{1}{A_4^{(k)}} \right). \]

For the remaining ratio, \( r = 3 \), we appeal one last time to Lemma 1. Since
\[ \frac{1}{2k-13k-l} > \frac{1}{2k-3k-l} > \frac{1}{2k-12k-l} \]
we are guaranteed the existence of an \( n \), \( 1 \leq n \leq k-1 \) with \( 3^n x \notin \left( \frac{1}{2k-3k-l}, \frac{1}{2k-13k-l} \right) \). \( k \)-good. We conclude by proving that \( x_0 = \frac{1}{2k-3k-l} \) is \( k \)-bad with respect to \( \left( \frac{1}{2k-3k-l}, \frac{1}{2k-13k-l} \right) \cup G_2^{(k)} \). It follows from (4) that
\[ \frac{1}{4k-m3k-l} 2^{k-2m+1} = \frac{1}{2k-13k-l}, \quad \frac{1}{4k-m3k-l} 2^{k-2m+2} = \frac{2}{2k-13k-l} > \frac{1}{2k3k-l-1} \]
and
\[ \frac{1}{4k-m3k-l} 2^{k-1} = \frac{4^m}{2k+13k-l} < \frac{1}{2k}. \]
This means that \( 2^ix_0 \in \left( \frac{1}{2k-3k-l}, \frac{1}{2k-13k-l} \right) \cup G_2^{(k)} \) for each \( 1 \leq i \leq k-1 \), so \( x_0 \) is \( k \)-bad with respect to that set. To sum up, in Case 3, we have shown that
\[ A_6^{(k)} = \begin{cases} \frac{1}{2}4k-m3k-l & \text{for even } k > 4, \\ 4k-m3k-l & \text{for odd } k. \end{cases} \]

This completes the proof of Theorem 1. \( \square \)

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**References**


