SOLUTION TO A PROBLEM DUE TO CHU AND KILIÇ

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Received: 2/26/22, Accepted: 4/5/22, Published: 5/20/22

Abstract

In 2021, Chu and Kiliç experimentally discovered a conjectured evaluation for the binomial sum

$$\sum_{k=0}^{n}(-1)^k\binom{2n-2k}{n-k}\binom{n+k}{2k}C_k$$

using Mathematica commands, and left it as an open problem to prove a closed-form evaluation for this hypergeometric sum, letting C_k denote the k^{th} Catalan number. For each case whereby n is a member of a given congruence class modulo 6, numerical evidence suggests that the above sum is always equal to a single hypergeometric expression, but it is not clear how to prove these conjectured evaluations using classical hypergeometric series, and, remarkably, the Maple implementation of the WZ method is not able to provide WZ proof certificates for any of the finite hypergeometric sum identities obtained, out of all congruence classes mod 6. In this article, we solve the open problem due to Chu and Kiliç concerning the above sum, using Zeilberger's algorithm in a nontrivial way. We also introduce further results, again via nontrivial applications of Zeilberger's algorithm, inspired by the Chu–Kiliç sum given above.

1. Introduction

Finite summations involving binomial coefficients are ubiquitous in both combinatorics and number theory, and the development of both classically oriented and computer-based tools to explicitly evaluate such summations forms a huge and active area of research. In the recent article [1], Chu and Kiliç made use of classical hypergeometric series identities to prove remarkable evaluations for finite sums involving Catalan numbers. The article [1] concluded with the open problem of proving an evaluation for

$$\sum_{k=0}^{n}(-1)^k\binom{2n-2k}{n-k}\binom{n+k}{2k}C_k, \tag{1}$$

and it is indicated in [1] that any attempts to prove this identity would be of great interest. This appears to be a difficult problem [1], and it is not clear as to how it could be possible to use classical hypergeometric series identities to evaluate (1). Furthermore, as we consider below, implementations of the WZ method in commercially available software cannot be used to prove experimentally discovered conjectured evaluations for (1). In this article, we solve the open problem given in [1] of proving evaluations for (1). Our proofs rely in a nontrivial way on Zeilberger's algorithm [3, §6].

Chu and Kiliç [1] claimed to have experimentally discovered a conjectured evaluation for (1) using Mathematica. This conjectured evaluation is given as a product of

$$\binom{\left\lfloor \frac{1}{3}(2n-1) \right\rfloor + 1}{\left\lfloor \frac{n}{3} \right\rfloor}\binom{2\left\lfloor \frac{1}{3}(2n-1) \right\rfloor + 1}{\left\lfloor \frac{1}{3}(2n-1) \right\rfloor} \tag{2}$$

and $\frac{(-1)^{\lfloor \frac{n}{3} \rfloor}}{n(n+1)}$ times an expression depending on the parity of n. Numerical evidence suggests that this conjectured evaluation is not quite right, and it appears that there may have been a minor typographical error in Chu and Kiliç's expression for (1) as given in [1]. However, this conjectured evaluation has led us to discover a correct closed form for (1). By considering the arguments of the floor function indicated in (2), together with the aforementioned parity condition, we need to consider $2 \times 3 = 6$ cases, as given by the congruence classes of n modulo 6.

2. Solution to a Problem Due to Chu and Kiliç

We begin with the congruence class of the form 0 (mod 6). Applying the mapping $n \mapsto 6n$ to (1), we have experimentally discovered the identity shown in (4) (cf. [1]). In the hope of applying the WZ method [3] to prove the binomial sum identity in (4), we set

$$F(n,k) := \frac{3(-1)^k(6n+1)\binom{6n+k}{2k}\binom{12n-2k}{6n-k}C_k}{2\binom{4n}{2n}\binom{8n-1}{4n-1}}, \tag{3}$$

as we would want to find a companion function G so that F and G form a WZ pair. Inputting

```
with(SumTools[Hypergeometric]):
```

into Maple, and then letting `f` be defined as $F(n,k)$, and setting `r := 1`, we find that Maple 2020 is not able to prove (4) via the WZ method. In particular, inputting

```
WZpair := WZMethod(f, r, n, k, 'cert'):
```

results in the error message indicated below.

```
Error, (in SumTools:-Hypergeometric:-WZMethod) WZ method fails
```

We remark that the integer sequences corresponding to the sums highlighted in (1) and in the below Theorem are not currently indexed in the OEIS [2]. In addition to CAS software not being able to produce a WZ proof certificate for (4), such software also cannot provide WZ proofs certificates for any of the other summation identities listed below, which is indicative of the remarkable nature about these hypergeometric identities.

Theorem 1. *The following binomial sum identities hold true for all positive integers n, and the last five such identities hold for $n \in \mathbb{N}_0$ (cf. [1]).*

$$\sum_{k=0}^{6n} (-1)^k \binom{6n+k}{2k}\binom{12n-2k}{6n-k} C_k = \frac{2\binom{4n}{2n}\binom{8n-1}{4n-1}}{18n+3} \tag{4}$$

$$\sum_{k=0}^{6n+1} (-1)^k \binom{6n+k+1}{2k}\binom{12n-2k+2}{6n-k+1} C_k = \frac{(2n+1)\binom{4n+1}{2n}\binom{8n+1}{4n}}{(3n+1)(6n+1)} \tag{5}$$

$$\sum_{k=0}^{6n+2} (-1)^k \binom{6n+k+2}{2k}\binom{12n-2k+4}{6n-k+2} C_k = \frac{2(n+1)\binom{4n+2}{2n}\binom{8n+3}{4n+1}}{3(2n+1)(3n+1)} \tag{6}$$

$$\sum_{k=0}^{6n+3} (-1)^k \binom{6n+k+3}{2k}\binom{12n-2k+6}{6n-k+3} C_k = -\frac{\binom{4n+2}{2n+1}\binom{8n+3}{4n+1}}{9n+6} \tag{7}$$

$$\sum_{k=0}^{6n+4} (-1)^k \binom{6n+k+4}{2k}\binom{12n-2k+8}{6n-k+4} C_k = -\frac{2(n+1)\binom{4n+3}{2n+1}\binom{8n+5}{4n+2}}{(3n+2)(6n+5)} \tag{8}$$

$$\sum_{k=0}^{6n+5} (-1)^k \binom{6n+k+5}{2k}\binom{12n-2k+10}{6n-k+5} C_k = -\frac{(2n+3)\binom{4n+4}{2n+1}\binom{8n+7}{4n+3}}{3(n+1)(6n+5)} \tag{9}$$

Proof. To prove (4), we apply Zeilberger's algorithm in the following manner, letting $F(n,k)$ be as in (3). Through the use of Zeilberger's algorithm, we can show that there exist polynomials $p_0(n)$, $p_1(n)$, $p_2(n)$, and $p_3(n)$ with integer coefficients together with a hypergeometric function $G(n,k)$ such that the identity

$$\begin{aligned} &p_3(n)F(n+3,k) + p_2(n)F(n+2,k) + p_1(n)F(n+1,k) + p_0(n)F(n,k) \\ &= G(n,k+1) - G(n,k) \end{aligned}$$

holds for $0 \leq k \leq n \in \{1,2,\ldots\}$. Verifying the four-term summation identity given above is, of course, a matter of finite computation. Explicit evaluations for the polynomials $p_i(n)$ and for the hypergeometric function G are given in the Mathematica notebook `6n.nb` associated with this article, and this notebook contains programs for numerically verifying that the above identities are correct. We write

$$b(n) = \sum_{k=0}^{6n} F(n,k)$$

to denote the binomial sum we want to evaluate, noting that $b(n)$ is independent of k. In view of the binomial coefficients in the numerator in our definition for F, we find that

$$b(n) = \sum_{k=0}^{m} F(n,k)$$

for $m > 6n$,

$$b(n+1) = \sum_{k=0}^{m} F(n+1,k)$$

for $m > 6(n+1)$, and so forth. So, we apply the operator $\sum_{k=0}^{6(n+3)} \cdot$ to the above identity involving the hypergeometric F- and G-functions, with the right-hand side of this identity telescoping under the application of $\sum_{k=0}^{6(n+3)} \cdot$. So, we obtain that

$$\begin{aligned} &p_3(n)b(n+3) + p_2(n)b(n+2) + p_1(n)b(n+1) + p_0(n)b(n) \\ &= G(n, 6n+19) - G(n,0) \end{aligned}$$

for all positive integers n. Equivalently,

$$p_3(n)b(n+3) + p_2(n)b(n+2) + p_1(n)b(n+1) + p_0(n)b(n) = 0 \tag{10}$$

for all $n \in \mathbb{N}$. We may verify the base cases whereby $b(1) = b(2) = b(3) = 1$. We may also verify that

$$p_3(n) + p_2(n) + p_1(n) + p_0(n) = 0 \tag{11}$$

for all $n \in \mathbb{N}$. So, since the sequences $(b(n) : n \in \mathbb{N})$ and $(1 : n \in \mathbb{N})$ satisfy the same recursion with polynomial coefficients indicated in (10) and (11), and satisfy the same base conditions, we may conclude that $b(n) = 1$ for all $n \in \mathbb{N}$.

Now, to prove (5), we proceed to set

$$F(n,k) := \frac{(-1)^k (3n+1)(6n+1)\binom{6n+k+1}{2k}\binom{12n-2k+2}{6n-k+1} C_k}{(2n+1)\binom{4n+1}{2n}\binom{8n+1}{4n}}. \tag{12}$$

Again, we apply Zeilberger's algorithm, via the Maple CAS, to prove this result. In this regard, we again input

```
with(SumTools[Hypergeometric]):
```

into Maple. We then set the expression `T` to be equal to the right-hand side of (12), and we input the following into Maple.

```
Zpair := Zeilberger(T, n, k, En):
```

In order to compute the required hypergeometric G-function corresponding to (12) according to Zeilberger's algorithm, we input the following.

```
G := Zpair[2]
```

To compute the polynomials corresponding to the p-functions in expressions as in (10), we input the following.

```
L := Zpair[1]
```

The corresponding data are provided in the `6nplus1` Mathematica notebook corresponding to this article. We again can show that an identity of the form

$$\begin{aligned} &p_3(n)F(n+3,k) + p_2(n)F(n+2,k) + p_1(n)F(n+1,k) + p_0(n)F(n,k) \\ &= G(n,k+1) - G(n,k) \end{aligned}$$

holds for the hypergeometric F-function defined in (12). In this case, the hypergeometric G-function corresponding to F and satistfying the above equality is as given in the `6nplus1` NB file corresponding to this article. The polynomials p satisfying the above equality are also given in this notebook file. Again, verifying the four-term summation identity indicated above is a finite computation. So, making use of the right-hand side of the above equality telescoping as we sum with respect to k, we may use the same recursive argument as before to show that

$$\sum_{k=0}^{6n+1} F(n,k) = 1$$

for all $n \in \mathbb{N}_0$.

Now, to prove (6), we first set

$$F(n,k) := \frac{3(-1)^k(2n+1)(3n+1)\binom{6n+k+2}{2k}\binom{12n-2k+4}{6n-k+2}C_k}{2(n+1)\binom{4n+2}{2n}\binom{8n+3}{4n+1}}. \tag{13}$$

Again, we apply Zeilberger's algorithm to find a corresponding G-function. Once again, we obtain, via Zeilberger's algorithm, an identity of the form

$$\begin{aligned} &p_3(n)F(n+3,k) + p_2(n)F(n+2,k) + p_1(n)F(n+1,k) + p_0(n)F(n,k) \\ &= G(n,k+1) - G(n,k) \end{aligned}$$

for the case whereby F is as (13). An explicit computation for the hypergeometric G-function is given in the provided `6plus2` notebook file, and explicit computations for the p-polynomials given above are also given in this NB document. So, we may mimic our proof of (4).

Now, as a first step toward proving (7), we set

$$F(n,k) := \frac{3(-1)^{k+1}(3n+2)\binom{6n+k+3}{2k}\binom{12n-2k+6}{6n-k+3}C_k}{\binom{4n+2}{2n+1}\binom{8n+3}{4n+1}}. \tag{14}$$

Again we obtain an identity of the form

$$\begin{aligned}&p_3(n)F(n+3,k)+p_2(n)F(n+2,k)+p_1(n)F(n+1,k)+p_0(n)F(n,k)\\&=G(n,k+1)-G(n,k)\end{aligned}$$

through Zeilberger's algorithm, letting F be as defined in (14). The required computations are given in the `6nplus3` NB file. Again, we may mimic our proof of (4).

In view of (8), we set

$$F(n,k):=\frac{(-1)^{k+1}(3n+2)(6n+5)\binom{6n+k+4}{2k}\binom{12n-2k+8}{6n-k+4}C_k}{2(n+1)\binom{4n+3}{2n+1}\binom{8n+5}{4n+2}}.$$

We may again use Zeilberger's algorithm according to the above F-function, in much the same way as in with our proof of (4). Relevant computations are given in the `6nplus4` NB file. As for the final binomial sum identity listed in the Theorem under consideration, Zeilberger's algorithm may once again be applied, in much the same way as above. The required computations are given in the `6nplus5` file. □

So, from our above proofs of the summation formulas in (4)–(9), we have completely solved the problem due to Chu and Kiliç on the binomial sum in (1).

The nontrivial nature about our application of Zeilberger's algorithm, as above, is reflected in the very complicated evaluations for the G-functions involved in our above proof. As a way of illustrating this, a TXT file containing Mathematica input for the evaluation for the G-function corresponding to (3) is about 230 KB in its size, whereas a TXT file containing Mathematica input for the one-line expression for F shown in (3) only takes up about 1 KB of disk space. So, in other words, writing out the evaluation for the hypergeometric G-function required in our proof of (4) takes about 230 times as much space compared to the one-line expression for F shown in (3).

The nontrivial nature about our application of Zeilberger's algorithm, as above, is also reflected in the complexity of the polynomials involved in our above proof.

Example 1. The polynomial $p_3(n)$ utilized in our proof of (4) is reproduced below explicitly, again with reference to the `6n.nb` file provided.

$$\begin{aligned}&-2411993230704362204226492825 6n^{34}\\&-13597611838095841926326853304 32n^{33}\\&-369311338522118958749221021286 40n^{32}\\&-6436385650769242030899185823252 48n^{31}\\&-80885871523432367012648690877726 72n^{30}\\&-780893482752653212240863627879383 04n^{29}\end{aligned}$$

$$
\begin{aligned}
&- 60252227684125194867071774517677 2608n^{28} \\
&- 38164430489724415108605406287117 68064n^{27} \\
&- 20228807190977321599275577298569 396224n^{26} \\
&- 91005559657577213124066273225087 123456n^{25} \\
&- 35125034721599100001715094987280 3774464n^{24} \\
&- 11727206124038999079294610646219 56792320n^{23} \\
&- 34084289899178546785833096989877 73132800n^{22} \\
&- 86655986953744145164499766978803 66284800n^{21} \\
&- 19341884066971851553684223948302 602731520n^{20} \\
&- 37999322231213995312313502573606 556139520n^{19} \\
&- 65819606013546101186922428519581 448232960n^{18} \\
&- 10060092183549789490383401256349 3373322240n^{17} \\
&- 13568731536052058543572970839415 4745873920n^{16} \\
&- 16138164600133420499521891456683 3552552960n^{15} \\
&- 16900097265747170240283661444128 8184945024n^{14} \\
&- 15546067119551144816404187594475 9700612288n^{13} \\
&- 12520438250336542150689072065897 4166069920n^{12} \\
&- 87901104890141308878518080258146 888615632n^{11} \\
&- 53493892257277351928301463547557 019679528n^{10} \\
&- 28017887770802882709821352464418 678029376n^{9} \\
&- 12514685929321414029926417403811 490728872n^{8} \\
&- 47115631551506842553285529256862 36476931n^{7} \\
&- 14724285354062568854401394534920 11522936n^{6} \\
&- 37425575043314699264542362814137 5939104n^{5} \\
&- 75215735749371880680572550079315 420056n^{4} \\
&- 11469957898864266470013845624369 391825n^{3} \\
&- 12432747969354918377451385741519 52100n^{2} \\
&- 85041712133127860500452600457920 000n \\
&- 27479569665227090595421877640000 00
\end{aligned}
$$

The unwieldy polynomial computation highlighted in Example 1 reflects the very elegant nature about the one-line binomial identity shown in (4). The polynomials required in our proofs of (5)–(9) are similarly unwieldy, with regard to Example 1.

3. Further Results

We generalize the above binomial sum evaluations by taking the first moments of the corresponding summands and evaluating the resultant sums in closed form. We discovered the following identities experimentally using Mathematica and the OEIS [2].

Theorem 2. *The following identities all hold for $n \in \mathbb{N}$. The last five such identities hold for all $n \in \mathbb{N}_0$.*

$$\sum_{k=0}^{6n}(-1)^k\binom{6n+k}{2k}\binom{12n-2k}{6n-k}kC_k = \frac{4(9n+1)\binom{4n}{2n}\binom{8n-1}{4n-1}}{3(6n+1)}$$

$$\sum_{k=0}^{6n+1}(-1)^{k+1}\binom{6n+k+1}{2k}\binom{12n-2k+2}{6n-k+1}kC_k = \frac{(2n+1)\binom{4n+1}{2n}\binom{8n+1}{4n}}{(3n+1)(6n+1)}$$

$$\sum_{k=0}^{6n+2}(-1)^{k+1}\binom{6n+k+2}{2k}\binom{12n-2k+4}{6n-k+2}kC_k = \frac{2(n+1)\binom{4n+2}{2n}\binom{8n+3}{4n+1}}{3(2n+1)(3n+1)}$$

$$\sum_{k=0}^{6n+3}(-1)^{k+1}\binom{6n+k+3}{2k}\binom{12n-2k+6}{6n-k+3}kC_k = \frac{(18n+11)\binom{4n+2}{2n+1}\binom{8n+3}{4n+1}}{3(3n+2)}$$

$$\sum_{k=0}^{6n+4}(-1)^{k}\binom{6n+k+4}{2k}\binom{12n-2k+8}{6n-k+4}kC_k = \frac{2(n+1)\binom{4n+3}{2n+1}\binom{8n+5}{4n+2}}{(3n+2)(6n+5)}$$

$$\sum_{k=0}^{6n+5}(-1)^{k}\binom{6n+k+5}{2k}\binom{12n-2k+10}{6n-k+5}kC_k = \frac{(2n+3)\binom{4n+4}{2n+1}\binom{8n+7}{4n+3}}{3(n+1)(6n+5)}$$

Proof. Zeilberger's algorithm may be applied to all of the above identities in virtually the same way as in Section 2. The required data are given in the "Generalization" Mathematica notebooks associated with this article. □

References

[1] W. Chu and E. Kiliç, Binomial sums involving Catalan numbers, *Rocky Mountain J. Math.* **51** (2021), 1221–1225.

[2] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, 2020. Available at `https://oeis.org`.

[3] M. Petkovšek, H. S. Wilf, and D. Zeilberger, $A = B$, A K Peters, Ltd., Wellesley, MA, 1996.