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ON THE GENERALIZED LEONARDO NUMBERS

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Abstract

In this paper, generalized Leonardo numbers $\mathcal{L}_{k,n}$ are defined. We also provide some properties of these numbers. A matrix representation of these numbers is defined and used to establish identities connecting these numbers. Finally, we define the incomplete generalized Leonardo numbers, and we establish a recurrence relation among these numbers as well as their properties.

1. Introduction

As is well known, the sequences of Fibonacci $\{F_n\}_{n\geq 0}$ and Lucas numbers $\{L_n\}_{n\geq 0}$ are defined, respectively, by

$$F_0 = 0, F_1 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad (n \geq 1)$$

$$L_0 = 2, L_1 = 1 \quad \text{and} \quad L_{n+1} = L_n + L_{n-1} \quad (n \geq 1).$$

The sequence of Leonardo numbers $\{Le_n\}_{n\geq 0}$ along with some of their properties is introduced by Catarino and Borges in [2]. Leonardo sequence is defined by the following recurrence relation

$$Le_0 = Le_1 = 1 \quad \text{and} \quad Le_{n+1} = Le_n + Le_{n-1} + 1 \quad (n \geq 1).$$

The relationship between Leonardo, Lucas, and Fibonacci numbers are expressed

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in the following identities, see [2],

$$Le_n = 2F_{n+1} - 1 \tag{1.1}$$

$$Le_n = L_{n+2} - F_{n+2} - 1 \tag{1.2}$$

$$Le_n = \frac{2}{5}(L_n + L_{n+2}) - 1, \tag{1.3}$$

and see more properties [1]-[4].

In this work, we will define the generalized Leonardo numbers and exhibit some properties of these numbers. We consider the generalized Leonardo sequence that is also an integer sequence related to both the Fibonacci and Lucas sequences.

2. Generalized Leonardo Numbers

We begin by defining a sequence of the generalized Leonardo numbers and establish its basic properties.

Definition 1. For a fixed positive integer k, define the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq 0}$ by

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k \quad (n \geq 2),$$

with the initial conditions $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$.

Let us look at some particular cases.

- If $k = 1$, the generalized Leonardo numbers are the classical Leonardo numbers $\mathcal{L}_{1,n} = Le_n$.
- If k is an odd integer, then $\mathcal{L}_{k,n}$ is an odd number for all n.
- If k is an even integer, then $\mathcal{L}_{k,n}$ is an even number, where $n \equiv 2 \pmod 3$, and otherwise, $\mathcal{L}_{k,n}$ is an odd number.

We establish a connection between the generalized Leonardo numbers and the Fibonacci and Lucas numbers.

Theorem 1. *For $n \geq 0$, we have*

$$\mathcal{L}_{k,n} = (k+1)F_{n+1} - k. \tag{2.1}$$

Proof. It is easy to check that Equation (2.1) holds for $n = 0, 1$. Assume it holds for $n(\geq 2)$. By definition and the inductive hypothesis, we obtain

$$\begin{aligned}
\mathcal{L}_{k,n+1} &= \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} + k \\
&= (k+1)F_{n+1} - k + (k+1)F_n - k + k \\
&= (k+1)F_{n+2} - k,
\end{aligned}$$

which shows that Equation (2.1) holds for $n + 1$, thereby proving the theorem. □

The relationship between the Fibonacci and Lucas numbers $L_n = F_{n+1} + F_{n-1}$, we get

$$\mathcal{L}_{k,n} = (k+1)\left(L_n - F_{n-1}\right) - k. \tag{2.2}$$

Particular case for $k = 1$, Equations (2.1) and (2.2) become Equations (1.1) and (1.2), respectively.

Since $F_{-n} = (-1)^{n+1} F_n$, we can derive the following identity for the generalized Leonardo numbers with negative subscripts.

Theorem 2. *For $n \geq 0$, we have*

$$\mathcal{L}_{k,-n} = \begin{cases} (-1)^n \mathcal{L}_{k,n-2}\ , & \text{if } n \text{ even} \\ (-1)^n \mathcal{L}_{k,n-2} - 2k\ , & \text{if } n \text{ odd} \end{cases}. \tag{2.3}$$

Taking $k = 1$ in Theorem 2, we get $Le_{-n} = (-1)^n (Le_{n-2} + 1) - 1$, (see [1, Theorem 2.1]).

We give the following summation formulas of the generalized Leonardo numbers by using Equation (2.1) and a well-known identity of the Fibonacci numbers.

Theorem 3. *For $n \geq 0$, we have*

(*i*) $\sum_{i=0}^{n} \mathcal{L}_{k,i} = \mathcal{L}_{k,n+2} - k(n+1) - 1$

(*ii*) $\sum_{i=0}^{n} \mathcal{L}_{k,2i} = \mathcal{L}_{k,2n+1} - kn$

(*iii*) $\sum_{i=0}^{n} \mathcal{L}_{k,2i+1} = \mathcal{L}_{k,2n+2} - k(n+2)$.

Proof. Since the proofs of all parts $(i) - (iii)$ are quite similar, we only give a proof for part (i). We have

$$\begin{aligned}
\sum_{i=0}^{n} \mathcal{L}_{k,i} &= \sum_{i=0}^{n} \left((k+1)F_{n+1} - k\right) \\
&= (k+1)\sum_{i=0}^{n} F_{n+1} - k(n+1) \\
&= (k+1)(F_{n+3} - 1) - k(n+1) \\
&= \mathcal{L}_{k,n+2} - k(n+1) - 1,
\end{aligned}$$

as desired. □

Taking $k = 1$ in Theorem 3, we get the identities of Proposition 3.1 in [2], respectively.

In [1], they defined the matrix Q associated with the Leonardo numbers by

$$Q = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

In fact, the matrix Q is also the matrix representation of the generalized Leonardo numbers. It is easy to verify that

$$Q^n = \frac{1}{k+1} \begin{bmatrix} \mathcal{L}_{k,n+2} - 1 & \mathcal{L}_{k,n+1} - 1 & \mathcal{L}_{k,n} - 1 \\ 1 - \mathcal{L}_{k,n} & 1 - \mathcal{L}_{k,n-1} & 1 - \mathcal{L}_{k,n-2} \\ 1 - \mathcal{L}_{k,n+1} & 1 - \mathcal{L}_{k,n} & 1 - \mathcal{L}_{k,n-1} \end{bmatrix}.$$

Theorem 4. *For $n \geq 1$, we have*

$$\begin{bmatrix} \mathcal{L}_{k,n+3} & \mathcal{L}_{k,n+2} & \mathcal{L}_{k,n+1} \\ \mathcal{L}_{k,n+2} & \mathcal{L}_{k,n+1} & \mathcal{L}_{k,n} \\ \mathcal{L}_{k,n+1} & \mathcal{L}_{k,n} & \mathcal{L}_{k,n-1} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{k,3} & \mathcal{L}_{k,2} & \mathcal{L}_{k,1} \\ \mathcal{L}_{k,2} & \mathcal{L}_{k,1} & \mathcal{L}_{k,0} \\ \mathcal{L}_{k,1} & \mathcal{L}_{k,0} & \mathcal{L}_{k,-1} \end{bmatrix} Q^n. \tag{2.4}$$

Proof. By induction on n, we see that Equation (2.4) holds for $n = 1$. Now assume Equation (2.4) is true for all integers $n \geq 1$. By the inductive hypothesis and the definition of $\mathcal{L}_{k,n}$, we obtain

$$\begin{aligned} \begin{bmatrix} \mathcal{L}_{k,3} & \mathcal{L}_{k,2} & \mathcal{L}_{k,1} \\ \mathcal{L}_{k,2} & \mathcal{L}_{k,1} & \mathcal{L}_{k,0} \\ \mathcal{L}_{k,1} & \mathcal{L}_{k,0} & \mathcal{L}_{k,-1} \end{bmatrix} Q^{n+1} &= \begin{bmatrix} \mathcal{L}_{k,3} & \mathcal{L}_{k,2} & \mathcal{L}_{k,1} \\ \mathcal{L}_{k,2} & \mathcal{L}_{k,1} & \mathcal{L}_{k,0} \\ \mathcal{L}_{k,1} & \mathcal{L}_{k,0} & \mathcal{L}_{k,-1} \end{bmatrix} Q^n Q \\ &= \begin{bmatrix} \mathcal{L}_{k,n+3} & \mathcal{L}_{k,n+2} & \mathcal{L}_{k,n+1} \\ \mathcal{L}_{k,n+2} & \mathcal{L}_{k,n+1} & \mathcal{L}_{k,n} \\ \mathcal{L}_{k,n+1} & \mathcal{L}_{k,n} & \mathcal{L}_{k,n-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{L}_{k,n+4} & \mathcal{L}_{k,n+3} & \mathcal{L}_{k,n+2} \\ \mathcal{L}_{k,n+3} & \mathcal{L}_{k,n+2} & \mathcal{L}_{k,n+1} \\ \mathcal{L}_{k,n+2} & \mathcal{L}_{k,n+1} & \mathcal{L}_{k,n} \end{bmatrix}, \end{aligned}$$

which shows that (2.4) holds for $n + 1$, thereby proving the theorem. □

Since $Q^{m+n} = Q^{m-1}Q^{n+1}$, equating $(1, 3)$-entry on both sides of this matrix equation, we get the following corollary.

Corollary 1. *For $m, n \geq 1$, we have*

$$\mathcal{L}_{k,m}\mathcal{L}_{k,n-1} + \mathcal{L}_{k,m-1}\mathcal{L}_{k,n} = \mathcal{L}_{k,m+1}\mathcal{L}_{k,n+1} - (k+1)\mathcal{L}_{k,m+n} - k. \tag{2.5}$$

Particular case for $k = 1$, Corollary 1 becomes Corollary 2.16 of [1].

Taking $m = n$ in Equation (2.5), we obtain the following identity.

$$\mathcal{L}_{k,2n} = \frac{1}{k+1}\left(\mathcal{L}_{k,n+1}^2 - 2\mathcal{L}_{k,n}\mathcal{L}_{k,n-1} - k\right). \tag{2.6}$$

When $k = 1$, Equation (2.6) is the example of Corollary 2.16 in [1].

3. The Incomplete Generalized Leonardo Numbers

In this section, we present the incomplete generalized Leonardo numbers which extend the incomplete Leonardo numbers of Catarino and Borges [3].

Definition 2. For a fixed non-negative integer k, define the incomplete generalized Leonardo numbers by

$$\mathcal{L}_{k,n}^{l} = (k+1)\sum_{i=0}^{l}\binom{n-i}{i} - k \quad (0 \le l \le \lfloor \frac{n}{2} \rfloor;\ n \ge 0).$$

We see that $\mathcal{L}_{k,n}^{0} = 1, \mathcal{L}_{k,n}^{1} = n(k+1) - k$ and $\mathcal{L}_{k,n}^{\lfloor \frac{n}{2} \rfloor} = \mathcal{L}_{k,n}$.

Theorem 5. *For a fixed non-negative integer k and $n \in \mathbb{N}$, the incomplete generalized Leonardo numbers satisfy the following recurrence equation:*

$$\mathcal{L}_{k,n+2}^{l+1} = \mathcal{L}_{k,n+1}^{l+1} + \mathcal{L}_{k,n}^{l} + k, \quad (0 \le l \le \frac{n-1}{2}). \tag{3.1}$$

Proof. We have

$$\begin{aligned} \mathcal{L}_{k,n+1}^{l+1} + \mathcal{L}_{k,n}^{l} + k &= (k+1)\sum_{i=0}^{l+1}\binom{n+1-i}{i} - k + (k+1)\sum_{i=0}^{l}\binom{n-i}{i} - k + k \\ &= (k+1)\sum_{i=0}^{l+1}\left(\binom{n+1-i}{i} + \binom{n+1-i}{i-1}\right) - k \\ &= (k+1)\sum_{i=0}^{l+1}\binom{n+2-i}{i} - k \\ &= \mathcal{L}_{k,n+2}^{l+1}, \end{aligned}$$

as desired. □

From Equation (3.1), we give the non-homogeneous recurrence equation of the incomplete generalized Leonardo numbers.

Theorem 6. *For a fixed non-negative integer k and $n \in \mathbb{N}$, we have*

$$\mathcal{L}_{k,n+2}^{l} = \mathcal{L}_{k,n+1}^{l} + \mathcal{L}_{k,n}^{l} + k - (k+1)\binom{n-l}{l}, \quad (0 \le l \le \frac{n-1}{2}). \tag{3.2}$$

Proof. By Equation (3.1) and the definition of $\mathcal{L}_{k,n}^{l}$, we have

$$\begin{aligned}\mathcal{L}_{k,n+2}^{l}-\mathcal{L}_{k,n+1}^{l}-\mathcal{L}_{k,n}^{l}-k &= \mathcal{L}_{k,n+1}^{l}+\mathcal{L}_{k,n}^{l-1}+k-\mathcal{L}_{k,n+1}^{l}-\mathcal{L}_{k,n}^{l}-k\\ &= \mathcal{L}_{k,n}^{l-1}-\mathcal{L}_{k,n}^{l}\\ &= (k+1)\sum_{i=0}^{l-1}\binom{n-i}{i}-(k+1)\sum_{i=0}^{l}\binom{n-i}{i}\\ &= (k+1)\binom{n-l}{l},\end{aligned}$$

as desired. □

Taking $k = 1$, Equation (3.2) becomes Equation (9) in [3]. Next, we give another property of the incomplete generalized Leonardo numbers.

Theorem 7. *For a fixed non-negative integer k and $n, s \in \mathbb{N}$, we have*

$$\mathcal{L}_{k,n+2s}^{l+s}=\sum_{i=0}^{s}\binom{s}{i}\mathcal{L}_{k,n+i}^{l+i}+(2^{s}-1)k, \quad (0\le l\le \frac{n-s}{2}). \tag{3.3}$$

Proof. By induction on s, we see that Equation (3.3) holds for $s = 1$. Now assume Equation (3.3) is true for all integers $s \geq 1$. By the inductive hypothesis and the definition of $\mathcal{L}_{k,n}^{l}$, we obtain

$$\begin{aligned}\sum_{i=0}^{s+1}\binom{s+1}{i}\mathcal{L}_{k,n+i}^{l+i} &= \sum_{i=0}^{s+1}\left[\binom{s}{i}+\binom{s}{i-1}\right]\mathcal{L}_{k,n+i}^{l+i}\\ &= \sum_{i=0}^{s}\binom{s}{i}\mathcal{L}_{k,n+i}^{l+i}+\sum_{i=0}^{s+1}\binom{s}{i-1}\mathcal{L}_{k,n+i}^{l+i}\\ &= \sum_{i=0}^{s}\binom{s}{i}\mathcal{L}_{k,n+i}^{l+i}+\sum_{i=0}^{s}\binom{s}{i}\mathcal{L}_{k,n+i+1}^{l+i+1}\\ &= \mathcal{L}_{k,n+2s}^{l+s}-(2^{s}-1)k+\mathcal{L}_{k,n+2s+1}^{l+s+1}-(2^{s}-1)k\\ &= \mathcal{L}_{k,n+2s+2}^{l+s+1}-k-(2^{s+1}-2)k\\ &= \mathcal{L}_{k,n+2s+2}^{l+s+1}-(2^{s+1}-1)k,\end{aligned}$$

which shows that (3.3) holds for $s + 1$, thereby proving the theorem. □

Taking $k = 1$ in Equation (3.3), we get

$$Le_{n+2s}^{l+s}=\sum_{i=0}^{s}\binom{s}{i}Le_{n+i}^{l+i}+2^{s}-1, \quad (0\le l\le \frac{n-s}{2}). \tag{3.4}$$

Equation (3.4) is a minor correction to Equation (10) of [3, Proposition 3], in which the term 2^s is missing.

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