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THE DIOPHANTINE EQUATION $\sum_{j=1}^{k} jf_j^p = l_n^q$

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Abstract

In this paper, we consider the Diophantine equation $F_1^p + 2F_2^p + \cdots + kF_k^p = L_n^q$ in positive integer variables (k, n, p, q), where F_i is the i^{th} term of the Fibonacci sequence and L_i the i^{th} term of its companion Lucas sequence. We prove that this equation has only finitely many nontrivial solutions.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by

$$F_0 = 0,\ F_1 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2}, \qquad \text{for } n \geq 2.$$

The initial terms of this sequence are

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

The Lucas sequence $(L_n)_{n\geq 0}$ is given by

$$L_0 = 2,\ L_1 = 1 \quad \text{and} \quad L_n = L_{n-1} + L_{n-2}, \qquad \text{for } n \geq 2.$$

The initial terms of this sequence are

$$2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ 199,\ 322,\ 521,\ 843,\ \ldots$$

Let $(\alpha, \beta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ be the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence. Binet's formula for F_n is

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad \text{for } n \geq 0. \tag{1.1}$$

This easily implies that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1}, \quad \text{for } n \geq 1. \tag{1.2}$$

It is easy to prove that

$$\frac{F_n}{F_{n+1}} \leq \frac{2}{3}, \text{ for } n \geq 2. \tag{1.3}$$

It is also easy to check that

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1, \text{ for } n \geq 0. \tag{1.4}$$

Binet's formula for L_n is

$$L_n = \alpha^n + \beta^n, \text{ for } n \geq 0. \tag{1.5}$$

Similarly, this implies the inequalities

$$\alpha^{n-1} \leq L_n \leq \alpha^{n+1}, \text{ for } n \geq 2. \tag{1.6}$$

Observe that

$$\begin{aligned} &F_1 = L_1, \quad F_1^2 = L_1. \\ &F_1 + 2F_2 = L_2, \quad F_1^2 + 2F_2^2 = L_2 \\ &F_1 + 2F_2 + 3F_3 = L_2^2. \end{aligned} \tag{1.7}$$

The natural question that arises is the following:

is $F_1^p + 2F_2^p + ... + kF_k^p$ a power of a Lucas number?

The aim of this paper is to give an answer to this question.

For a quick history, the Diophantine equation

$$\sum_{j=1}^{k} jF_j^p = F_n^q$$

has been studied in 2018 by G. Soydan, L. Németh, and L. Szalay [8], where F_i is the i^{th} Fibonacci number. They solved this equation for $(p,q) \in \{(1,1),(1,2),(2,1),(2,2)\}$. Further, they conjectured that the only non-trivial solutions are given by:

$$F_4^2 = 9 = F_1 + 2F_2 + 3F_3,\ F_8 = 21 = F_1 + 2F_2 + 3F_3 + 4F_4,\ F_4^3 = 27 = F_1^3 + 2F_2^3 + 3F_3^3.$$

In 2019, K. Gueth, F. Luca and L. Szalay [3] confirmed the conjecture, for $\max\{p, q\} \leq 10$. This result was again improved by Altassan and Luca [1] who proved that if such an equation is satisfied, then $\max\{k, n, p, q\} \leq 10^{2500}$. The second and third authors of this paper studied a similar equation, where the Fibonacci sequence is replaced by the Pell sequence (see [9]).

In this paper, we investigate the Diophantine equation

$$F_1^p + 2F_2^p + \cdots + kF_k^p = L_n^q. \tag{1.8}$$

We consider the solution given by

$$F_1^p = 1 = L_1^q,$$

as well as the solutions given by

$$F_1^p + 2F_2^p = 3 = L_2,$$

as trivial solution to (1.8). We will ignore such solutions. So, we consider $k \geq 3$, which implies that $n \geq 2$.

Note that even the case $p = 0$ leads to some interesting equations. Indeed, $p = 0$ gives

$$\sum_{j=1}^{k} j = L_n^q,$$

i.e.,

$$\frac{k(k+1)}{2} = \binom{k+1}{2} = L_n^q.$$

All perfect powers in binomial coefficients have been found by Győry [4]. His result implies that $q \leq 2$ i.e., $q = 1$ or $q = 2$.

For $q = 1$, we get a Lucas number which is a triangular number ($L_n = k(k+1)/2$) and the only such are $L_1 = 1,\ L_2 = 3,$ and $L_{18} = 5778$ (see [6]). The corresponding values of k are respectively $k = 1$, $k = 2$, and $k = 107$. So the only solutions of our equation (1.8) in this case are $(k, n, p, q) \in \{(1, 1, 0, 1), (2, 3, 0, 1), (107, 18, 0, 1)\}$.

For $q = 2$, we get $(2k+1)^2 - 2(2L_n)^2 = 1$. So, $L_n = P_{2m}/2$ is the half of an even Pell number (or a balancing number) and it is known that there is no balancing number which is a term of the Lucas sequence. This is a result due to K. Liptai in 2006 [5].

In this paper, we consider the case of positive variables and prove the following result.

Theorem 1. *The Diophantine Equation* (1.8) *has only finitely many solutions in positive integers variables* k, n, p, q, *with* $k \geq 3$. *Such solutions satisfy* $\max\{k, n, p, q\} < 10^{2100}$.

Our proof uses techniques from Diophantine approximations such as lower bounds for linear forms in logarithms of algebraic numbers that we will recall later.

2. Proof of Theorem 1

Recall that we are studying the Diophantine equation

$$\sum_{j=1}^{k} jF_j^p = L_n^q.$$

Put

$$s := \sum_{j=1}^{k} jF_j^p = L_n^q \quad \text{and} \quad x := \log s. \tag{2.1}$$

Our purpose is to bound x. Observe that

$$(\alpha^{k/3})^p \leq (\alpha^{k-2})^p < F_k^p < s$$

and

$$s < k\left(\sum_{j=1}^{k} F_j\right)^p < k(F_{k+2})^p < k\alpha^{(k+1)p} < \alpha^{(k+1)p+3\log k} < \alpha^{3kp},$$

where we used (1.4) to get that $\sum_{j=1}^{k} F_j < F_{k+2}$. Since $\log\alpha \in (1/3, 1/2)$, one can see that

$$kp/3 < 3\log s \quad \text{and} \quad 2\log s < 3kp.$$

We deduce that

$$\frac{2}{3}\log s < kp < 9\log s,$$

which leads to

$$10^{-1}x < kp < 10x.$$

Similarly, we have

$$(\alpha^{n/2})^q \leq (\alpha^{n-1})^q < s = L_n^q \leq (\alpha^{n+1})^q < \alpha^{2nq}$$

to obtain

$$x < nq < 10x.$$

We record this as a lemma.

Lemma 1. *With the notations given in* (2.1)*, if* (k, n, p, q) *is a nontrivial solution to* (1.8)*, then*

$$10^{-1}x < kp < 10x \quad \textit{and} \quad x < nq < 10x.$$

Next, we will look for positive constants t_1, t_2, and t_3 such that

$$k > x^{t_1}, \quad p > x^{t_2}, \quad \text{and} \quad n > x^{t_3}.$$

Let (k, n, p, q) be a nontrivial solution of the Diophantine Equation (1.8).

2.1. Value of t_1

In this subsection, we will prove the following result.

Lemma 2. *If $x > 10^{150}$, then $k > x^{1/4}$.*

Proof. Assume that $x > 10^{150}$ and that $k \le x^{1/4}$ in order to get a contradiction. Using the method in [1], we have

$$\begin{aligned} L_n^q = s &= \sum_{j=1}^{k} jF_j^p = kF_k^p \left(1 + \sum_{j=1}^{k-1} \left(\frac{k-j}{k}\right)\left(\frac{F_{k-j}}{F_k}\right)^p\right) \\ &= kF_k^p + kF_k^p \sum_{j=1}^{k-1} \left(\frac{k-j}{k}\right)\left(\frac{F_{k-j}}{F_k}\right)^p . \end{aligned} \tag{2.2}$$

Then, one obtains

$$|L_n^q - kF_k^p| = kF_k^p \sum_{j=1}^{k-1} \left(\frac{k-j}{k}\right)\left(\frac{F_{k-j}}{F_k}\right)^p$$

and dividing both sides by kF_k^p, we get

$$\begin{aligned} \left|\left(kF_k^p\right)^{-1} L_n^q - 1\right| &= \sum_{j=1}^{k-1} \left(\frac{k-j}{k}\right)\left(\frac{F_{k-j}}{F_k}\right)^p < 2\sum_{j\ge 1}\left(\frac{2}{3}\right)^{jp} \\ &< \frac{2}{1.5^p}\sum_{j\ge 0}\left(\frac{2}{3}\right)^j = \frac{6}{1.5^p}, \end{aligned} \tag{2.3}$$

where we used (1.3), as well as the fact that $\displaystyle\sum_{j\ge 0}\left(\frac{2}{3}\right)^j = 3.$

Before completing the proof of the current lemma, let us recall some necessary results. For any non-zero algebraic number η of degree d over $\mathbb{Q}$ whose minimal polynomial over $\mathbb{Z}$ is $a_0 \prod_{i=1}^{d}\left(X - \eta^{(i)}\right)$ with $a_0 > 0$, we denote by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\left\{1, \left|\eta^{(i)}\right|\right\}\right)$$

the usual absolute logarithmic height of η. We recall Theorem 9.4 of [2], which is a modified version of a result of Matveev [7].

Theorem 2. *Let $\alpha_1, \ldots, \alpha_r$ be real algebraic numbers and let $b_1, \ldots, b_r$ be nonzero integers. Let D be the degree of the number field $\mathbb{Q}(\alpha_1, \ldots, \alpha_r)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j \ge \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\} \quad \text{for} \quad j = 1, \ldots, r.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_r|\}.$$

If $\Lambda = \prod_{j=1}^{r} \alpha_j^{b_j} - 1 \neq 0$, *then*

$$\left| \prod_{j=1}^{r} \alpha_j^{b_j} - 1 \right| \geq \exp(-1.4 \cdot 30^{r+3} \cdot r^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_r).$$

Put

$$\Lambda_1 := (kF_k^p)^{-1} L_n^q - 1.$$

We check that $\Lambda_1 \neq 0$. Indeed, from equation (1.8), we have

$$0 < \sum_{j=1}^{k-1} jF_j^p = -(kF_k^p - L_n^q),$$

which implies that

$$\Lambda_1 > 0.$$

We will use Matveev's theorem to get a lower bound for Λ_1. Put

$$r := 3,\ \alpha_1 := k,\ \alpha_2 := F_k,\ \alpha_3 := L_n, \quad b_1 := -1, \quad b_2 := -p,\ b_3 := q.$$

Note that $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}$. Thus, we take $D := 1$. Since

$$h(\alpha_1) = \log k < \log(10x),\ h(\alpha_2) = \log F_k < 0.5k, \ \text{ and } h(\alpha_3) = \log L_n < n,$$

we take $A_1 := \log(10x)$, $A_2 := 0.5k$, $A_3 := n$. Finally, by Lemma 1, $B := 10x > \max |b_i|$. Hence, Matveev's theorem implies that

$$\log |\Lambda_1| \geq -1.4 \times 30^6 \times 3^{4.5} \times 1^2 \times (1 + \log 1)(1 + \log(10x))(\log(10x))(0.5k)(n).$$

So, we obtain

$$\log |\Lambda_1| > -1.44 \times 10^{10} kn \left(\log(10x)\right)^2, \tag{2.4}$$

where we used the fact that $1 + \log(10x) < 2\log(10x)$. Thus, Inequalities (2.3) and (2.4) imply that

$$p < 3.56 \times 10^{11} kn \left(\log(10x)\right)^2.$$

By Lemma 1, we observe that $n < 10x/q < 100kp/q$. So, we have $p < 3.56 \times 10^{11} k(100kp/q) \left(\log(10x)\right)^2$, which leads to

$$q < 3.56 \times 10^{13} k^2 \left(\log(10x)\right)^2. \tag{2.5}$$

Since $k \leq x^{1/4}$, we get that $q < 3.56 \times 10^{13} x^{1/2} \left(\log(10x)\right)^2$. Thus, we obtain

$$n > x/q > 2.8 \times 10^{-14} x^{1/2} / \left(\log(10x)\right)^2 > x^{1/3},$$

as $x > 10^{150}$. Hence, we have $q < 10x/n < 10x^{2/3} < 10n^2 < n^3$. The inequalities $n > x^{1/3} > 10^{50}$ lead to

$$\frac{q}{\alpha^{2n}} < \frac{n^3}{\alpha^{2n}} < \frac{1}{\alpha^n},$$

where we used the fact that $n^3 < \alpha^n$ for $n \geq 19$.

On the other hand, we have

$$L_n^q = (\alpha^n + \beta^n)^q = \alpha^{nq}\left(1 + \frac{(-1)^n}{\alpha^{2n}}\right)^q.$$

If n is odd, then

$$\begin{aligned} 1 > \left(1 + \frac{(-1)^n}{\alpha^{2j}}\right)^q = \left(1 - \frac{1}{\alpha^{2n}}\right)^q &= \exp\left(q \log\left(1 - \frac{1}{\alpha^{2n}}\right)\right) \\ &> \exp\left(-\frac{q}{\alpha^{2n}}\right) > \exp\left(-\frac{1}{\alpha^n}\right) \\ &> 1 - \frac{2}{\alpha^n} \end{aligned}$$

because $\frac{1}{\alpha^n} < \alpha^{-10^{150}}$ is very small. If n is even, then

$$\begin{aligned} 1 < \left(1 + \frac{(-1)^n}{\alpha^{2n}}\right)^q &= \exp\left(q \log\left(1 + \frac{1}{\alpha^{2n}}\right)\right) \\ &< \exp\left(\frac{q}{\alpha^{2n}}\right) < \exp\left(\frac{1}{\alpha^n}\right) \\ &< 1 + \frac{2}{\alpha^n}, \end{aligned}$$

again as $\frac{1}{\alpha^n} < \alpha^{-10^{150}}$. Hence, we obtain

$$L_n^q = \alpha^{nq}\left(1 + \frac{(-1)^n}{\alpha^{2n}}\right)^q = \alpha^{nq}\left(1 + \zeta_{q,n}\right), \qquad \text{with} \quad |\zeta_{q,n}| < \frac{2}{\alpha^n}.$$

We deduce that

$$\alpha^{nq}\left(1 + \zeta_{q,n}\right) = kF_k^p\left(1 + \sum_{j=1}^{k-1}\left(\frac{k-j}{k}\right)\left(\frac{F_{k-j}}{F_k}\right)^p\right).$$

We then have

$$\begin{aligned} |kF_k^p - \alpha^{nq}| &= \left|\alpha^{nq}\zeta_{q,n} - kF_k^p \sum_{j=1}^{k-1}\left(\frac{k-j}{k}\right)\left(\frac{F_{k-j}}{F_k}\right)^p\right| \\ &\leq \alpha^{nq}\left|\zeta_{q,n}\right| + kF_k^p \sum_{j=1}^{k-1}\left(\frac{k-j}{k}\right)\left(\frac{F_{k-j}}{F_k}\right)^p \\ &< \frac{2\alpha^{nq}}{\alpha^n} + \frac{6kF_k^p}{1.5^p} \end{aligned}$$

and dividing both sides of the above inequality by α^{nq}, we get

$$\left|kF_k^p\alpha^{-nq}-1\right| \quad < \quad \frac{2}{\alpha^n}+\frac{kF_k^p}{\alpha^{nq}}\frac{6}{1.5^p}. \tag{2.6}$$

We check that $kF_k^p/\alpha^{nq} < 5/3$. One can see that $p > 10^{111}$. Indeed, by Lemma 1 and using the assumption that $k \le x^{1/4}$, we have

$$p > 10^{-1}x/k > 10^{-1}x^{3/4} > 10^{-1}\left(10^{150}\right)^{3/4} > 10^{111}.$$

Since then, p and n are both very large, we have

$$\frac{2}{\alpha^n} < \frac{1}{4}, \quad \text{and} \quad \frac{6}{1.5^p} < \frac{1}{4}.$$

Then, Inequality (2.6) leads to

$$\left|\frac{kF_k^p}{\alpha^{nq}}-1\right| < \frac{1}{4}+\frac{1}{4}\frac{kF_k^p}{\alpha^{nq}}.$$

In particular, one can see that

$$\frac{kF_k^p}{\alpha^{nq}} < 1+\frac{1}{4}+\frac{1}{4}\frac{kF_k^p}{\alpha^{nq}} = \frac{5}{4}+\frac{1}{4}\frac{kF_k^p}{\alpha^{nq}},$$

which implies that

$$\frac{3}{4}\frac{kF_k^p}{\alpha^{nq}} < \frac{5}{4},$$

i.e., $\frac{kF_k^p}{\alpha^{nq}} < 5/3$. Then, Inequality (2.6) implies that

$$\left|kF_k^p\alpha^{-nq}-1\right| < \frac{2}{\alpha^n}+\frac{10}{1.5^p} < \frac{12}{1.5^{\min\{p,n\}}}, \tag{2.7}$$

as $\alpha > 1.5$.

Put

$$\Lambda_2 := kF_k^p\alpha^{-nq}-1.$$

We check that $\Lambda_2 \neq 0$. Indeed, if $\Lambda_2 = 0$, then $\alpha^{nq} = kF_k^p \in \mathbb{Z}$, which is a contradiction since no power of α of nonzero integer exponent can be an integer. Thus, $\Lambda_2 \neq 0$. We use Matveev's theorem with

$$r := 3,\ \alpha_1 := k,\ \alpha_2 := F_k,\ \alpha_3 := \alpha, \quad b_1 := 1, \quad b_2 := p,\ b_3 := -nq.$$

Note that $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}(\sqrt{5})$. Thus, we take $D := 2$. We know that

$$h(\alpha_1) = \log k < \log(10x),\ h(\alpha_2) = \log F_k < 0.5k, \ \text{ and } h(\alpha_3) = (\log\alpha)/2,$$

then we take

$$A_1 := 2\log(10x),\ A_2 := k,\ A_3 := \log\alpha.$$

Here again, $B := 10x$. Hence, Matveev's theorem implies that

$$\log|\Lambda_2| > -1.87 \times 10^{12} k \left(\log(10x)\right)^2. \tag{2.8}$$

Now, Inequalities (2.7) and (2.8) imply that

$$\min\{p, n\} < 4.7 \times 10^{12} k \left(\log(10x)\right)^2. \tag{2.9}$$

Assume that $\min\{p, n\} = n$. Then, we have

$$n < 4.7 \times 10^{12} k \left(\log(10x)\right)^2. \tag{2.10}$$

Inequalities (2.5) and (2.10) together with Lemma 1 imply that

$$10^{-1}x < nq < 8.5 \times 10^{25} k^3 \left(\log(10x)\right)^4 < 8.5 \times 10^{25} x^{3/4} \left(\log(10x)\right)^4.$$

Then, we obtain

$$x^{1/4} < 8.5 \times 10^{26} \left(\log(10x)\right)^4,$$

which gives $x < 10^{149}$. This is a contradiction.

Assume that $\min\{p, n\} = p$. Then, from Inequality (2.9), we get

$$p < 4.7 \times 10^{12} k \left(\log(10x)\right)^2.$$

Now, using Lemma 1, we finally obtain

$$10^{-1}x/k < p < 4.7 \times 10^{12} k \left(\log(10x)\right)^2.$$

Then, one sees that

$$x < 4.7 \times 10^{13} k^2 \left(\log(10x)\right)^2.$$

As $k \leq x^{1/4}$, this leads to $x^{1/2} < 4.7 \times 10^{13} \left(\log(10x)\right)^2$ i.e.,

$$x^{1/4} < 7 \times 10^6 \left(\log(10x)\right).$$

We deduce that $x < 10^{36}$, which is again a contradiction. This completes the proof of Lemma 2. □

2.2. Value of t_2

The goal of this subsection is the proof of the next lemma.

Lemma 3. *If $x > 10^{1520}$, then $p > x^{1/9}$.*

Proof. As for the proof of Lemma 2, we assume that $x > 10^{1520}$ and that $p \leq x^{1/9}$ in order to get a contradiction.

Since $x > 10^{1520} > 10^{150}$, by Lemma 2, we have

$$k/2 > (1/2)x^{1/4} > 5 \times 10^{379}.$$

Using Lemma 1 together with the fact that $k/2 > (1/2)x^{1/4}$, we have

$$p < 10x/k < 10k^3 < 100(k/2)^3 < (k/2)^4.$$

Put $l = \lfloor k/2 \rfloor + 1$. One can certainly see that $l > k/2 > p^{1/4}$. Let $j \in \{l, \ldots, k\}$. We have

$$\frac{p}{\alpha^{2j}} \leq \frac{(k/2)^4}{\alpha^{2l}} < \frac{(k/2)^4}{\alpha^{k+2}} < \frac{1}{\alpha^{k/2}},$$

where we used the inequality $(k/2)^4 < \alpha^{k/2+2}$, for $k \geq 50$. So from Inequality (19) in [1], we deduce that

$$s = \frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2}\left(1 + \tau_{k,p}\right),$$

where $|\tau_{k,p}| < 30/\alpha^{k/2}$. This means that

$$\left|\frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2} - s\right| < \frac{30\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2\alpha^{k/2}}. \tag{2.11}$$

So our equation (1.8) leads to

$$\left|\frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2} - L_n^q\right| < \frac{30\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2\alpha^{k/2}}.$$

Dividing both sides of the above inequality by L_n^q, we obtain

$$\begin{aligned}\left|\alpha^{(k+1)p}\left(\frac{k\alpha^p - (k+1)}{5^{p/2}(\alpha^p - 1)^2}\right)L_n^{-q} - 1\right| &< \frac{30\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2\alpha^{k/2}L_n^q} \\ &< \frac{30\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2\alpha^{k/2}k\alpha^{(k-1)p}}, \end{aligned} \tag{2.12}$$

where we used the fact that $L_n^q = s > kF_k^p > k\alpha^{k-2}$. Now, we have

$$\begin{aligned}\frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2k\alpha^{(k-1)p}} &= \frac{\alpha^{2p}(k\alpha^p - (k+1))}{k5^{p/2}(\alpha^p - 1)^2} < \frac{\alpha^{3p}}{5^{p/2}(\alpha^p - 1)^2} \\ &< \frac{\alpha^{3p}}{\alpha^{1.6p}(\alpha^p - 1)^2} < 6,\end{aligned}$$

where we used the fact that $\sqrt{5} > \alpha^{1.6}$ as well as the fact that $\alpha^{3t} - 6\alpha^{1.6t}(\alpha^t - 1)^2 < 0$, for every real number $t \geq 1$. So, Inequality (2.12) leads to

$$\left|\alpha^{(k+1)p}\left(\frac{k\alpha^p - (k+1)}{5^{p/2}(\alpha^p - 1)^2}\right)L_n^{-q} - 1\right| < \frac{180}{\alpha^{k/2}}. \tag{2.13}$$

Put

$$\Lambda_3 = \alpha^{(k+1)p}\left(\frac{k\alpha^p-(k+1)}{5^{p/2}(\alpha^p-1)^2}\right)L_n^{-q}-1.$$

We will prove that $\Lambda_3 \neq 0$, under the assumption $x > 10^{1520}$. Assuming that $\Lambda_3 = 0$, we get

$$k\alpha^p-(k+1) = \Delta_p^2 5^{p/2}\alpha^{-(k+1)p}L_n^q,$$

where $\Delta_p = \alpha^p - 1$. Taking the norms in $\mathbb{Q}(\sqrt{5})$, we get

$$\left|\left((-1)^p - L_p + 1\right)k^2 + (2-L_p)k + 1\right| = 5^p L_n^{2q}\left|(-1)^p - L_p + 1\right|^2.$$

If p is even, we get

$$\left|\left(L_p - 2\right)k^2 + (L_p-2)k - 1\right| = 5^p L_n^{2q}\left|L_p - 2\right|^2. \tag{2.14}$$

So $L_p - 2$ divides $(L_p-2)\,k^2 + (L_p-2)k - 1$, which implies that $L_p - 2$ divides 1, i.e., $L_p - 2 = \pm 1$. Thus, $p \in 1, 2$, and only $p = 2$ is convenient and the left-hand side of (2.14) is smaller than $2k^2$ since $L_p - 2 = \pm 1$. So, we obtain

$$2k^2 > 5^p L_n^{2q}\left|L_p - 2\right|^2 = 5^p L_n^{2q} \geq 5L_n^{2q},$$

showing that $L_n^q < k$, which is false since $L_n^q = kF_k^p + (k-1)F_{k-1}^p + ... + 2F_2^p + F_1^p > k$.

If p is odd, we get

$$\left|L_p k^2 + (L_p-2)k - 1\right| = 5^p L_n^{2q} L_p^2 \tag{2.15}$$

and the left-hand side of (2.15) is smaller than $2k^2L_p^2$. So, we obtain

$$2k^2L_p^2 > \left|L_p k^2 + (L_p-2)k - 1\right| = 5^p L_n^{2q}L_p^2 \geq 5L_n^{2q}L_p^2,$$

showing again that $L_n^q < k$, which is a contradiction.

Hence, we have $\Lambda_3 \neq 0$. Now, we can again apply Matveev's Theorem, with

$$r := 3,\ \alpha_1 := \alpha,\ \alpha_2 := \frac{k\alpha^p-(k+1)}{5^{p/2}(\alpha^p-1)^2},\ \alpha_3 := L_n,$$

$$b_1 := (k+1)p, \quad b_2 := 1,\ b_3 := -q, \quad D := 2.$$

One can see that

$$h(\alpha_1) = (\log\alpha)/2,\ h(\alpha_3) = \log L_n < 0.5n.$$

Now, for $h(\alpha_2)$, it is well-known that for any algebraic numbers η and γ and any integer s, we have

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma),$$
$$h(\eta+\gamma) \leq h(\eta) + h(\gamma) + \log 2,$$
$$h(\eta^s) \leq |s|\, h(\eta).$$

Using these properties, we have

$$\begin{aligned} h(\alpha_2) &\leq h(k\alpha^p - (k+1)) + 2h(\alpha^p - 1) + h(5^{p/2}) \\ &\leq h(k) + ph(\alpha) + h(k+1) + 2h(\alpha^p) + ph(5^{1/2}) + 3\log 2 \\ &\leq (3\log\alpha/2)p + 2\log k + 4\log 2 \leq 10p(\log(10x)). \end{aligned}$$

We can then take $A_1 := \log\alpha$, $A_2 := 20p(\log(10x))$, $A_3 := n$, $B = 20x$. Hence, Matveev's theorem implies that

$$\log|\Lambda_3| > -1.87\times 10^{13} np\left(\log(10x)\right)^2, \tag{2.16}$$

and together with (2.13), we get that

$$k < \frac{2}{\log\alpha}\left(1.87\times 10^{13} pn\left(\log(10x)\right)^2 + \log 180\right) < 7.8\times 10^{13} pn\left(\log(10x)\right)^2. \tag{2.17}$$

Since $p \leq x^{\frac{1}{9}}$ and $k > x^{\frac{1}{4}}$, it follows that

$$n > 10^{-13}\frac{k}{7.8\left(p\left(\log(10x)\right)^2\right)} > \frac{1.28\times 10^{-14} x^{\frac{1}{4}-\frac{1}{9}}}{\left(\log(10x)\right)^2} = \frac{1.28\times 10^{-14} x^{\frac{5}{36}}}{\left(\log(10x)\right)^2} > x^{\frac{1}{8}},$$

because $x > 10^{1520}$. That is, $x < n^8$ and then we have

$$q < \frac{10x}{n} < 10n^7 < n^8 \qquad \text{since} \qquad n > x^{1/8} > 10.$$

Thus, one obtains

$$L_n^q = \alpha^{nq}\left(1 + \frac{(-1)^n}{\alpha^{2n}}\right)^q = \alpha^{nq}\left(1 + \zeta_{q,n}\right), \qquad \text{with} \quad |\zeta_{q,n}| < \frac{2}{\alpha^n}.$$

Now, using $s = L_n^q$ and (2.11), we get

$$\begin{aligned} & \left|\frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2} - \alpha^{nq}\right| \\ = & \left|\frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2} - \alpha^{nq}\left(1 + \zeta_{q,n}\right) + \alpha^{nq}\zeta_{q,n}\right| \\ = & \left|\frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2} - s + \alpha^{nq}\zeta_{q,n}\right| \\ < & \frac{30\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2\alpha^{k/2}} + \alpha^{nq}\zeta_{q,n} < \frac{180}{\alpha^{k/2}} + \alpha^{nq}|\zeta_{q,n}|. \end{aligned}$$

Dividing both sides by α^{nq} and using the fact that $|\zeta_{q,n}| < \frac{2}{\alpha^n}$, we obtain

$$\left|\alpha^{p(k+1)-qn}\left(\frac{(k\alpha^p - (k+1))}{(\alpha^p - 1)^2}\right)5^{-\frac{p}{2}} - 1\right| < \frac{200}{\alpha^{\min\left\{\frac{k}{2},n\right\}}}. \tag{2.18}$$

Put
$$\Lambda_4 = \alpha^{p(k+1)-qn}\left(\frac{(k\alpha^p-(k+1))}{(\alpha^p-1)^2}\right)5^{-\frac{p}{2}}-1.$$
We prove that $\Lambda_4 \neq 0$. Indeed, if $\Lambda_4 = 0$, then $k\alpha^p-(k+1) = 5^{p/2}\Delta_p^2\alpha^{qn-(k+1)p}$, and taking the norms in $\mathbb{Q}(\sqrt{5})$, we get the equation
$$\left|((-1)^p - L_p + 1)\,k^2 + (2-L_p)k+1\right| = 5^p\left|(-1)^p - L_p + 1\right|^2,$$
which has no solutions with $k \geq 3$ from [3]. So, $\Lambda_4 \neq 0$. Again, we will apply again Matveev's Theorem, with
$$r := 3,\ \alpha_1 := \alpha,\ \alpha_2 := \frac{k\alpha^p-(k+1)}{(\alpha^p-1)^2},\ \alpha_3 := \sqrt{5},$$
$$b_1 := (k+1)p-qn, \quad b_2 := 1,\ b_3 := -p, \quad D := 2.$$
We have
$$h(\alpha_1) = (\log\alpha)/2, \quad h(\alpha_2) < 10p(\log(10x)), \quad \text{ and } h(\alpha_3) = (\log 5)/2,$$
and we take $A_1 := \log\alpha$, $A_2 := 20p(\log(10x))$, $A_3 := \log 5$, $B = 20x$. Hence, Matveev's theorem implies that
$$\log|\Lambda_4| > -3.01\times 10^{13}p\left(\log(10x)\right)^2, \tag{2.19}$$
and together with (2.18), we get
$$\begin{aligned}\min\left\{\frac{k}{2},n\right\} &< \frac{1}{\log\alpha}\left(3.01\times 10^{13}p\left(\log(10x)\right)^2+\log 200\right)\\ &< 6.3\times 10^{13}p\left(\log(10x)\right)^2.\end{aligned}$$
If $\min\{k/2,n\} = k/2$, then
$$k < 1.26\times 10^{14}p\left(\log(10x)\right)^2.$$
Since $k > 10^{-1}x/p$ by Lemma 1, we get $x/p < 1.26\times 10^{15}p\left(\log(10x)\right)^2$, which implies that $p > 2.82\times 10^{-8}x^{1/2}/(\log(10x))$.

The assumption, $p \leq x^{\frac{1}{9}}$ implies that $x^{\frac{1}{9}} > 2.82\times 10^{-8}x^{1/2}/(\log(10x))$ i.e., $x^{7/18}/(\log(10x)) < 3.55\times 10^8$. We deduce that $x < 10^{27}$, which is false.

If $\min\{k/2,n\} = n$, then we obtain
$$n < 6.3\times 10^{13}p\left(\log(10x)\right)^2. \tag{2.20}$$
Now, from inequalities (2.17), (2.20) and Lemma 2, we get
$$x^{\frac{1}{4}} < k < 5\times 10^{27}p^2\left(\log(10x)\right)^4 \leq 5\times 10^{27}x^{\frac{2}{9}}\left(\log(10x)\right)^4,$$
i.e.,
$$\frac{x^{\frac{1}{36}}}{\left(\log(10x)\right)^4} < 5\times 10^{27},$$
which leads to $x < 10^{1510}$. This is another contradiction and finishes the proof of Lemma 3. □

2.3. Value of t_3

The aim of this subsection is to prove the following lemma.

Lemma 4. *If $x > 10^{2099}$, then $n > x^{1/10}$.*

Proof. Suppose that $x > 10^{2099}$. Then, by Lemma 2, we have $k > x^{1/4} > 10^{524}$, and using Lemma 1, we obtain $p < 10x/k < 10k^3 < k^4$, which implies that

$$\frac{p}{\alpha^{2k}} < \frac{k^4}{\alpha^{2k}} < \frac{1}{\alpha^k},$$

since $k^4 < \alpha^k$, for $k \geq 28$.

We now write

$$F_k^p = \frac{\alpha^{kp}}{5^{p/2}}\left(1 - \frac{(-1)^k}{\alpha^{2k}}\right)^p$$

and use the similar arguments as in Lemma 3 to obtain

$$F_k^p = \frac{\alpha^{kp}}{5^{p/2}}\left(1 - \frac{(-1)^k}{\alpha^{2k}}\right)^p = \frac{\alpha^{kp}}{5^{p/2}}(1 + \zeta_{k,p}), \qquad |\zeta_{k,p}| < \frac{2}{\alpha^k}.$$

We finally go back to (2.2) and use Inequality (2.3) to get

$$\left|k\alpha^{pk}5^{-\frac{p}{2}}L_n^{-q} - 1\right| < \frac{10}{1.5^{\min\{p,k\}}}.$$

Similarly as for Λ_2, we easily check that $k\alpha^{pk}5^{-\frac{p}{2}}L_n^{-q} - 1 \neq 0$. We can use once again Matveev's theorem to obtain

$$\min\{p, k\} < 3.3 \times 10^{14} n(\log(10x))^2.$$

Since $x > 10^{2099}$, by Lemmas 2 and 3, $\min\{p, k\} > x^{\frac{1}{9}}$ and then we get

$$n > \frac{3.03 \times 10^{-15} x^{\frac{1}{9}}}{(\log(10x))^2} > x^{\frac{1}{10}},$$

for $x > 10^{2099}$ and the proof of Lemma 4 is complete. □

2.4. Completing the Proof of Theorem 1

We start this subsection by numerically bounding x. Precisely, we want to prove that $x \leq 10^{2099}$. Assume that $x > 10^{2099}$. Then, by Lemma 4, we have $q < 10x/n < 10n^9 < n^{10}$. So, our main Equation (1.8) becomes

$$\frac{k\alpha^{pk}}{5^{\frac{p}{2}}}\left(1 + \zeta_{p,k}\right) = L_n^q = \alpha^{nq}\left(1 + \zeta_{q,n}\right),$$

with $|\zeta_{p,k}| < \frac{6}{1.5^{\min\{p,k\}}}$ and $|\zeta_{q,n}| < \frac{2}{\alpha^n}$. As $x > 10^{2099}$, by Lemmas 2, 3 and 4, we have $\min\{n,p,k\} > x^{1/10} > 10^{209}$. So, the three variables k, p and n are large and then after rearranging similarly as in Lemmas 2 and 3, we obtain

$$\left|k\alpha^{pk-qn}5^{-p/2} - 1\right| < \frac{10}{1.5^{\min\{n,p,k\}}}.$$

Put

$$\Lambda_5 := k\alpha^{pk-qn}5^{-p/2} - 1.$$

We will distinguish two cases.

`Case 1:` $\Lambda_5 \neq 0$. In this case, applying Matveev's theorem, we obtain

$$\min\{n,p,k\} < 7.43 \times 10^{12}(\log(10x))^2.$$

Observe that by Lemmas 2, 3 and 4, $\min\{n,p,k\} > x^{\frac{1}{10}}$, so that $x^{\frac{1}{10}} < 7.43 \times 10^{12}(\log(10x))^2$, which gives $x < 10^{182}$, a contradiction to $x > 10^{2099}$.

`Case 2:` $\Lambda_5 = 0$. In this case, we get $\alpha^{kp-nq} = 5^{p/2}/k$. Squaring this, we get $\alpha^{2(kp-nq)} = 5^p/k^2$. The only possibility is that both sides of the above equality are 1. Thus, $kp = nq$ and $k = 5^{p/2}$. In particular, p is even and

$$p = \frac{2\log k}{\log 5} < 2\log k < \log(10x), \tag{2.21}$$

where we used once again Lemma 1 to get $k < 10x$. Since $x > 10^{2099}$, from Lemma 3, we have $p > x^{1/9}$. Hence, inequality (2.21) leads to

$$\frac{x^{1/9}}{\log(10x)} < 2,$$

which implies that $x < 10^{18}$. This is again a contradiction. So, $x \leq 10^{2099}$, as expected.

Finally, by Lemma 1, $\max\{k,n,p,q\} < 10x \leq 10^{2100}$, which completes the proof of Theorem 1.

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