A GENERALIZATION OF AN INEQUALITY BY GRAHAM

Curtis Cooper
School of Computer Science and Mathematics, University of Central Missouri, Warrensburg, Missouri
cooper@ucmo.edu

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Abstract

Let $b \geq 2$, i, and n be nonnegative integers. Let $w(i)$ be the base b digital sum of i and let

$$W(n) = \sum_{i=0}^{n} w(i).$$

Graham proved that for $b = 2$ and for positive integers n_1 and $n_2 \geq n_1$,

$$W(n_1 - 1) + W(n_2 - 1) + n_1 < W(n_1 + n_2 - 1) + 1.$$

We will prove that this same inequality holds for any $b \geq 2$.

1. Introduction

Let $b \geq 2$, i, and n be nonnegative integers. Let $w(i)$ be the base b digital sum of i and let

$$W(n) = \sum_{i=0}^{n} w(i).$$

Graham [2] gave an elegant proof that for $b = 2$ and for positive integers n_1 and $n_2 \geq n_1$, we have

$$W(n_1 - 1) + W(n_2 - 1) + n_1 < W(n_1 + n_2 - 1) + 1.$$

Allouche [1] gave a more standard proof of this result and suggested some generalizations. We will prove that this inequality holds for any $b \geq 2$.

2. Notation and Definitions

We begin with some notation and definitions.

To help visualize the nonnegative integers in base b, it is convenient to arrange these numbers according to their size and base b digital sum. In this infinite array, row 0 contains the nonnegative integers with base b digital sum 0, row 1 contains the nonnegative integers with base b digital sum 1, etc. Column 0 contains the nonnegative integers 0, 1, …, $b-1$, column 1 contains the nonnegative integers b, $b+1$, …, $2b-1$, etc.

Here is the initial segment of the infinite array for $b = 2$.

0																		
1	2	4		8				16								32		
	3	5	6	9	10	12		17	18	20		24				33	34	36
			7		11	13	14		19	21	22	25	26	28			35	37
							15				23		27	29	30			
															31			

Here is the initial segment of the infinite array for $b = 10$.

0															
1	10									100					
2	11	20								101	110				
3	12	21	30							102	111	120			
4	13	22	31	40						103	112	121	130		
5	14	23	32	41	50					104	113	122	131	140	
6	15	24	33	42	51	60				105	114	123	132	141	150
7	16	25	34	43	52	61	70			106	115	124	133	142	151
8	17	26	35	44	53	62	71	80		107	116	125	134	143	152
9	18	27	36	45	54	63	72	81	90	108	117	126	135	144	153
	19	28	37	46	55	64	73	82	91	109	118	127	136	145	154
		29	38	47	56	65	74	83	92		119	128	137	146	155
			39	48	57	66	75	84	93			129	138	147	156
				49	58	67	76	85	94				139	148	157
					59	68	77	86	95					149	158
						69	78	87	96						159
							79	88	97						
								89	98						
									99						

Now, we need a definition for the nonnegative integers in an interval.

Definition 1. Let $m_1 \leq m_2$ be nonnegative integers. Let

$$[m_1..m_2] = \{m_1, m_1 + 1, \ldots, m_2\}.$$

Next, we define a function δ associated with w and a one-to-one mapping $\varphi\colon [0..r] \to [s..s+r]$. This function will be essential in the proof of Graham's inequality for an arbitrary base.

Definition 2. Let r and s be nonnegative integers. Let $\varphi\colon [0..r] \to [s..s+r]$ be a one-to-one map. Then

$$\delta(\varphi) = \min_{0 \le i \le r} (w(\varphi(i)) - w(i)).$$

3. Crucial Lemma and Its Applications

We now present the following lemma. This lemma will help in proving a generalization of Graham's inequality.

Lemma 1. *Let $b \ge 2$, r, and s be nonnegative integers. Then, there exists a one-to-one mapping*

$$\varphi\colon [0..r] \to [s..s+r]$$

such that

(i) if $s \le r$, then $\delta(\varphi) \ge 0$;

(ii) if $s > r$, then $\delta(\varphi) \ge 1$.

Before we prove Lemma 1, we will demonstrate how we can apply Lemma 1 in three different ways. We will use these applications in the induction step in the proof of Lemma 1. Lemma 1 states that for base b, there exists a one-to-one map $\varphi\colon [0..r] \to [s..s+r]$ such that if $s \le r$, then $\delta(\varphi) \ge 0$, and if $s > r$, then $\delta(\varphi) \ge 1$. The first way we can apply Lemma 1 is just as it is given in the lemma. We will state the second way to apply Lemma 1 in the following corollary.

Corollary 1. *Let $b \ge 2$, r, and s be nonnegative integers. Assume that there exists a one-to-one mapping $\varphi\colon [0..r] \to [s..s+r]$ such that if $s \le r$, then $\delta(\varphi) \ge 0$, and if $s > r$, then $\delta(\varphi) \ge 1$. Let $a \ge 0$ and $k > 0$ be integers. Then, there exists a*

$$\varphi'\colon [ab^k..ab^k + r] \to [ab^k + s..ab^k + s + r] \subseteq [ab^k..b^{k+1} - 1]$$

such that φ' is one-to-one and if $s \le r$, then $\delta(\varphi') \ge 0$, and if $s > r$, then $\delta(\varphi') \ge 1$.

Proof. Assume that $s > r$. We first note that

$$w(ab^k + i) = w(a) + w(i)$$

for $i \in [0..b^k - 1]$. By assumption, there exists a one-to-one map $\varphi\colon [0..r] \to [s..s+r]$ such that $\delta(\varphi) \ge 1$. Then, we define the one-to-one map

$$\varphi'\colon [ab^k..ab^k + r] \to [ab^k + s..ab^k + s + r]$$

given by the formula

$$\varphi'(ab^k + i) = ab^k + \varphi(i).$$

In addition, using $\delta(\varphi) \geq 1$, for each $i \in [0..r]$, we have

$$\begin{aligned} w(\varphi'(ab^k + i)) - w(ab^k + i) &= w(ab^k + \varphi(i)) - w(ab^k + i) \\ &= w(a) + w(\varphi(i)) - w(a) - w(i) \\ &= w(\varphi(i)) - w(i) \geq 1. \end{aligned}$$

So φ' is a one-to-one mapping such that $\delta(\varphi') \geq 1$. Without the assumption $s > r$, we have $\delta(\varphi') \geq 0$. □

For an example of Corollary 1, let $b = 4$, $a = 2$, $k = 2$, $r = 3$, and $s = 16$. Then, there exists a one-to-one mapping $\varphi\colon [32..35] \to [48..51]$ such that $\delta(\varphi) \geq 1$. To describe the third way to apply Lemma 1, we need another definition and some results.

Definition 3. Let n be a nonnegative integer. Let $t(n)$ be the number of terminating $(b-1)$s in the base b representations of the numbers in $[0..n-1]$.

For base 10, the function t is called the terminating nines function. It is known [3] that

$$t(n) = \sum_{i \geq 1} \left\lfloor \frac{n}{b^i} \right\rfloor$$

and that

$$w(n) = n - (b-1)t(n).$$

In addition, for any positive integer k and $i \in [0..b^k - 1]$, we have

$$\begin{aligned} t(i) + t(b^k - 1 - i) &= \frac{b^k - 1}{b - 1} - k, \\ w(i) + w(b^k - 1 - i) &= (b-1)k. \end{aligned} \tag{1}$$

It can also be shown, by translating (1), that for integers $a \geq 0$, $1 \leq a_1 \leq b$, and $k > 0$, and for integers $0 \leq i \leq a_1 b^{k-1} - 1$, we have

$$w(ab^k + i) + w(ab^k + a_1 b^{k-1} - 1 - i) = 2w(a) + a_1 + (b-1)(k-1). \tag{2}$$

We will use the properties of the functions t and w in the set-up for a corollary to apply Lemma 1.

Corollary 2. *Let $b \geq 2$, r, and s be nonnegative integers. Assume that there exists a one-to-one mapping $\varphi\colon [0..r] \to [s..s+r]$ such that if $s \leq r$, then $\delta(\varphi) \geq 0$, and* *if $s > r$, then $\delta(\varphi) \geq 1$. Let $a \geq 0$, $1 \leq a_1 \leq b$, $k > 0$, and $s + r < b^k$ be integers. Then, there exists a one-to-one map*

$$\begin{aligned}\varphi'' \colon &[ab^k + a_1 b^{k-1} - 1 - (s+r)..ab^k + a_1 b^{k-1} - 1 - s] \\ &\to [ab^k + a_1 b^{k-1} - 1 - r..ab^k + a_1 b^{k-1} - 1]\end{aligned}$$

such that φ'' is one-to-one and if $s \leq r$, then $\delta(\varphi'') \geq 0$, and if $s > r$, then $\delta(\varphi'') \geq 1$.

Proof. Assume that $s > r$. Let

$$\begin{aligned}\varphi' \colon &[ab^k + a_1 b^{k-1} - 1 - r..ab^k + a_1 b^{k-1} - 1] \\ &\to [ab^k + a_1 b^{k-1} - 1 - (s+r)..ab^k + a_1 b^{k-1} - 1 - s]\end{aligned}$$

given by the formula

$$\varphi'(ab^k + a_1 b^{k-1} - 1 - i) = ab^k + a_1 b^{k-1} - 1 - \varphi(i),$$

where $i \in [0..r]$. In addition, by (2) and $\delta(\varphi) \geq 1$, for each $i \in [0..r]$, we have

$$\begin{aligned}&w(\varphi'(ab^k + a_1 b^{k-1} - 1 - i)) - w(ab^k + a_1 b^{k-1} - 1 - i) \\ &= w(ab^k + a_1 b^{k-1} - 1 - \varphi(i)) - w(ab^k + a_1 b^{k-1} - 1 - i) \\ &= 2w(a) + a_1 + (b-1)(k-1) - w(ab^k + \varphi(i)) - 2w(a) - a_1 \\ &\quad - (b-1)(k-1) + w(ab^k + i) \\ &= w(ab^k + i) - w(ab^k + \varphi(i)) \\ &= w(i) - w(\varphi(i)) \leq \text{-}1.\end{aligned}$$

Therefore, we let

$$\begin{aligned}\varphi'' = \varphi'^{-1} \colon &[ab^k + a_1 b^{k-1} - 1 - (s+r)..ab^k + a_1 b^{k-1} - 1 - s] \\ &\to [ab^k + a_1 b^{k-1} - 1 - r..ab^k + a_1 b^{k-1} - 1].\end{aligned}$$

Then φ'' is a one-to-one map and satisfies $\delta(\varphi'') \geq 1$.

With the assumption $s > r$, we have $\delta(\varphi'') \geq 0$. □

For an example of Corollary 2, let $b = 3$, $a = 4$, $a_1 = 2$, $k = 2$, $r = 1$, and $s = 2$. Then, there exists a one-to-one mapping $\varphi \colon [38..39] \to [40..41]$ such that $w(\varphi(i)) - w(i) \geq 1$.

4. Proof of Lemma 1

The proof is by induction on r.

`Base Step`. $r = 0$.

For $s = 0$, the one-to-one map $\varphi(0) = 0$ has the property that $w(\varphi(0)) = w(0) = 0$ and $w(0) = 0$ so $\delta(\varphi) = 0 \geq 0$. For $s \geq 1$, the one-to-one map $\varphi(0) = s$ has the property that $w(\varphi(0)) = w(s)$ and $w(0) = 0$ so $\delta(\varphi) = w(s) \geq 1$. Therefore, the base step is shown.

`Induction Step`. Assume the lemma holds for all nonnegative integers less than $r > 0$. We will prove the inductive thesis/statement of the lemma for r.

If $s \leq r$, then the two sets $[0..r]$ and $[s..s+r]$ overlap. Partition them into $[0..s-1] \cup [s..r]$ and $[s..r] \cup [r+1..s+r]$. Using the induction hypothesis, we apply Lemma 1(ii) so that there is a one-to-one map $\varphi\colon [0..s-1] \to [r+1..s+r]$ with $\delta(\varphi) \geq 1$. Define the map $\varphi'\colon [0..r] \to [s..s+r]$ by

$$\varphi'(i) = \begin{cases} \varphi(i), & \text{if } i \in [0..s-1]; \\ i, & \text{if } i \in [s..r]. \end{cases}$$

Then φ' is one-to-one and satisfies $\delta(\varphi') \geq 0$. This establishes (i) in the case $s \leq r$.

If $s > r$, then we consider two cases. Let k be the smallest integer such that $r < b^k$. The first case is when $[s..s+r]$ does not contain a multiple of b^k. The second case is when $[s..s+r]$ contains a multiple of b^k.

`Case 1`. There is no multiple of b^k in $[s..s+r]$.

Let ab^k be the largest multiple of b^k such that $ab^k < s$. Let $\varphi\colon [0..r] \to [ab^k..ab^k + r]$ for $i \in [0..r]$ by $\varphi(i) = ab^k + i$. We know that $w(\varphi(i)) = w(a) + w(i)$ for all $i \in [0..r]$. Thus, $\delta(\varphi) = w(a) \geq 0$. We have two subcases, when the intervals $[ab^k..ab^k + r]$ and $[s..s+r]$ overlap and they do not overlap.

`Subcase 1.1`. The two intervals overlap.

Then $a \geq 1$ since if $a = 0$ and they overlap, this would contradict $s > r$. Therefore, $\delta(\varphi) = w(a) > 0$. Consider the interval of overlap, $[m_1..m_2]$, where $m_1 \leq m_2$. If $[m_1..m_2] = [s..s+r]$, then φ is one-to-one and $\delta(\varphi) = w(a) > 0$ and we are done. If not, then by applying the induction hypothesis of Lemma 1 and Corollary 1, we let $ab^k + r_1 = m_1 - 1$ and $ab^k + s_1 = m_2 + 1$. Then, there exists a one-to-one map

$$\varphi'\colon [ab^k..ab^k + r_1] \to [ab^k + s_1..ab^k + s_1 + r_1]$$

such that $\delta(\varphi') \geq 1$. Now, let

$$\varphi''\colon [ab^k..ab^k + r_1] \to [ab^k + s_1..ab^k + s_1 + r_1]$$

be defined by

$$\varphi''(i) = \begin{cases} \varphi' \circ \varphi(i), & \text{if } i \in [0..r_1]; \\ \varphi(i), & \text{if } i \in [r_1 + 1..r]. \end{cases}$$

The mapping φ'' is one-to-one and satisfies $\delta(\varphi'') \geq 1$.

Subcase 1.2. The two intervals do not overlap.

Let t be the largest multiple of b^{k-1} in $[s..s+r]$. Then $t = ab^k + a_1 b^{k-1}$, where $1 \le a_1 < b$. If $t = s$, then because there is no multiple of b^k in $[s..s+r]$, the mapping $\varphi\colon [0..r] \to [t..t+r]$ given by the formula $\varphi(i) = t + i$ is one-to-one and $\delta(\varphi) = w(a) + a_1$ and we are done. Therefore, we can assume $t \ne s$.

We proceed to apply the induction hypothesis twice by dividing the sets $[ab^k..ab^k + r]$ and $[s..s+r]$. Partition them into

$$\begin{aligned}&[ab^k..ab^k + r_1] \cup [ab^k + r_1 + 1..ab^k + r]\\ \text{and } &[ab^k + a_1 b^{k-1} - 1 - r_2..ab^k + a_1 b^{k-1} - 1] \cup [s_1..s_1 + r_1],\end{aligned}$$

where $r_1 = s + r - t$, $s_1 = t$, $r_2 = r - r_1 - 1$, $s_2 = a_1 b^{k-1} - 1 - r$. Note that $s_1 + r_1 = s + r$, $s_1 = t$, $ab^k + a_1 b^{k-1} - 1 = t - 1$, and $ab^k + a_1 b^{k-1} - 1 - r_2 = s$. Also, note that $ab^k + r = ab^k + a_1 b^{k-1} - 1 - s_2$ and $ab^k + r_1 + 1 = ab^k + a_1 b^{k-1} - 1 - (s_2 + r_2)$. Because $t \ne s$, these partition sets are nonempty. So we can apply the induction hypothesis twice. By applying the induction hypothesis of Lemma 1 and Corollary 1, there exists a one-to-one map

$$\varphi'\colon [ab^k..ab^k + r_1] \to [s_1..s_1 + r_1]$$

such that $\delta(\varphi') \ge 1$. Note that $s_2 + r_2 < b^k$. Thus, applying the induction hypothesis of Lemma 1 and Corollary 2, there exists a one-to-one map

$$\varphi''\colon [ab^k + r_1 + 1..ab^k + r] \to [ab^k + a_1 b^{k-1} - 1 - r_2..ab^k + a_1 b^{k-1} - 1]$$

such that $\delta(\varphi'') \ge 0$. Now, let $\varphi'''\colon [0..r] \to [s..s+r]$ be defined by

$$\varphi'''(i) = \begin{cases} \varphi' \circ \varphi(i), & \text{if } i \in [0..r_1];\\ \varphi'' \circ \varphi(i), & \text{if } i \in [r_1 + 1..r].\end{cases}$$

The mapping φ''' is one-to-one and satisfies $\delta(\varphi''') \ge 1$.

Case 2. There exists a multiple of b^k in $[s..s+r]$.

We first note that this multiple of b^k is unique since $r < b^k$. Let ab^k be this multiple and let $ab^k = a_1 b^{k_1}$ where a_1 and b are relatively prime. If $s = a_1 b^{k_1}$, then the map $\varphi\colon [0..r] \to [a_1 b^{k_1}..a_1 b^{k_1} + r]$ given by the formula $\varphi(i) = a_1 b^{k_1} + i$ is one-to-one and $\delta(\varphi) = w(a_1)$ and we are done. So we assume $s \ne a_1 b^{k_1}$. Now let $\varphi\colon [0..r] \to [(a_1 - 1)b^{k_1}..(a_1 - 1)b^{k_1} + r]$ for $i \in [0..r]$ be defined by the formula $\varphi(i) = (a_1 - 1)b^{k_1} + i$. Thus, $\delta(\varphi) = w(a_1 - 1)$.

We proceed to apply the induction hypothesis twice by dividing the two sets $[(a_1 - 1)b^{k_1}..(a_1 - 1)b^{k_1} + r]$ and $[s..s+r]$. Let $t = a_1 b^{k_1}$. Partition the sets into

$$\begin{aligned}&[(a_1 - 1)b^{k_1}..(a_1 - 1)b^{k_1} + r_1] \cup [(a_1 - 1)b^{k_1} + r_1 + 1..(a_1 - 1)b^{k_1} + r]\\ \text{and } &[a_1 b^{k_1} - 1 - r_2..a_1 b^{k_1} - 1] \cup [s_1..s_1 + r_1],\end{aligned}$$

where $r_1 = s + r - t$, $s_1 = t$, $r_2 = r - r_1 - 1$, and $s_2 = b^{k_1} - r - 1$. Note that $s_1 + r_1 = s + r$, $s_1 = t$, $a_1 b^{k_1} - 1 = t - 1$, and $a_1 b^{k_1} - 1 - r_2 = s$. Also, note that $(a_1 - 1)b^{k_1} + r = a_1 b^{k_1} - 1 - s_2$ and $(a_1 - 1)b^{k_1} + r_1 + 1 = a_1 b^{k_1} - 1 - (s_2 + r_2)$. Because $s \neq a_1 b^{k_1}$, each of these partition sets is nonempty. Since $s_2 + r_2 < b^{k_1}$, we can apply the induction hypothesis twice. If $a_1 - 1 = 0$, then φ is the identity map and all the sets in the partition are disjoint. So the maps we will create from the induction hypothesis will have their δs greater than or equal to 1. And if $a_1 - 1 > 0$, all the maps we will create from the induction hypothesis will have their δs greater than or equal to 0. By applying the induction hypothesis of Lemma 1 and Corollary 1, there exists a one-to-one map

$$\varphi' \colon [(a_1 - 1)b^{k_1}..(a_1 - 1)b^{k_1} + r_1] \to [s_1..s_1 + r_1]$$

such that $\delta(\varphi') \geq 1$. And applying the induction hypothesis of Lemma 1 and Corollary 2, there exists a one-to-one map

$$\varphi'' \colon [(a_1 - 1)b^{k_1} + r_1 + 1..(a_1 - 1)b^{k_1} + r] \to [a_1 b^{k_1} - 1 - r_2..a_1 b^{k_1} - 1]$$

such that $\delta(\varphi'') \geq 0$. Now, let $\varphi''' \colon [0..r] \to [s..s + r]$ be defined by

$$\varphi'''(i) = \begin{cases} \varphi' \circ \varphi(i), & \text{if } i \in [0..r_1]; \\ \varphi'' \circ \varphi(i), & \text{if } i \in [r_1 + 1..r]. \end{cases}$$

The mapping φ''' is one-to-one and satisfies $\delta(\varphi''') \geq 1$.

This completes the induction step. Therefore, the lemma is true. □

5. Main Result

We are now ready to state and prove the theorem.

Theorem 1. *Let $b \geq 2$, n_1, and $n_1 \leq n_2$ be positive integers. Then*

$$W(n_1 - 1) + W(n_2 - 1) + n_1 < W(n_1 + n_2 - 1) + 1.$$

Proof. Since $0 < n_1 \leq n_2$, by Lemma 1, there exists a one-to-one mapping $\varphi \colon [0..n_1 - 1] \to [n_2..n_2 + n_1 - 1]$ such that $\delta(\varphi) \geq 1$. Thus, $w(\varphi(i)) - w(i) \geq 1$ for $0 \leq i \leq n_1 - 1$. Then

$$\begin{aligned} W(n_2 + n_1 - 1) - W(n_2 - 1) + 1 &> W(n_2 + n_1 - 1) - W(n_2 - 1) \\ &= \sum_{i=n_2}^{n_2+n_1-1} w(i) = \sum_{i=0}^{n_1-1} w(\varphi(i)) \\ &\geq \sum_{i=0}^{n_1-1} (w(i) + 1) = W(n_1 - 1) + n_1. \end{aligned}$$

This completes the proof. □

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