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SUM AND MOD SUM GRAPHS OF ARITHMETIC SETS

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Abstract

The concept of a sum graph was introduced by Harary in 1990. Since then, many articles have been written on the classification of sum graphs and the determination of the sum number of elements from several well-known graph classes. Since non-trivial sum graphs cannot be connected, Boland et al., also in 1990, introduced a generalization of sum graphs, the so-called mod sum graphs. As for the other concept, several authors investigated the mod sum number of elements from well-known graph classes. In this article, we take a different route. For some arithmetically interesting sets, for example, arithmetic progressions and prime numbers, we characterize the induced sum graphs and mod sum graphs in most cases. Furthermore, we prove density results and provide intriguing future research options.

1. Introduction

Given a finite set A of positive integers, the *sum graph* of A, denoted by $G(A)$, is the graph with vertex set A and edge set $\{ab : a, b \in A, a \neq b, a + b \in A\}$. For any positive integer n, the graph $G_n(A)$ with vertex set A and edge set

$$\{ab : a, b \in A, a \neq b, \text{ there exists some } c \in A \text{ with } a + b \equiv c \mod n\}$$

is called the *mod sum graph* of A modulo n.

Sum and mod sum graphs have received a lot of attention in the last three decades since their introduction in 1990 by Harary [4] and Boland et al. [2]. Usually, research is done by considering certain, often connected, graphs and determining whether they are sum graphs by giving an appropriate set A or how many vertices have to be added, which is the so-called *sum number*, until the resulting graphs are sum graphs. A good overview over known results is given in [3], hence we will only cite publications which are directly related to our findings. We take a different route, namely we start with arithmetic sets of some interesting structure and determine their corresponding sum and mod sum graphs. This point of view appears to be new, but, in our opinion, at least equally thrilling.

To the best of our knowledge, in all previous articles on mod sum graphs, there is a restriction on the size of n: the modulus was always chosen to be bigger than $\max(A)$. The reason is that if $n < \max(A)$, then some of the elements of A could coincide as vertices of $G_n(A)$. Unless mentioned otherwise, we will not identify these vertices but rather treat them as distinct. If we indeed have $n > \max(A)$, then the edge set of $G_n(A)$ can also be written in the form

$$\{ab : a, b \in A, a \neq b, a + b \in A \text{ or } a + b - n \in A\}.$$

In this article, we will assume that $n > \max(A)$ for some types of arithmetic sets, but for others we will not restrict n. The reason is that for certain arithmetic sets the graph $G_n(A)$ has interesting properties if we do not assume $n > \max(A)$. Note that if $n < \max(A)$ and if $a_1, a_2 \in A$ are integers with $a_1 \equiv a_2 \mod n$, then the neighborhoods of a_1 and a_2 coincide. This fact will be used frequently in this paper. Furthermore, call two sets V, W of vertices *(non-)adjacent* if any vertex of V is (non-)adjacent to any vertex of W.

Note that for $n \geq \max(A) + \max(A \setminus \{\max(A)\})$, we have $G(A) = G_n(A)$, hence sum graphs are special mod sum graphs. We will state this well-known and trivial fact as a proposition since we will use it in some of the forthcoming proofs. Note that in this article, we use $0 \notin \mathbb{N}$ and the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proposition 1. *Let A be a subset of $\mathbb{N}$ and let a_1, a_2 be the two largest elements in A. If $n \geq a_1 + a_2$, we have $G_n(A) = G(A)$.*

Proof. For any two vertices r, s, we have $r + s \leq a_1 + a_2 \leq n$. Hence, either $r + s$ will not be reduced modulo n or it will be reduced to $0 \notin A$. Thus, we do not get any additional edges in $G_n(A)$ compared to $G(A)$. □

The following graph classes and operations will occur in this paper. We denote by P_k the path graph and K_k the complete graph on k vertices. For two graphs G and H, the graph $G \cup H$ is the (disjoint) union of G and H. For a graph G, we denote the complement of G by $\overline{G}$. Furthermore, we will use the rather standard notations $[m, n] = \{i \in \mathbb{Z} : m \leq i \leq n\}$ and $[n] = [1, n]$. The degree of a vertex v in the graph G will be denoted by $d_G(v)$ and its neighborhood by $N_G(v)$. We might omit the index in unambiguous situations.

Apart from the results given in Section 7, we consider an arithmetically interesting infinite set A, that is, a set with some kind of arithmetic structure. Let A_k denote the set of the k smallest elements of A.[1] When considering $G_n(A_k)$, we can ask several questions:

- What structure does $G_n(A_k)$ have?
- Fix k and consider the sequence $(G_n(A_k))_n$. What can be said about this sequence, for example about the number of edges? If n is sufficiently large, we have $G_n(A_k) = G(A_k)$; if $n = 1$, the graph $G_n(A_k)$ is complete. For each n, we have $E(G(A_k)) \subset E(G_n(A_k))$.
- Fix n and consider the sequence $(G_n(A_k))_k$. What can be said about this sequence, for example about the proportion of edges compared with $\binom{|A_k|}{2} = \binom{k}{2}$?

We will answer some of these questions for certain sets A, leaving others for future research.

2. Arithmetic Progressions

Arithmetic progressions contain much additive structure. Since sum graphs are some sort of additive objects, it seems natural to start with these sequences.

2.1. Sum Graph of an Arithmetic Progression

In this section, we consider arithmetic progressions

$$A^{(a,d)} = \{x \in \mathbb{N} : x = a + jd \text{ for some } j \in \mathbb{N}_0\}.$$

As an example, the sum graph $G(A_{10}^{(6,2)})$ is displayed in Figure 1.

For two vertices $a + id, a + jd$ of $G(A_k^{(a,d)})$ to be adjacent, some $\ell \in \mathbb{N}$ with

$$(a + id) + (a + jd) = a + \ell d \quad \Longrightarrow \quad a = (\ell - (i + j))d$$

needs to exist. This is only possible if $d \mid a$. More precisely, $G(A_k^{(a,d)})$ has edges if and only if $d \mid a$ and $k \geq 2$. Hence, we obtain the following result.

Proposition 2. *Let a, d, k be positive integers with $d \nmid a$. Then we have $G(A_k^{(a,d)}) = \overline{K_k}$.*

[1] Note that usually the notation A_k denotes the set of elements of A that are smaller or equal to k. Since with our notation most results are much more appealing, we will use this instead.

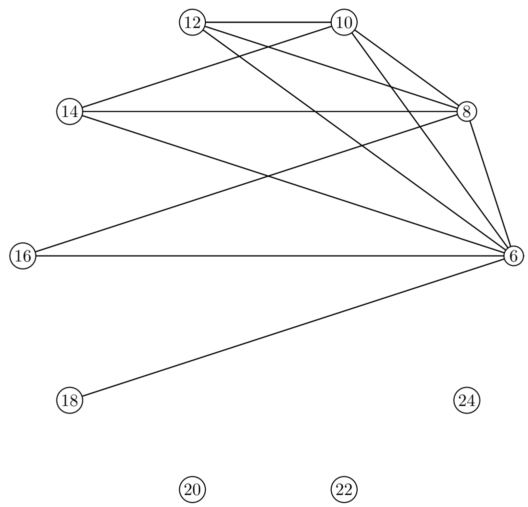

Figure 1: The sum graph $G(A_{10}^{(6,2)})$.

In view of the previous proposition, we consider only $d \mid a$ for the remainder of this subsection. It is well known that sum graphs which are induced by A and cA, where c is some positive integer, are isomorphic (see for example [9]). This fact implies

$$G(A_k^{(a,d)}) = G(A_k^{(a/d,1)}).$$

Hence, it suffices to investigate the structure of sum graphs induced by sets of consecutive positive integers. This has been done before in part in [9]. Using and slightly modifying arguments used in that paper, we obtain the following results, which are still phrased rather general.

Let $a = h \cdot d$, then the vertex a of $G(A_k^{(a,d)})$ has neighbors $a + jd$ for $1 \leq j \leq k - 1 - h$. In view of the nice additive structure of arithmetic progressions, the following terms in the sequence have, apart from a repetition in the middle for $a + id$ with $i \in \left\{\lfloor \frac{k-1}{2} \rfloor, \lceil \frac{k}{2} \rceil\right\}$ (since no loops are allowed), exactly one neighbor less than the previous one, that is,

$$N(a + id) = \left\{a + jd \in A^{(a,d)} : i \neq j \text{ and } 0 \leq j \leq k - 1 - h - i\right\}. \tag{1}$$

Using Equation (1) and the aforementioned [9] (Theorem 2.1), we obtain the following structural description of the considered sum graphs.

Proposition 3. *Let a, d, k be positive integers. If $d \mid a$, say $a = h \cdot d$, then $G(A_k^{(a,d)})$ is the unique graph with degree sequence*

$$0, 0, \ldots, 0, 1, 2, \ldots, \left\lceil \frac{t}{2} \right\rceil - 1, \left\lceil \frac{t}{2} \right\rceil, \left\lceil \frac{t}{2} \right\rceil, \left\lceil \frac{t}{2} \right\rceil + 1, \ldots, t - 1, t,$$

where $t = k - 1 - h$. *Moreover,* $G(A_k^{(a,d)})$ *has exactly* $\left\lfloor \frac{1}{4}(t+1)^2 \right\rfloor$ *edges.*

2.2. Mod Sum Graphs of an Arithmetic Progression

In this section, we examine mod sum graphs of arithmetic progressions, where we will assume that the modulus n fulfils $n > \max(A_k^{(a,d)})$. Similarly as before, the simple computation

$$(a + id) + (a + jd) = n + (a + \ell d) \quad \Longrightarrow \quad n = a + (i + j - \ell)d \tag{2}$$

helps characterize those moduli for which sum graph and mod sum graph coincide.

Proposition 4. *We have*

$$G_n(A_k^{(a,d)}) \neq G(A_k^{(a,d)})$$

if and only if $n = a + md$ *for some* $m \in [k, 2k-3]$ *and* $k \geq 2$.

Proof. If $n = a + md$ for some $m \in [k, 2k-3]$ and $k \geq 2$, then $a + (k-1)d$ and $a + (m-k+1)d$ are adjacent in $G_n(A_k^{(a,d)})$ since their sum equals a modulo n, but not in $G(A_k^{(a,d)})$.

If, on the other hand, we have $G_n(A_k^{(a,d)}) \neq G(A_k^{(a,d)})$, then i, j, ℓ fulfilling (2) exist. Note that $k \geq 2$ is certainly necessary. Since $i \neq j$ and $i, j, \ell \in [0, k-1]$ as well as $n \geq a + kd$ hold, the number $i + j - \ell$ has to lie between k and $2k - 3$. □

Combining this result with the previous ones, we obtain the following characterization.

Theorem 1. *Let* a, d, k, n *be positive integers and* $n > a + (k-1)d$. *If* $n > a + (2k-3)d$ *or* $n \notin A^{(a,d)}$, *then* $G_n(A_k^{(a,d)}) = G(A_k^{(a,d)})$. *Otherwise, we have*

$$G_n(A_k^{(a,d)}) \cong \begin{cases} \left([0, k-1], \left\{ij : i \neq j, i + j \geq \frac{n-a}{d}\right\}\right) & \text{if } d \nmid a, \\ \left([0, k-1], \left\{ij : i \neq j, i + j \geq \frac{n-a}{d} \text{ or } i + j \leq k-1\right\}\right) & \text{if } d \mid a. \end{cases}$$

To obtain the neighborhood of each vertex, let $n = a + (2(k-1) - r)d$ with $r \in [k-2]$. Denoting by N' the set of additional neighbors compared to the sum graph, a simple computation yields

$$\begin{aligned} N'(a + id) &= \{a + jd : 2(k-1) - r - i \leq j \leq k-1, i \neq j\} \\ &= \begin{cases} \emptyset & \text{if } i < k - 1 - r, \\ \{a + (k-\ell)d : \ell \in [i - k + r + 2] \setminus \{k - i\}\} & \text{if } i \geq k - 1 - r. \end{cases} \end{aligned}$$

Therefore, we get

$$|N'(a+id)| = \begin{cases} 0 & \text{if } i < k-1-r, \\ i-k+r+2 & \text{if } k-1-r \le i < k-1-\lfloor \frac{r}{2} \rfloor, \\ i-k+r+1 & \text{if } i \ge k-1-\lfloor \frac{r}{2} \rfloor. \end{cases}$$

Using the handshaking lemma, it is straightforward to compute the number of additional edges:

$$\frac{1}{4}(r^2+2r)+x,$$

where $x = \frac{1}{4}$ if r is odd and $x = 0$ otherwise. For example, if $k = 10, a = 6, d = 2, r = 3$ (and consequently $n = 36$), then we get the four additional edges, which are shown in Figure 2 in red.

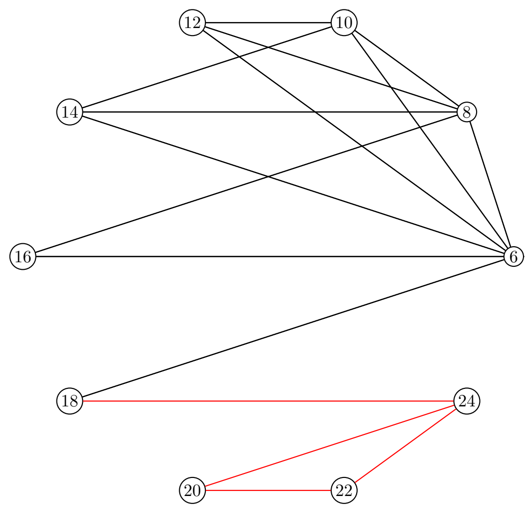

Figure 2: The mod sum graph $G_{36}(A_{10}^{(6,2)})$.

3. Second Order Linear Recurrences

Linear recurrences of order 2 are particularly interesting for (mod) sum graphs since they are basically defined via linear combinations of exactly two parts of some given set.

3.1. Fibonacci Numbers

By f_k we denote the k-th Fibonacci number starting with $f_1 = 1, f_2 = 2$. Although this is the sequence given in Fibonacci's original work, usually the starting values 1 and 1 or 0 and 1 are used. For our purposes the chosen definition is more convenient. We also define $F_k = \{f_i : 1 \leq i \leq k\}$.

Proposition 5. *For every $k \in \mathbb{N}$, we have $G(F_k) = P_{k-1} \cup K_1$.*

Proof. First of all, by definition of the Fibonacci numbers, f_i and f_{i+1} are adjacent in $G(F_k)$ for every $i \in [k-1]$. Since $f_i + f_j < f_{j+1}$ for $1 \leq i < j-1 \leq k-2$, there are no more edges. □

Let us turn our attention to mod sum graphs now. As an example, the mod sum graph $G_{178}(F_{11})$ is displayed Figure 3.

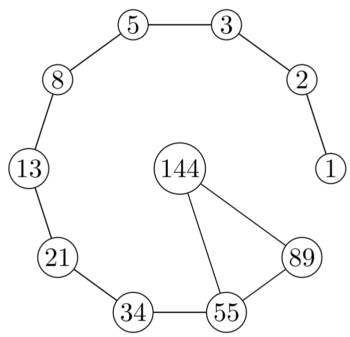

Figure 3: The mod sum graph $G_{178}(F_{11})$.

Again, we will assume $n > \max(F_k) = f_k$. The following results will be very helpful in determining the structure of these graphs.

Lemma 1. *Edges from $G_n(F_k)$ which do not belong to $G(F_k)$ are incident to f_k.*

Proof. Since $f_i + f_j < f_k$ for $i < j < k-1$, the sums $f_i + f_j$ are smaller than n and will therefore not be reduced. □

Lemma 2. *The graph $G_n(F_k)$ contains at most k edges.*

Proof. There are $k-2$ edges from $G(F_k)$ which also belong to $G_n(F_k)$. Let $i < j < k$ be positive integers. For f_i and f_j being adjacent to f_k, there need to exist positive integers $p < q$ such that

$$f_i + f_k = n + f_p \qquad \text{and} \qquad f_j + f_k = n + f_q$$

hold (note $p < i$ and $q < j$). These equations imply

$$f_i + f_q = f_j + f_p. \tag{3}$$

Because of $p < i < j$, we have $|j - p| \geq 2$. From Zeckendorf's theorem [11] we would get $\{i, q\} = \{j, p\}$ if $|i - q| \geq 2$. But this is not possible because of $p < i < j$ and

$q < j$. Hence, we have $|i - q| \leq 1$. If $|i - q| = 1$, then $f_j + f_p$ would be a Fibonacci number which cannot be true. Therefore, we get $i = q$. Now we obtain $p = i - 2$ and $j = i + 1$ as a solution of Equation (3). This solution is unique by Zeckendorf's theorem. Furthermore, we get $n = f_k + f_{i-1}$. Thus, there cannot be more than two additional edges in $G_n(F_k)$ compared to $G(F_k)$. □

Theorem 2. *Given $n, k \in \mathbb{N}$ with $n > f_k$, we have, denoting by E_k the edge set of $G(F_k)$,*

$$G_n(F_k) = \begin{cases} (F_k, E_k \cup \{f_{i+1}f_k, f_{i+2}f_k\}) & \text{if } n = f_k + f_i \text{ for some } i \in [k-3], \\ (F_k, E_k \cup \{f_{i+1}f_k\}) & \text{if } n = f_k + f_{i+1} - f_j \\ & \text{for some } i \in [k-2] \text{ and } j \in [i], \\ P_{k-1} \cup K_1 & \text{otherwise.} \end{cases}$$

Proof. If $n = f_k + f_i$ for some $i \in [k-3]$, then the statement follows from

$$f_{i+1} + f_k = f_{i-1} + f_i + f_k \equiv f_{i-1}, \qquad f_{i+2} + f_k = f_i + f_{i+1} + f_k \equiv f_{i+1}$$

and Lemma 2.

In case of $f_k + f_i < n < f_k + f_{i+1}$ for some $i \in [k-2]$, there can be at most one additional edge since two additional edges are only possible if n is the sum of two Fibonacci numbers (see proof of Lemma 2). Since $f_{i+1} + f_k \equiv f_j$ holds in the second case, f_{i+1} and f_k are adjacent.

For all other n, no additional edges exist because of the fact that $n = f_k + f_i - f_j$ is a necessary condition for f_i and f_k to be adjacent. □

The number of occurrences of the graph types given in Theorem 2 are $k - 3$, $\frac{(k-1)(k-2)}{2}$, and $f_{k-1} - 1 - (\frac{k^2}{2} - \frac{k}{2} - 2)$; note that $k - 3 + \frac{(k-1)(k-2)}{2} = \frac{k^2}{2} - \frac{k}{2} - 2$. Hence, the proportion of graphs $P_{k-1} \cup K_1$ among $G_n(F_k)$ with $n \in [f_k + 1, f_{k-1} + f_k - 1]$ grows to 1 for $k \to \infty$.

3.2. Other Starting Values

The recursion of the Fibonacci numbers has been considered for other starting values than 1 and 2 in the past. One example would be the well-known Lucas numbers starting with 1 and 3 as its first two terms, or, more commonly, with 2 and 1. The arguments in the previous subsection can be used to determine the sum and mod sum graphs of such generalizations[2] of the Fibonacci numbers. To make it more explicit, we start with positive integers $a < b$ and consider the sequence $(s_k)_{k \in \mathbb{N}}$ with $s_1 = a, s_2 = b$ and $s_{i+2} = s_i + s_{i+1}$ for $i \in \mathbb{N}$. Furthermore, let $S_k = \{s_i : 1 \leq i \leq k\}$.

Similarly as for the Fibonacci sequence, we can determine the structure of $G(S_k)$.

[2]Note that many, very different, generalizations of Fibonacci numbers exist.

Proposition 6. *For every $k \in \mathbb{N}$, we have $G(S_k) = P_{k-1} \cup K_1$.*

To generalize Theorem 2, we, as a matter of fact, only need a weak version of Zeckendorf's Theorem.

Lemma 3. *Given a positive integer n, there is at most one pair of indices $i < j$ such that $n = s_i + s_j$ holds.*

Proof. Suppose we had

$$n = s_i + s_j = s_h + s_\ell$$

for $i < j, h < \ell$, and $j \leq \ell$. Since, if $j < \ell$,

$$n = s_h + s_\ell = s_h + s_{\ell-1} + s_{\ell-2} \geq s_h + s_j + s_i = s_h + n$$

would yield a contradiction, we obtain $j = \ell$. This implies $s_i = s_h$ and hence $i = h$. □

Using this lemma instead of Zeckendorf's Theorem, we can easily adjust the proofs of Lemma 2 and Theorem 2 to obtain the following results.

Lemma 4. *The graph $G_n(S_k)$ contains at most k edges.*

Theorem 3. *Given $n, k \in \mathbb{N}$ with $n > s_k$, we have, denoting by E_k the edge set of $G(S_k)$,*

$$G_n(S_k) = \begin{cases} (S_k, E_k \cup \{s_{i+1}s_k, s_{i+2}s_k\}) & \text{if } n = s_k + s_i \text{ for some } i \in [k-3], \\ (S_k, E_k \cup \{s_{i+1}s_k\}) & \text{if } n = s_k + s_{i+1} - s_j \\ & \text{for some } i \in [k-2] \text{ and } j \in [i], \\ P_{k-1} \cup K_1 & \text{otherwise.} \end{cases}$$

4. Prime Numbers

In the next three sections, we will not necessarily assume $n > \max(A_k)$ for the set A_k we consider. Let $\mathbb{P}_k$ be the set of the first k primes $2 = p_1 < p_2 < \cdots < p_k$. Figure 4 shows the sum graph $G(\mathbb{P}_8)$.

Proposition 7. *The graph $G(\mathbb{P}_k)$ is a star with some additional isolated vertices. More precisely, the centre of the star is the vertex 2, the leafs are the primes $p \in \mathbb{P}_k$ such that $(p, p+2)$ is a twin prime with $p + 2 \leq p_k$, and all other vertices are isolated.*

Proof. Since every prime greater than 2 is odd, the sum of two such primes can never be a prime. Hence, two vertices can only be adjacent if one of them is the vertex 2. This gives the result. □

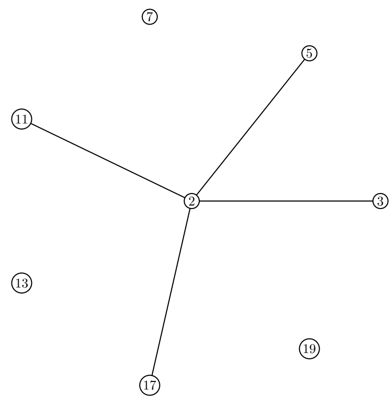

Figure 4: The sum graph $G(\mathbb{P}_8)$.

According to the preceding result, the number of edges of $G(\mathbb{P}_k)$ is determined by the number of twin primes up to p_k. Let $\pi_2(p_k)$ denote this number. Currently, no asymptotics for $\pi_2(x)$ are known, but the first Hardy-Littlewood conjecture implies $\pi_2(x) \sim 2C_2\frac{x}{(\ln x)^2}$ for some constant C_2. Since the prime number theorem yields $p_k \sim k \ln k$, the asymptotic number of edges of $G(\mathbb{P}_k)$, assuming the first Hardy-Littlewood conjecture, is $2C_2\frac{k\ln k}{(\ln(k\ln k))^2}$.

Let us now consider mod sum graphs. Figure 5 shows the mod sum graph $G_{19}(\mathbb{P}_8)$.

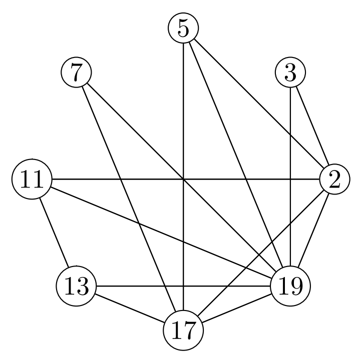

Figure 5: The mod sum graph $G_{19}(\mathbb{P}_8)$.

The next two results deal with prime moduli.

Proposition 8. *Let p be a prime. If k is sufficiently large, the graph $G_p(\mathbb{P}_k)$ is a complete graph.*

Proof. We need to show that given two primes $q_1, q_2 \le p_k$, there is a prime $q \le p_k$

with $q_1 + q_2 \equiv q \mod p$. This is equivalent to $q = q_1 + q_2 + jp$ for some $j \in \mathbb{Z}$. We consider two cases.

- Suppose that $q_1 + q_2$ is a multiple of p. Then we can choose $q = p$ as long as $p_k \geq p$.
- Suppose that $q_1 + q_2$ is not a multiple of p. Since p is a prime, $q_1 + q_2$ and p are coprime and Dirichlets theorem on primes in arithmetic progressions yields the existence of such a prime q if j is sufficiently large, that is, if k is sufficiently large. Note that choosing k sufficiently large for given q_1, q_2 may yield more primes in $\mathbb{P}_k$ that have to be considered. But since the magnitude of k depends only on the residue class of $q_1 + q_2$ modulo p, we can choose k such that the above argument is valid for all $q_1, q_2 \leq p_k$.

□

Combining these two results about sum graphs and mod sum graphs for prime moduli, we have the following situation.

Let p be a fixed prime and consider the sequence $(G_p(\mathbb{P}_k))_k$. For large k, we have shown that the mod sum graph $G_p(\mathbb{P}_k)$ is a complete graph. For smaller k, this structural property does not need to hold. We might ask the following questions.

- Can anything be said about the smallest k_0 such that $G_p(\mathbb{P}_k)$ is a complete graph for all $k \geq k_0$?
- What can be said about the proportion of edges of the graphs $G_p(\mathbb{P}_k)$ with $1 \leq k < k_0$, compared with the number $\binom{k}{2}$ of possible edges?

The following proposition, which immediately follows from Propositions 1 and 7, gives a partial answer to the first question.

Proposition 9. *Let p be a prime and let $k \geq 2$. If $p \geq p_k + p_{k-1}$, then $G_p(\mathbb{P}_k)$ is not a complete graph.*

Certainly we could also fix k and consider the sequence $(G_n(\mathbb{P}_k))_n$ for arbitrary n. Since $G_1(\mathbb{P}_k)$ is trivially a complete graph and $G_n(\mathbb{P}_k) = G(\mathbb{P}_k)$ holds for sufficiently large n, we could consider the sequence of the number of edges of $G_n(\mathbb{P}_k)$. Although this number decreases in the long run, the sequence need not be monotonic (see Figure 6).

If n is not a prime, the following result shows that $G_n(\mathbb{P}_k)$ is not a complete graph (see Figure 7 for the graph $G_{10}(\mathbb{P}_8)$).

Proposition 10. *If n is composite, the graph $G_n(\mathbb{P}_k)$ is complete if and only if $k = 1$.*

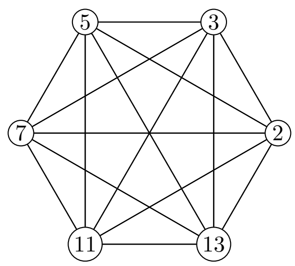

Figure 6: The mod sum graphs $G_3(\mathbb{P}_6), G_5(\mathbb{P}_6), G_7(\mathbb{P}_6)$.

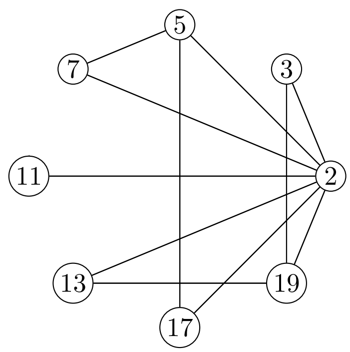

Figure 7: The mod sum graph $G_{10}(\mathbb{P}_8)$.

Proof. If $k = 1$, the graph is trivially complete for all n. If $k = 2$, the graph is complete if and only $5 \equiv 2, 3 \mod n$. This is possible if and only if n is 1, 2 or 3, but these are not composite integers. Hence, let $k \geq 3$ from now on.

If n is odd, we have $n \geq 9$. Let p_* be the smallest prime divisor of n. We choose some $q \in \{3, 5\} \setminus \{p_*\}$; note that $q \in \mathbb{P}_k$ holds. First, let us consider the case $p_* \leq p_k$ (then $p_* \in \mathbb{P}_k$). Then we have $p_* + q < n$, since $p_* \leq \sqrt{n}$, and $2 \mid (p_* + q)$. This implies $p_* + q \notin \mathbb{P}$ and, moreover, that p_* and q are not adjacent in $G_n(\mathbb{P}_k)$. If $p_* > p_k$, we have $n \geq p_*^2 > 2p_* > 2p_k \geq p_k + p_{k-1}$, and, by Propositions 1 and 7, $G_n(\mathbb{P}_k)$ is not complete.

For even n, we distinguish three cases. If n is some power of 2, then 3 and 5 cannot be adjacent. If $n = 6$, it can easily be verified that $G_6(\mathbb{P}_3)$ is not complete (2 and 5 are not adjacent). For $k \geq 4$, the vertices 3 and 7 are not adjacent, since $3 + 7 \equiv 4 \mod 6$ and $4 \notin \mathbb{P}$. If n is neither a power of 2 nor equal to 6, we have $n \geq 10$. Let p_* be the smallest odd prime divisor of n. Now we can argue as in the case of odd n. □

5. Powers of a Fixed Integer

In this section, we consider the sum graph and mod sum graph for the set consisting of powers of a fixed integer. Let $a \in \mathbb{N}$ with $a \geq 2, POW(a) := \{a^m : m \in \mathbb{N}_0\}$ and let $POW(a)_k$ denote the first k elements of $POW(a)$.

Proposition 11. *The graph $G(POW(a)_k)$ has only isolated vertices.*

Proof. Two vertices $v = a^m, w = a^s$ are adjacent if and only if $a^m + a^s = a^r$ for some r. We can assume $m < s$, yielding $a^m(1 + a^{s-m}) = a^r$. Since the right-hand side is a power of a and the left-hand side is not, no two vertices are adjacent. □

Let us now consider the mod sum graph. Figure 8 shows some mod sum graphs covering some of the cases of Theorem 4.

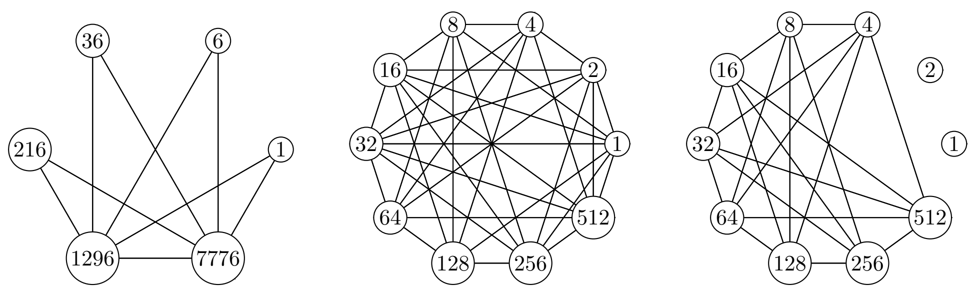

Figure 8: The mod sum graphs $G_{16}(POW(6)_6)$ (left), $G_5(POW(2)_{10})$ (middle), and $G_{20}(POW(2)_{10})$ (right).

Theorem 4. *Let $a \geq 2$ and $a = \prod_{i=1}^{r} p_i^{e_i}$ with distinct primes p_i. If k is sufficiently large (such that all residue classes V_i are non-empty), the mod sum graph $G_n(POW(a)_k)$ has the following structure.*

1. *If $n = \prod_{i=1}^{r} p_i^{f_i}$ with $f_i \in \mathbb{N}_0$, two vertices v, w are adjacent if and only if $n|v$ or $n|w$.*

2. *If $(a, n) = 1$, we can partition the vertex set of $G_n(POW(a)_k)$ into $\mathrm{ord}_n(a)$*[3] *sets V_i such that:*

   - *$|V_i| - |V_j| \in \{-1, 0, 1\}$ for all i, j;*
   - *for each $i \neq j$, either V_i is adjacent to V_j or these two sets are non-adjacent;*
   - *V_i is complete if 2 is a power of a modulo n. If this is not the case, V_i is an independent set.*

[3]Here $\mathrm{ord}_n(a)$ denotes the multiplicative order of a modulo n, that is, $\mathrm{ord}_n(a) := \min\{d \in \mathrm{N} : a^d \equiv 1 \mod n\}$.

3. *If $n = b \cdot c$ with $(a, c) = 1$ and $b = \prod_{i=1}^{r} p_i^{f_i}$ with $f_i \in \mathbb{N}_0$ we can partition the vertex set of $G_n(POW(a)_k)$ into $\max_{1\leq i\leq r}\{\lceil \frac{f_i}{e_i} \rceil\} + \mathrm{ord}_c(a)$ sets V_i such that:*
   - *exactly $\max_{1\leq i\leq r}\{\lceil \frac{f_i}{e_i} \rceil\}$ of the sets have exactly one element. For all other sets V_i, V_j, we have $|V_i| - |V_j| \in \{-1, 0, 1\}$;*
   - *for each $i \neq j$, either V_i is adjacent to V_j or these two sets are non-adjacent;*
   - *for each set V_i consisting of at least two elements we have the following: V_i is complete if 2 is a power of a modulo c. If this is not the case, V_i is an independent set.*

Proof. Let A be the submonoid of $(\mathbb{Z}/n\mathbb{Z}, \cdot)$ generated by a. Partition the set $POW(a)_k$ into $|A|$ subsets $V_1, \ldots, V_{|A|}$ according to the residue of a^m modulo n. This partition, distinguished by the three cases, has the following structural properties.

1. If $n = \prod_{i=1}^{r} p_i^{f_i}$ with $f_i \in \mathbb{N}_0$, we have $a^m \equiv 0 \mod n$ if and only if $me_i \geq f_i$ for all i, hence $|A| = 1 + \max_{1\leq i\leq r}\{\lceil \frac{f_i}{e_i} \rceil\}$. If we had $a^m \equiv a^{\ell+m} \not\equiv 0 \mod n$, we would also get $a^m \equiv a^{j\ell+m}$ for all j, in contradiction to the observation above. Hence, the sets V_i corresponding to the non-zero residue classes have exactly one element. Every other element lies in the set V_i corresponding to the residue class 0.

2. If $(a, n) = 1$, we have $|A| = \mathrm{ord}_n(a)$ and each set V_i has either $\lfloor \frac{k}{|A|} \rfloor$ or $\lfloor \frac{k}{|A|} \rfloor + 1$ elements.

3. Let $n = \prod_{i=1}^{r} p_i^{f_i} \prod_{j=1}^{m} q_j^{g_j}$ with $q_j \neq p_i$. Combining the previous two cases, using the Chinese remainder theorem, yields the existence of sets V_i with exactly one element, namely those corresponding to the powers a^m where $m < \max_{1\leq i\leq r}\{\lceil \frac{f_i}{e_i} \rceil\}$. The number of elements in every other set V_i is equally distributed as in the second case.

Now, we take a look at the adjacencies in the respective cases.

1. If $n = \prod_{i=1}^{r} p_i^{f_i}$ with $f_i \in \mathbb{N}_0$, every vertex in the set corresponding to the residue class 0 is adjacent to every other vertex. Any other two vertices are not adjacent: If $v = a^m, w = a^\ell$ with $\ell < m$ then $v + w = a^\ell(1 + a^{m-\ell})$. This is not 0 modulo n and it is a power of a modulo n if and only if $1 + a^{m-\ell}$ is a power of a modulo n. But this would imply $1 \equiv a^s - a^{m-\ell} \mod n$ for some s, hence a would be invertible modulo n. Since this is not the case, v and w are not adjacent.

2. We still have to prove the last statement. Each set V_i either induces a complete subgraph or an independent set of $G_n(POW(a)_k)$: Let $v, w \in V_i$. Then we

have $v + w \equiv 2a^\ell$ for some ℓ. Thus, v and w are adjacent if and only if $2 \equiv a^r \mod n$ for some $r \in \mathbb{N}$.

3. Again, we only have to verify the last statement. Let $v, w \in V_i$. Then $v + w \equiv 2a^\ell$ holds for some ℓ. Using the Chinese remainder theorem, we have to solve the system $2a^\ell \equiv a^r \mod b, 2a^\ell \equiv a^r \mod c$ for some $r \in \mathbb{N}$. Note that since V_i has at least two elements, we have $a^\ell \equiv 0 \mod b$, so the first congruence is solvable. The second congruence can be dealt with as in the previous case.

□

6. Squares

In this section, we will allow the vertex 0.[4] When considering sum graphs of the squares, it is straightforward to traverse between the sum graph with vertex 0 and the one without vertex 0. When considering mod sum graphs, we will easily get results about the sum graph not including 0 from the results about the sum graph including 0.

Let $Q_{0,k} := \{n^2 : n \in \{0, \dots, k\}\}$ (note $|Q_{0,k}| = k + 1$), $Q_k := Q_{0,k} \setminus \{0\}$ and let q_i be the i-th element of $Q_{0,k}$. Figure 9 shows the graph $G(Q_{0,17})$.

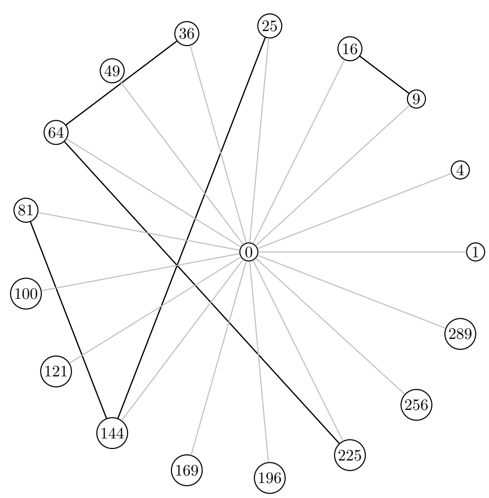

Figure 9: The sum graph $G(Q_{0,17})$. Edges incident to the vertex 0 are drawn in grey for better visibility.

[4]This is usually not done when investigating sum graphs. The research branch that allows this label, as well as negative integers, deals with so-called integral sum graphs.

Proposition 12. *Every vertex in $G(Q_{0,k})$ is adjacent to the vertex 0. Two non-zero vertices u, v are adjacent if and only if there is a Pythagorean triple (u, v, w) with $w \leq k$.*

Proof. Certainly every vertex $v \neq 0$ is adjacent to 0. Since two non-zero vertices u, v are adjacent if and only if $u^2 + v^2 = w^2$ for some $w^2 \in Q_{0,k}$, we get the result. □

Proposition 12 leaves a few questions about the structure of $G(Q_{0,k})$ unanswered, in particular concerning the structure of the graph $G(Q_k) = G(Q_{0,k}) - \{0\}$. Note that $G(Q_k) = G(Q_{0,k} \setminus \{0\})$ holds; this gives the "standard" sum graph. Since for each a there are only finitely many Pythagorean triples with leg a, we have indeed a finite set of edges. The following result is concerned with the number of total edges and the maximum degree.

Theorem 5.

1. *The number of edges in $G(Q_k)$ is*

$$\frac{1}{\pi} k \log k + Bk + O(k^{\frac{1}{2}} \exp(-c(\log k)^{\frac{3}{5}} (\log \log k)^{-\frac{1}{5}}))$$

   for some constant B and some $c > 0$.

2. *The maximum degree $\Delta(G(Q_k))$ of $G(Q_k)$ is at least $\left\lceil \frac{\lfloor \log_2 k \rfloor - 1}{2} \right\rceil$.*

Proof.

1. The number of edges in $G(Q_k)$ is equal to the number of Pythagorean triples (a, b, c) with $c \leq k$. Using the estimate of [8] gives the result.

2. Fix $N \in \mathbb{N}$ and let $x, y \in \mathbb{N}_0$ with $N = x + y, x > y$. Further, let $a = 2^{2x} - 2^{2y}, b = 2 \cdot 2^x \cdot 2^y$, and $c = 2^{2x} + 2^{2y}$. Note that $a^2 + b^2 = c^2$ and $b = 2 \cdot 2^N$. Hence, the number of appearances of 2^{N+1} as a leg of a Pythagorean triple is at least the number of different partitions $N = x + y$ with $x > y$, which is $\lceil \frac{N}{2} \rceil$. Since each of these Pythagorean triples yields an edge incident to the vertex with label 2^{2N+2}, we get

$$\Delta(G(Q_k)) \geq d(2^{2\lfloor \log_2 k \rfloor}) \geq \left\lceil \frac{\lfloor \log_2 k \rfloor - 1}{2} \right\rceil.$$

□

Remark 1.

- The result about the number of edges in $G(Q_k)$ can be strengthened to $\frac{1}{\pi} k \log k + Bk + O(k^{\frac{53}{116} + \varepsilon})$ if we assume the Riemann hypothesis, compare [5].

- The exact determination of the maximum degree appears to be difficult. One possible route of success might be the following. Results on the number of occurrences of an integer n as a leg of a Pythagorean triangle exist (see [10]). Using these results, it seems possible to determine an integer $r \leq k$ having the most occurrences as a leg of a Pythagorean triangle in the following way: Let n_1 be the largest primorial less than or equal to k and define $k_1 := \frac{k}{n_1}$. Now let n_2 be the largest primorial less than or equal to k_1. Proceed inductively until there is no primorial less than or equal to k_i. We conjecture that r is given by the product of all these primorials. The problem when considering edges in the graph $G^*(Q_k)$ is to keep control of the length of the hypotenuse: The above mentioned results include all Pythagorean triples with a as a leg, but we may only count those which have hypotenuse at most k.

- When looking at graphs $G(Q_{0,k})$ for "small" k, one might suspect that $G(Q_{0,k})$ is always a forest. This is not the case: If $k \geq 267$, the graph $G(Q_{0,k})$ contains triangles. To find a triangle in $G(Q_{0,k})$ we need a, b, c such that $a^2 + b^2, b^2 + c^2, c^2 + a^2$ are squares. This is related to Euler bricks: An *Euler brick* is a cuboid with integer edge and face diagonal lengths. The smallest Euler brick was discovered by Halcke in 1719 and has side lengths $44, 117, 240$ and face diagonal lengths $125, 244, 267$. There are infinitely many Euler bricks, and Saunderson [6] found a parametric solution that gives infinitely many, but not all, Euler bricks. Until today it is unknown whether higher-dimensional Euler bricks exist. Such bricks would yield cycles in $G(Q_{0,k})$.

Now we turn our attention to the mod sum graph $G_n(Q_{0,k})$. We deal mainly with prime moduli and consider only mod sum graphs where $n \leq k^2$ and k is large enough to ensure that $Q_{0,k}$ contains at least one square from each residue class of squares. Figure 10 shows some mod sum graphs for $Q_{0,k}$.

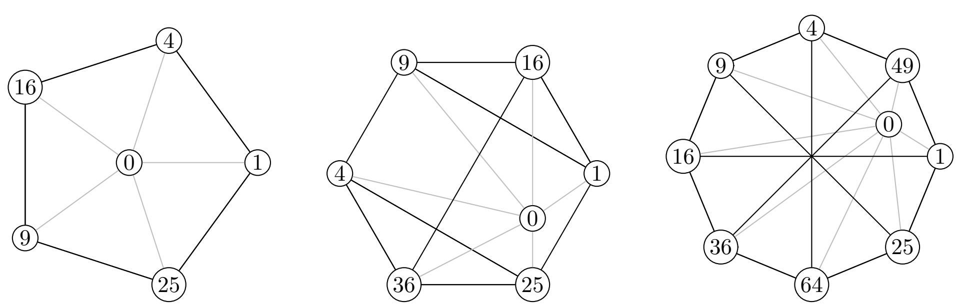

Figure 10: The mod sum graphs $G_{11}(Q_{0,k})$ (for $k \geq 5$), $G_{13}(Q_{0,k})$ (for $k \geq 6$), and $G_{17}(Q_{0,k})$ (for $k \geq 8$). For reasons of simplicity in the presentation, each residue class set V_i is only shown as a single vertex.

Proposition 13. *Let p be an odd prime and let $k \geq \frac{p-1}{2}$. Then the graph $G_p(Q_{0,k})$ has the following structure.*

- *The vertex set V can be partitioned into $\frac{p+1}{2}$ subsets $V_0, V_i, i = 1, \ldots, \frac{p-1}{2}$ corresponding to the residue 0 and the $\frac{p-1}{2}$ quadratic residues modulo p.*
- *The vertex set V_0 is a clique.*
- *If $p \equiv \pm 1 \mod 8$, each vertex set V_i is a clique. Otherwise, the sets V_i are independent sets.*
- *The set V_0 is adjacent to every set V_i.*
- *Each set V_i is adjacent to exactly $n_0(p)$ sets V_0, V_j, with $j \neq i$, and non-adjacent to every other, where*

$$n_0(p) = \begin{cases} \frac{p-1}{4}, & \text{if } p \equiv 1 \mod 8, \\ \frac{p+1}{4}, & \text{if } p \equiv 3 \mod 8, \\ \frac{p+3}{4}, & \text{if } p \equiv 5 \mod 8, \\ \frac{p-3}{4}, & \text{if } p \equiv 7 \mod 8. \end{cases}$$

Proof. Partition the set of squares into sets V_0, V_i according to the respective residue modulo p. Since $k \geq \frac{p-1}{2}$, the number of these sets is the number of squares modulo p, which is $\frac{p+1}{2}$. It is clear that V_0 is a clique and that V_0 is adjacent to every set V_i. Let $v, w \in V_i$ with $v \equiv w \equiv m \mod p$ for some positive integer m. Then V_i is a clique if and only if $v + w$ is a square modulo p and V_i is an independent set in the other case. Since m is a square modulo p, $v + w \equiv 2m$ is a square modulo p if and only if 2 is a quadratic residue modulo p, hence if and only if $p \equiv \pm 1 \mod 8$.

It remains to compute the number $n_0(p)$. According to [1], the number $\tilde{n}(p)$ of squares x^2 modulo p such that $x^2 + c$ is again a square modulo p is given by

$$\tilde{n}(p) = \begin{cases} \frac{p+3}{4}, & \text{if } p \equiv 1 \mod 4, \\ \frac{p+1}{4}, & \text{if } p \equiv 3 \mod 4 \end{cases}$$

if c is a square modulo p. If $p \equiv \pm 1 \mod 8$, we have to exclude the residue $2c$. From this, we easily obtain the values for $n_0(p)$ given in the statement. □

As an almost immediate corollary, we can deduce the structure of the graph $G_p(Q_k) = G_p(Q_{0,k} \setminus \{0\})$.

Corollary 1. *Let p be an odd prime and let $k \geq \frac{p-1}{2}$. Then the graph $G_p(Q_k)$ has the following structure.*

- *The vertex set V can be partitioned into $\frac{p-1}{2}$ subsets $V_i, i = 1, \ldots, \frac{p-1}{2}$ corresponding to the $\frac{p-1}{2}$ quadratic residues modulo p.*

- *If* $p \equiv \pm 1 \mod 8$, *each vertex set* V_i *is a clique. Otherwise, the sets* V_i *are independent sets.*
- *Each set* V_i *is adjacent to exactly* $n(p)$ *sets* $V_j (j \neq i)$ *and non-adjacent to every other, where*

$$n(p) = \begin{cases} \frac{p-9}{4}, & \textit{if } p \equiv 1 \mod 8, \\ \frac{p-3}{4}, & \textit{if } p \equiv 3 \mod 8, \\ \frac{p-5}{4}, & \textit{if } p \equiv 5 \mod 8, \\ \frac{p-7}{4}, & \textit{if } p \equiv 7 \mod 8. \end{cases}$$

Proof. Using Proposition 13, we only need to compute the number $n(p)$. This can be done by examining which sets V_j, including the case $j = 0$, are adjacent to V_i in $G_p(Q_{0,k})$ but not in $G_p(Q_k)$. This is the case if and only if one of the following two situations occurs.

- $j = 0$. Since each V_i is adjacent to V_0 in $G_p(Q_{0,k})$, this gives exactly one set V_j (namely the set V_0) that is adjacent to V_i in $G_p(Q_{0,k})$ but not in $G_p(Q_k)$.
- $j \neq 0$. Then V_i and V_j are adjacent in $G_p(Q_{0,k})$ but not in $G_p(Q_k)$ if and only if for $v \in V_i, w \in V_j$ we have $v + w \equiv 0 \mod p$. This happens exactly if -1 is a quadratic residue modulo p, that is, if and only if $p \equiv 1 \mod 4$.

Thus, we get

$$n(p) = \begin{cases} n_0(p) - 2, & \text{if } p \equiv 1 \mod 4, \\ n_0(p) - 1, & \text{if } p \equiv 3 \mod 4. \end{cases}$$

Together with the formula for $n_0(p)$ in Proposition 13 this yields the desired statement. □

Remark 2.

- The graph $G_2(Q_{0,k})$ is a complete graph.
- If n is composite, some of the above results still hold. We can still partition the vertex set into suitable subsets[5] V_0, V_i such that V_0 is a clique and is adjacent to every other set V_i. The sets V_i are cliques if and only if 2 is a square modulo n, that is, if and only if n can be written as $n = 2^\varepsilon \prod p_i^{e_i}$ where $\varepsilon \in \{0, 1\}$ and each odd prime divisor p_i of n satisfies $p_i \equiv \pm 1 \mod 8$. For composite moduli, it seems harder to determine the number of adjacent sets, though. Moreover, the graphs are, in general, less symmetric, see Figure 11.

[5]The number of these subsets is equal to the number of squares modulo n which can be computed with the formulas in [7].

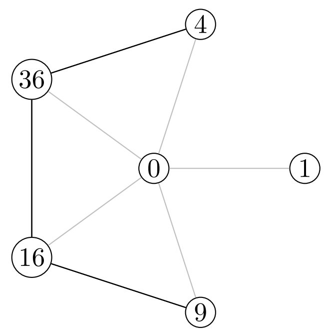

Figure 11: The mod sum graph $G_{24}(Q_{0,k})$ for $k \geq 6$. For reasons of simplicity in the presentation, each residue class set V_i is only shown as a single vertex.

- The graph $G_n(Q_{0,k})$ shares some connections with *quadratic residue Cayley graphs* QRC_n. These are graphs whose vertex sets are the residues modulo n and two vertices a, b are adjacent whenever $a - b$ is a quadratic residue modulo n. If we identify two vertices in the graph $G_n(Q_{0,k})$ with the same residues modulo n, which makes sense since properties depend only on the residue modulo n, then the vertex set of $G_n(Q_{0,k})$ is a subset of the vertex set of QRC_n. If further -1 is a quadratic residue modulo n, then the graph $G_n(Q_{0,k})$ is a subgraph of QRC_n and one could try to deduce properties of one of these graphs via the other.

Remark 3. For $\ell > 2$, let $Q^\ell_{0,k} := \{n^\ell : n \in \{0, \dots, k\}\}$. According to Fermat's Last Theorem, there are no non-trivial solutions to the equation $a^\ell + b^\ell = c^\ell$, hence $G(Q^\ell_{0,k})$ is a star with centre 0.

To consider the mod sum graph $G_n(Q^\ell_{0,k})$ one would need to examine n-th power residues. We will leave this problem as future research.

7. Density Results

Extremal results for sum graphs, such as the maximum number of edges, have been investigated in [9]. It is also well known that mod sum graphs on more than one vertex cannot be complete [2]. Note that this holds true when $n > \max(A)$. As we have already seen in Proposition 8, this is not necessarily true when $n < \max(A)$. Nevertheless, it is easy to prove that $G_{k+1}([k]) = K_k \setminus M$, where M is a maximum matching of K_k (see for example [2, Observation 1]). Hence, mod sum graphs can have many more edges than sum graphs over the same set A.

We believe that the examination of connections between the natural density[6]

[6]Using the notation $A(m) = \{a \in A : a \leq m\}$, the natural density α of A is defined via

of A and certain graph theoretical properties of $G(A_k)$ or $G_n(A_k)$ is an intriguing route of research. Since, to the best of our knowledge, nothing has been done before in this area, we begin with rather basic results.

Let A be a set of positive integers of natural density α. We denote by a_1 the smallest element of A, by a_2 the second smallest and so on. Hence, we have $A_k = \{a_i : i \in [k]\}$ and $|A_k| \approx \alpha \cdot \max(A_k)$ (for large k).

For the remainder of this section, we consider the number of edges in $G(A_k)$. Since the sum of two odd numbers is even, the tipping point for the natural density of A is $\frac{1}{2}$, as the next two results show.

Proposition 14. *Let A be a set of positive integers of natural density α. The number of edges of $G(A_k)$ lies between 0 and $\lfloor \frac{1}{4}(k-1)^2 \rfloor$. Moreover, for $\alpha = \frac{1}{2}$ both extremal values occur.*

Proof. The first statement immediately follows from [9, Theorem 2.1]. To prove the second statement consider the set of odd positive numbers (minimum number of edges) and the set of even positive numbers (maximum number of edges). □

Proposition 15. *Let A be a set of positive integers of natural density $\alpha > \frac{1}{2}$. Then, for sufficiently large k, the graph $G(A_k)$ contains at least one edge.*

Proof. Denote by a the smallest element of A and consider the boxes $B_h = \{h \cdot a + j \in A : j \in [0, a-1]\}$. If we find two consecutive boxes B_h, B_{h+1} containing together at least $a+1$ elements, then, by the pigeonhole principle, some $i \in [0, a-1]$ with $h \cdot a + i, (h+1) \cdot a + i \in A$ exists. Let k_0 be the integer such that $a_{k_0} = (h+1) \cdot a + i$, then, for every $k \geq k_0$, the graph $G(A_k)$ contains the edge $\{a, h \cdot a + i\}$.

It suffices to prove $A(n) \geq \frac{n}{2} + a$ for sufficiently large n since then, again by the pigeonhole principle, two such consecutive boxes exist. To this end, let $\alpha = \frac{1}{2} + \beta$ for some $\beta > 0$, n_0 such that $\frac{a}{n} < \beta$ for all $n \geq n_0$, and $\varepsilon = \beta - \frac{a}{n_0}$. Then

$$\left| \frac{A(n) - \frac{n}{2} - a}{n} - \left(\beta - \frac{a}{n}\right) \right| = \left| \frac{A(n)}{n} - \alpha \right| \underbrace{<}_{n \geq n(\varepsilon)} \varepsilon$$

implies $A(n) \geq \frac{n}{2} + a$ for $n \geq \max\{n_0, n(\varepsilon)\}$. □

8. Open Questions

Since this article gives a new perspective on sum graphs and mod sum graphs, several options for future research arise. Some of the many open questions, which could be examined, were already mentioned in the previous sections. More will be given in this final part of the paper.

$\frac{A(m)}{m} \to \alpha$ as $m \to \infty$, if this limit exists.

Sum graphs and mod sum graphs of other arithmetically interesting sets and sequences, such as the set of numbers that can be written as a sum of two squares, the set of the values of the Euler totient function, or geometric progressions, should be investigated. In particular, higher order linear recurrences, in view of the results in Sections 2 and 3, might be of interest. Furthermore, for higher powers, which we briefly mentioned in Section 6, the induced mod sum graph should be determined.

For these other sets, as well as the sets considered here, one could try to determine more properties (such as connectedness and diameter) and characteristic numbers (such as clique number and chromatic number) of the respective sum graph and mod sum graph. It would also be interesting to get the asymptotic number of edges where we cannot determine them exactly, for example in $G_n(\mathbb{P}_k)$.

In Sections 4, 5, and 6, the mod sum graph has been considered without the restriction $n > \max(A_k)$. One could try to consider the mod sum graph of these sets when the modulus is restricted.

In some of the obtained results, the graphs $G_n(A_k)$ were explicitly given for sufficiently large k. One could examine the graphs $G_n(A_k)$ where k is not large enough for the respective statements to be true.

In the last section, we got just a glimpse of possible results about (mod) sum graphs given by sets of certain density. It would be most desirable to obtain stronger structural statements for $G(A_k)$ depending on the density of A.

Other statistical questions of interest might be the following.

- How large can the difference of edges of the sum graph and a mod sum graph be at most?
- What is the mean number of edges over all mod sum graphs, with bounded n, for given A?

As a generalization of sum graphs and mod sum graphs, we can consider the following setting: Let $(M, \circ)$ be a *commutative magma*, that is, a non-empty set M equipped with a commutative binary operation. Let $A \subset M$ be a non-empty finite subset. Then we can define a graph $G = (V, E)$ as follows: the set of vertices is A and two vertices v, w are adjacent if and only if $v \circ w \in A$. If $(M, \circ) = (\mathbb{N}, +)$, this gives exactly the sum graph of A; if $(M, \circ) = (\mathbb{Z}/n\mathbb{Z}, +)$ for some n, this gives the mod sum graph $G_n(A)$. This type of graph can be examined, either in general or for other interesting commutative magmas, such as $(\mathbb{N}, \cdot)$, also known as product graph, or $(\mathbb{Z}/n\mathbb{Z}, \cdot)$, also known as mod product graph.

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