POLYGONAL BALANCING NUMBERS I

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Abstract

Balancing numbers are certain integers defined relative to a Diophantine equation between triangular numbers. Many properties, relations and generalizations of balancing numbers have appeared in the literature. We present, in two parts, the basic theory of polygonal balancing numbers. In this part, we define s-agonal-balancing, s-agonal Lucas-balancing and s-agonal-counterbalancing numbers in terms of three conditions. The first condition requires s-agonal-balancing numbers and s-agonal Lucas-balancing numbers to be among the solutions of a Diophantine Pell equation for $s \geq 3$. Groups of transformations that act on solutions of the s-agonal-balancing Diophantine equation are defined, and the orbits of solutions under the group actions are determined. Moreover, connections among the solutions of the family of s-agonal-balancing numbers for all values of s will be demonstrated. These connections allow us to delineate all solutions of the s-agonal-balancing Diophantine equations.

1. Introduction

Behera and Panda [2], essentially, defined a balancing number as an integer B such that

$$1 + 2 + \cdots + (B - 1) = (B + 1) + (B + 2) + \cdots + M,$$

for some M, which we call a counterbalancing number. For example, since $1+\cdots+5 = 7+8$, we see that $(B, M) = (6, 8)$ is a balancing, counterbalancing number pair. This is equivalent to the Diophantine equation involving triangular numbers given by

$$\frac{(B-1)B}{2} + \frac{B(B+1)}{2} = \frac{M(M+1)}{2}, \text{ or } B^2 = \frac{M(M+1)}{2}. \quad (1)$$

Solving Equation (1) for the positive counterbalancing number M, we have that

$$M = \frac{-1+\sqrt{8B^2+1}}{2}. \quad (2)$$

Thus, $8B^2+1$ must be an odd, perfect square for B to be a balancing number. Panda and Behera also found that a particular function and its inverse,

$$f(x) = 3x + \sqrt{8x^2+1} \text{ and } f^{-1}(x) = 3x - \sqrt{8x^2+1},$$

provide the next and previous balancing numbers from a given one:

$$B_{k+1} = f(B_k) = 3B_k + \sqrt{8B_k^2+1} \text{ and } B_{k-1} = f^{-1}(B_k) = 3B_k - \sqrt{8B_k^2+1}. \quad (3)$$

From Equations (3), they deduced the fundamental recurrence relation

$$B_{k+1} = 6B_k - B_{k-1}, \text{ with } B_0 = 0 \text{ and } B_1 = 1.$$

Panda [9] introduced the Lucas-balancing numbers, defined by

$$C_k = \sqrt{8B_k^2+1}, \text{ with } C_0 = 1 \text{ and } C_1 = 3, \quad (4)$$

as an analog of the Lucas numbers relative to the Fibonacci numbers. From Equation (2), we see that $C_k = 2M_k+1$. Thus, rather than basing an analysis of balancing numbers on B and C we may use M rather than C if it is to our advantage.

Using Definition 4 and that of the function $f(x)$, we see that

$$B_{k+1} = 3B_k + C_k. \quad (5)$$

Although he does not explicitly refer to the Lucas-balancing numbers, Liptai [6] showed that the balancing numbers and Lucas-balancing numbers satisfy the Pell equation

$$C^2 - 8B^2 = 1. \quad (6)$$

Using Definition 4 and Equation (5) we may express C_{k+1} as a linear combination of B_k and C_k.

Theorem 1. *For $k \geq 0$, we have that*

$$C_{k+1} = 3C_k + 8B_k. \quad (7)$$

Moreover, since $64B_k^2 = 8C_k^2 - 8$, we may define a function taking one Lucas-balancing number C_k to the next, C_{k+1}.

Corollary 1. *We have* $C_{k+1} = g(C_k) = 3C_k + \sqrt{8C_k^2 - 8}$*, where*

$$g(y) = 3y + \sqrt{8y^2 - 8}. \tag{8}$$

The equations, functions, recursions from Equation (1) to Equation (8) provide the foundations for subsequent studies of balancing numbers and Lucas-balancing numbers, and their properties.

Panda [8] generalized the balancing number concept to arbitrary sequences. In particular, he considered balancing the sequence of odd integers:

$$1 + 3 + \cdots + (2b - 3) = (2b + 1) + (2b + 3) + \cdots + (2m - 1).$$

This leads to, what we call, the square-balancing equation:

$$(b - 1)^2 + b^2 = m^2. \tag{9}$$

In this case, we must make a distinction between a square-balancing number B, given by $2b - 1$, and the square-balancing index b corresponding to $B = 2b - 1$. Similarly, we must distinguish the square-counterbalancing number $M = 2m - 1$, from the square-counterbalancing index m. As with the balancing numbers, the square-balancing Equation (9) along with analogs of Equations (2) through (6), provide the basis for exploring the square-balancing numbers and associated Lucas-square-balancing numbers.

Our aim is to generalize the problems of balancing the triangular and square equations to balancing sequences for polygonal numbers of any number of sides. That is, for a given number of sides, $s \geq 3$, we balance a sequence that yields a general balancing equation involving the formula for s-sided polygonal numbers. A balancing number for the general s-agonal balancing equation will be called an s-agonal-balancing number. Corresponding to the triangular and square cases, we will develop the analogs of Equations (1) through (6). These will provide a general theory for the family of all s-agonal-balancing numbers.

For each s, greater than or equal to three, the s-agonal-balancing numbers are a result of balancing certain arithmetic sequences. The general problem of balancing a single arithmetic sequence $(an + b)_{n \geq 0}$, resulting in the, so called, (a, b)-type balancing numbers, was first introduced by Kovács, Liptai, and Olajos [4]. Rather than considering just one sequence of balancing numbers, we show that the entire family of s-agonal-balancing numbers and associate s-agonal Lucas-balancing numbers are interrelated across values of s.

2. Fundamental Properties of Balancing, Lucas-Balancing, Counterbalancing and Pell Numbers

We list some well-known properties and identities of balancing and Lucas-balancing numbers together with counterbalancing number properties that will be used in later sections. First, we observe a relationship between Lucas-balancing and counterbalancing numbers. From Equations (2) and (4), we see that

$$M_k = \frac{-1+C_k}{2} \text{ or } C_k = 2M_k + 1. \tag{10}$$

The well-known recurrence relations for B_k and C_k are

$$B_{k+1} = 6B_k - B_{k-1} \text{ and } C_{k+1} = 6C_k - C_{k-1}, \tag{11}$$

where $B_0 = 0$, $B_1 = 1$ and $C_0 = 1$, $C_1 = 3$. From Equations (10) and (11), we obtain the recurrence for the counterbalancing numbers given by

$$M_{k+1} = 6M_k - M_{k-1} + 2, \tag{12}$$

where $M_0 = 0$ and $M_1 = 1$. Solving the recurrences, with their respective initial conditions, we obtain the Binet formulas for B_k, C_k and M_k:

$$B_k = \frac{(3+\sqrt{8})^k - (3-\sqrt{8})^k}{2\sqrt{8}}, \tag{13}$$

$$C_k = \frac{(3+\sqrt{8})^k + (3-\sqrt{8})^k}{2}, \tag{14}$$

$$M_k = \frac{(3+\sqrt{8})^k + (3-\sqrt{8})^k - 2}{4}. \tag{15}$$

By extending the Recurrences 11 and 12 backwards, the respective initial conditions imply that

$$B_{-k} = -B_k,\ C_{-k} = C_k,\ \text{and } M_{-k} = M_k. \tag{16}$$

for all integers k.

From the Binet forms shown in Equations (13) and (14), and the Extensions 16, we may derive De Moivre's identity for balancing numbers, (see Panda [9]).

Theorem 2. *For $k \in \mathbb{Z}$, we have*

$$\left(C_k + B_k\sqrt{8}\right)^n = C_{kn} + B_{kn}\sqrt{8}.$$

Using the pair of recurrences in Equations (5) and (7), we may express the actions of $f(x)$ and $g(y)$ in matrix form. That is, we have a Brahmagupta matrix representation of the transition between one (B_k, C_k) pair and the next.

Theorem 3. *For all $k \in \mathbb{Z}$, we have*

$$\begin{bmatrix} B_{k+1} \\ C_{k+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} B_k \\ C_k \end{bmatrix}, \text{ and } \begin{bmatrix} B_k \\ C_k \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}^k \begin{bmatrix} B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} C_k & B_k \\ 8B_k & C_k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

since $B_0 = 0$ and $C_0 = 1$.

The later matrix form was used by Ray [11] to prove the identities

$$B_{k+\ell} = C_k B_\ell + B_k C_\ell \text{ and } C_{k+\ell} = C_k C_\ell + 8B_k B_\ell.$$

It is useful to express the fundamental transformation in Theorem 3 in terms of B_k and M_k.

Theorem 4. *For all $k \in \mathbb{Z}$, we have*

$$\begin{bmatrix} B_{k+1} \\ M_{k+1} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} B_k \\ M_k \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Proof. We make appropriate substitutions of the form $C = 2M + 1$ in the transformation, then express the result as a transformation on B_k and M_k:

$$\begin{bmatrix} B_{k+1} \\ C_{k+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} B_k \\ C_k \end{bmatrix}$$

becomes

$$\begin{bmatrix} B_{k+1} \\ 2M_{k+1} + 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} B_k \\ 2M_k + 1 \end{bmatrix},$$

which may be expressed in the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} B_{k+1} \\ M_{k+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} B_k \\ M_k \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

Solving for $[B_{k+1}, M_{k+1}]^T$, we have

$$\begin{aligned} \begin{bmatrix} B_{k+1} \\ M_{k+1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} B_k \\ M_k \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \left(\begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} B_k \\ M_k \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

□

Finally, Table 1 displays the first few values near index zero of the sequences of balancing (A001109), Lucas-balancing (A001541), and counterbalancing (A001108) numbers.

k	-2	-1	0	1	2	3	4	5	6
B_k	-6	-1	0	1	6	35	204	1189	6930
C_k	17	3	1	3	17	99	577	3363	19601
M_k	8	1	0	1	8	49	288	1681	9800

Table 1: Balancing, Lucas-balancing and counterbalancing numbers

3. Defining s-agonal-Balancing Numbers

For ordinary balancing numbers, we saw that the defining Diophantine Equation (1) reduces to solving $T_{B-1} + T_B = T_M$, where T_n denotes the nth triangular number. Balancing the sequence of odd natural numbers led to the square-balancing equation $(b-1)^2 + b^2 = m^2$. To generalize, we want to balance a sequence to obtain the analogous equation for s-sided polygonal numbers. That is, for each $s \geq 3$, we wish to balance a sequence to obtain the *s-agonal-balancing equation*

$$P(s, b-1) + P(s, b) = P(s, m), \tag{17}$$

where $P(s, n)$ the s-sided polygonal number of order n given by the well-known formula

$$P(s, n) = \sum_{i=1}^{n} \big((s-2)i - (s-3)\big) = \frac{1}{2}n\big((s-2)n - (s-4)\big). \tag{18}$$

Hence, we may balance the sequence $\big((s-2)i-(s-3)\big)_{i\geq 1}$ at an index b to obtain the s-agonal-balancing Equation (17). For b satisfying the s-agonal-balancing Equation (17), we call b an *s-agonal-balancing index* with corresponding *s-agonal-balancing number* given by $B = (s-2)b-(s-3)$. Additionally, the *s-agonal-counterbalancing index* m corresponds to the *s-agonal-counterbalancing number* $M = (s-2)m-(s-3)$.

Substituting the expression for $P(s, n)$ from Formula (18) into the s-agonal-balancing Equation (17), we obtain

$$\frac{b-1}{2}\big((s-2)(b-1) - (s-4)\big) + \frac{b}{2}\big((s-2)b - (s-4)\big) = \frac{m}{2}\big((s-2)m - (s-4)\big).$$

Making the substitutions $b = (B + (s-3))/(s-2)$ and $b - 1 = (B-1)/(s-2)$ and simplifying, we have

$$\frac{1}{(s-2)}\left(2B^2 + 2(s-3)\right) = (s-2)m^2 - (s-4)m.$$

Solving this quadratic equation for m, and selecting the positive root to assure that m is positive, we obtain an explicit relationship between m and B given by

$$m = \frac{(s-4) + \sqrt{8B^2 + s^2 - 8}}{2(s-2)}. \tag{19}$$

In order for m and B to be integers, Equation (19) requires the following three conditions for B to be an s-agonal-balancing number:

$$8B^2 + s^2 - 8 \text{ is a perfect square} \tag{20}$$
$$(s-4) + \sqrt{8B^2 + s^2 - 8} \text{ is divisible by } 2(s-2) \text{ and} \tag{21}$$
$$B \text{ must be of the form } (s-2)b - (s-3). \tag{22}$$

The first Condition (20) requires $8B^2 + s^2 - 8 = C^2$, for some integer C. That is, $x = B$ and $y = C$ satisfy the s-agonal Pell equation

$$y^2 - 8x^2 = s^2 - 8. \tag{23}$$

The second Condition (21) expresses the requirement that the s-agonal-counterbalancing index m in Equation (19) must be integral. Finally, the third Condition, (22), is needed because not all solutions of the s-agonal Pell Equation, (23), are of the correct form to be a member of the sequence that we are balancing. Since $C = \sqrt{8B^2 + s^2 - 8}$, Condition (21) may be expressed in the form

$$(s-4) + C \text{ is divisible by } 2(s-2).$$

For a given s-agonal-balancing number B and s-agonal-counterbalancing index m, we call C the corresponding *s-agonal Lucas-balancing number*. Additionally, we see that there is a direct relationship between the s-agonal-counterbalancing number $M = (s-2)m - (s-3)$, its s-agonal-counterbalancing index m and the s-agonal Lucas-balancing number C given by

$$m = \frac{(s-4) + C}{2(s-2)} = \frac{M + (s-3)}{(s-2)}. \tag{24}$$

Specifically, we have

$$C = 2(s-2)m - (s-4) \text{ and } C = 2M + (s-2).$$

We define the s-agonal-balancing, s-agonal Lucas-balancing and s-agonal-counterbalancing numbers by these three conditions.

Definition 5. For any $s \geq 3$, we say a triple (B, C, M) of integers, with $C > 0$ and $M > 0$ is an *s-agonal-balancing triple* if it satisfies the three conditions

$$C^2 - 8B^2 = s^2 - 8 \tag{25}$$
$$C \equiv -(s-4) \equiv s \pmod{2(s-2)} \tag{26}$$
$$B \equiv -(s-3) \equiv 1 \pmod{(s-2)}. \tag{27}$$

We note that, as a consequence of Equation (24), Equation (26) may be rewritten in terms of M as

$$M \equiv 1 \pmod{(s-2)}. \tag{28}$$

Both reflect the requirement that m be an integer. Importantly, this allows both congruence Conditions (26) and (27) to be expressed using the same modulus $s-2$.

For a specific $s \geq 3$, we use $B^{[s]}$ to denote an s-agonal-balancing number, $C^{[s]}$ for the corresponding s-agonal Lucas-balancing number and $M^{[s]}$ will denote the corresponding s-agonal-counterbalancing number. For completeness, we denote by $b^{[s]}$ and $m^{[s]}$ the corresponding s-agonal-balancing index and the s-agonal-counterbalancing index, respectively.

Finally, the first few s-agonal-balancing, s-agonal Lucas-balancing and s-agonal-counterbalancing numbers for $s = 3, 4, 5, 13, 20$ are shown in Tables (2) through (6). Each was computed by searching for values of b and m satisfying the s-agonal balancing Equation (17). For certain small values of s, the sequences $\left(B_k^{[s]}\right)$, $\left(C_k^{[s]}\right)$, $\left(M_k^{[s]}\right)$, $\left(b_k^{[s]}\right)$ and $\left(m_k^{[s]}\right)$ appear in the OEIS; $B_k^{[3]} = b_k^{[3]}$: A001109, $C_k^{[3]}$: A001541, $M_k^{[3]} = m_k^{[3]}$: A001108; $B_k^{[4]}$: A002315, $C_k^{[4]}$: A077445, $m_k^{[4]}$: A001653; $C_k^{[5]}$: A144796, $m_k^{[5]}$: A144941; $m_k^{[6]}$: A251602; $m_k^{[7]}$: A133327 and $m_k^{[8]}$: A251896.

k	0	1	2	3	4	5	6	7
$B_k^{[3]}$	1	6	35	204	1189	6930	40391	235416
$C_k^{[3]}$	3	17	99	577	3363	19601	114243	665857
$M_k^{[3]}$	1	8	49	288	1681	9800	57121	332928

Table 2: Triangle-balancing, Lucas triangle-balancing, triangle-counterbalancing numbers

k	0	1	2	3	4	5	6	7
$B_k^{[4]}$	1	7	41	239	1393	8119	47321	275807
$C_k^{[4]}$	4	20	116	676	3940	22964	133844	780100
$M_k^{[4]}$	1	9	57	337	1969	11481	66921	390049

Table 3: (Square-balancing, Lucas square-balancing, square-counterbalancing numbers

4. Transformations between Solutions

We proceed by solving the family $y^2 - 8x^2 = s^2 - 8$ of s-agonal Pell equations. We will see that the solutions of an individual s-agonal Pell equation directly involve the (ordinary) balancing numbers and Lucas-balancing numbers. Moreover, we will

k	0	1	2	3	4	5
$B_k^{[5]}$	1	76	1597	87712	1842937	101219572
$C_k^{[5]}$	5	215	4517	248087	5212613	286292183
$M_k^{[5]}$	1	106	2257	124042	2606305	143146090

Table 4: Pentagonal-balancing, Lucas pentagonal-balancing, pentagonal-counterbalancing numbers

k	0	1	2	3	4	5
$B_k^{[13]}$	1	199	5776	410840	4300106581	305861412979
$C_k^{[13]}$	13	563	16337	1162031	12162538093	865106716883
$M_k^{[13]}$	1	276	8163	581010	6081269041	432553358436

Table 5: Triskaidecagonal-balancing, Lucas triskaidecagonal-balancing, triskaidecagonal-counterbalancing numbers

k	0	1	2	3	4
$B_k^{[20]}$	1	693577	65585233	6201795601	1065886915567897
$C_k^{[20]}$	20	1961732	185503052	17541326900	3014783463904292
$M_k^{[20]}$	1	980857	92751517	8770663441	1507391731952137

Table 6: Icosagonal-balancing, Lucas icosagonal-balancing, icosagonal-counterbalancing numbers

show that the entire family of solutions may be generated from one trivial solution from each s-agonal Pell equation. All other solutions are generated by a simple symmetry and a cross-connection between solutions for certain distinct values of s. The cross-connection will involve a companion Pell equation for the s-agonal Pell equation family.

For any $s \geq 3$ the *trivial solution* to the s-agonal Pell equation is given by

$$x_0^{[s]} = 1 \text{ and } y_0^{[s]} = s.$$

As with ordinary balancing numbers we may define functions that map a solution of the general s-agonal Pell Equation (23) to another solution. In particular, we define the pair of functions f and g by

$$f(x;s) = 3x + \sqrt{8x^2 + (s^2-8)} \text{ and } g(y;s) = 3y + \sqrt{8y^2 - 8(s^2-8)}.$$

It is easy to see that the inverses of f and g, have corresponding forms:

$$f^{-1}(x;s) = 3x - \sqrt{8x^2 + (s^2-8)} \text{ and } g^{-1}(y;s) = 3y - \sqrt{8y^2 - 8(s^2-8)}.$$

The forms for the k-fold compositions f^k and g^k directly involve the (ordinary) balancing numbers. By extending the recursions for the balancing numbers B_k and the Lucas-balancing numbers C_k to negative values of k, as in Section 2, the k-fold compositions of the inverses also have similar forms.

Theorem 6. *For $k \in \mathbb{N}$, the k-fold compositions of $f(x;s)$ and of $g(y;s)$ are given by*

$$f^k(x;s) = C_k x + B_k\sqrt{8x^2+(s^2-8)} \text{ and } g^k(y;s) = C_k y + B_k\sqrt{8y^2-8(s^2-8)}.$$

Similarly, the k-fold compositions of $f^{-1}(x;s)$ and $g^{-1}(y;s)$ are

$$\left(f^{-1}\right)^k(x;s) = C_k x - B_k\sqrt{8x^2+(s^2-8)}$$

and

$$\left(g^{-1}\right)^k(y;s) = C_k y - B_k\sqrt{8y^2-8(s^2-8)}.$$

Proof. To show that $f^k(x;s) = C_k x + B_k\sqrt{8x^2+(s^2-8)}$, for all $k \in \mathbb{N}$, we need only prove that $f\left(f^k(x;s);s\right) = C_{k+1}x + B_{k+1}\sqrt{8x^2+(s^2-8)}$, and the result would follow by induction, since $B_1 = 1$ and $C_1 = 3$. We first simplify the argument $D = 8(f^k(x;s))^2 + (s^2-8)$ in the radical. We have

$$\begin{aligned} D &= 8\left(C_k x + B_k\sqrt{8x^2+(s^2-8)}\right)^2 + (s^2-8) \\ &= 8\left(C_k^2x^2 + 2C_kB_kx\sqrt{8x^2+(s^2-8)} + B_k^2(8x^2+(s^2-8)\right) + (s^2-8) \\ &= 8C_k^2x^2 + 16B_kC_kx\sqrt{8x^2+(s^2-8)} + 8^2B_k^2x^2 + (8B_k^2+1)(s^2-8). \end{aligned}$$

Since the ordinary balancing and Lucas-balancing numbers B_k and C_k satisfy the Pell equation $8B_k^2 + 1 = C_k^2$, we see that

$$\begin{aligned} D &= C_k^2\left(8x^2+(s^2-8)\right) + 2(C_k\sqrt{8x^2+(s^2-8)})\,(8B_kx) + 8^2B_k^2x^2 \\ &= \left(C_k\sqrt{8x^2+(s^2-8)} + 8B_kx\right)^2. \end{aligned}$$

Thus, we find that

$$\begin{aligned} f\left(f^k(x;s);s\right) &= 3f^k(x;s) + \sqrt{8\left(f^k(x;s)\right)^2 + (s^2-8)} \\ &= 3\left(C_kx + B_k\sqrt{8x^2+(s^2-8)}\right) + C_k\sqrt{8x^2+(s^2-8)} + 8B_kx \\ &= (3C_k + 8B_k)\,x + (3B_k + C_k)\sqrt{8x^2+(s^2-8)}. \end{aligned}$$

As noted in Section 2, $C_{k+1} = 3C_k + 8B_k$ and $B_{k+1} = 3B_k + C_k$, hence, we have

$$f^{k+1}(x;s) = f\left(f^k(x;s);s\right) = C_{k+1}x + B_{k+1}\sqrt{8x^2+(s^2-8)}.$$

All other cases follow in similar fashion. □

The combined action of $f^k(x;s)$ and $g^k(y;s)$ may be represented by exactly the same matrix multiplication as the ordinary balancing maps $f(x)$ and $g(y)$ in Section 2.

Corollary 2. *For any solution (x, y) of the s-agonal Pell equation, and for any $k \in \mathbb{Z}$, we have*

$$f^k(x;s) = \ \ C_k x + B_k y \tag{29}$$

$$g^k(y;s) = 8B_k x + C_k y, \tag{30}$$

where we have suppressed the apparent dependence of f^k on y, and of g^k on x. In matrix form, we see that

$$\begin{bmatrix} f^k(x;s) \\ g^k(y;s) \end{bmatrix} = \begin{bmatrix} C_k & B_k \\ 8B_k & C_k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}^k \begin{bmatrix} x \\ y \end{bmatrix}.$$

Proof. Let (x, y) be a solution of the s-agonal Pell equation $y^2 - 8x^2 = s^2 - 8$. Solving this equation in two ways, we have

$$y = \sqrt{8x^2 + (s^2 - 8)} \text{ and } 8x = \sqrt{8y^2 - 8(s^2 - 8)}.$$

Equations (29) and (30) now follow from Theorem 6 by substitution and Theorem 3. □

Since the matrix form for $[f^k(x;s), g^k(y;s)]^T$ is exactly the same as for the ordinary balancing generator functions $[f(x), g(y)]^T$, only the initial conditions distinguish solutions to the s-agonal Pell equations for different values of s.

Definition 7. We denote the *generator transformation* F defined by the pair of functions $f(x;s)$ and $g(y;s)$ by

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = F \begin{bmatrix} x \\ y \end{bmatrix}.$$

Also, the *inverse generator transformation* corresponding to $f^{-1}(x;s)$ and $g^{-1}(y;s)$ is given by

$$F^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & -1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = F^{-1} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Note that, for convenience, we also denote the matrices representing the F and F^{-1} transformation by F and F^{-1}, respectively.

We may restate the result of Corollary 2 in terms of the transformation F and its matrix form.

Corollary 3. *In matrix form, we have that*

$$F^k \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}^k \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C_k & B_k \\ 8B_k & C_k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The minimal polynomial for the generator matrix F provides recursive relations for solutions of the s-agonal Pell equation.

Lemma 1. *The minimal polynomial for F is $\chi(\lambda) = \lambda^2 - 6\lambda + 1$.*

Proof. We need only show that $F^2 = 6F - I$:

$$F^2 = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}^2 = \begin{bmatrix} 17 & 6 \\ 48 & 17 \end{bmatrix} = \begin{bmatrix} 18 & 6 \\ 48 & 18 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 6 \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} - I = 6F - I.$$

□

For a specific s value, a pair of functions $\left(f^k(x; s), g^k(y; s)\right)$ maps a solution of the s-agonal Pell equation to another solution.

Theorem 8. *If (x, y) is a solution of the s-agonal Pell equation $y^2 - 8x^2 = s^2 - 8$, then $\left(f^k(x; s), g^k(y; s)\right)$ is also a solution.*

Proof. From Corollary 2, we see that

$$\begin{aligned}
(g^k(y; s))^2 - 8(f^k(x; s))^2 &= (C_k y + 8B_k x)^2 - 8\,(C_k x + B_k y)^2 \\
&= \left(C_k^2 y^2 + 16 C_k B_k yx + 64 B_k^2 x^2\right) \\
&\qquad - 8\left(C_k^2 x^2 + 2C_k B_k yx + B_k^2 y^2\right) \\
&= \left(C_k^2 - 8B_k^2\right) y^2 - 8\left(C_k^2 - 8B_k^2\right) x^2.
\end{aligned}$$

The pair $(X, Y) = (B_k, C_k)$ satisfies $Y^2 - 8X^2 = 1$, the (ordinary) Pell Equation (6) for balancing numbers. Hence, we have

$$(g^k(y; s))^2 - 8(f^k(x; s))^2 = y^2 - 8x^2 = s^2 - 8.$$

□

Corollary 4. *Suppose (x_0, y_0) is a solution to the s-agonal Pell equation. Let $(x_k, y_k) = (f^k(x_0, s), g^k(y_0; s))$ be solutions generated by $f^k(x; s)$ and $g^k(y; s)$ from (x_0, y_0). Then, we have*

$$x_{k+1} = 6x_k - x_{k-1} \text{ and } y_{k+1} = 6y_k - y_{k-1}.$$

Proof. Since the generator matrix F satisfies $F^2 = 6F - I$, we have

$$\begin{aligned}
\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} &= F^{k+1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = F^2 \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} = 6F \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} - I \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} \\
&= 6 \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} = \begin{bmatrix} 6x_k - x_{k-1} \\ 6y_k - y_{k-1} \end{bmatrix}.
\end{aligned}$$

□

Example 9. Given $s \geq 3$, the trivial solution $(x_0^{[s]}, y_0^{[s]}) = (1, s)$ generates sequences $(x_k^{[s]})_{k\geq 0}$ and $(y_k^{[s]})_{k\geq 0}$ that satisfy the recurrence relations

$$x_{k+1}^{[s]} = 6x_k^{[s]} - x_{k-1}^{[s]} \text{ and } y_{k+1}^{[s]} = 6y_k^{[s]} - y_{k-1}^{[s]},$$

where we define $x_1^{[s]}$ and $y_1^{[s]}$ by

$$\begin{bmatrix} x_1^{[s]} \\ y_1^{[s]} \end{bmatrix} = F \begin{bmatrix} x_0^{[s]} \\ y_0^{[s]} \end{bmatrix} = \begin{bmatrix} 3x_0^{[s]} + y_0^{[s]} \\ 8x_0^{[s]} + 3y_0^{[s]} \end{bmatrix} = \begin{bmatrix} 3+s \\ 8+3s \end{bmatrix}.$$

That is, our initial conditions on the recurrence relations are

$$x_0^{[s]} = 1,\ x_1^{[s]} = s+3 \text{ and } y_0^{[s]} = s,\ y_1^{[s]} = 3s+8, \text{ respectively.}$$

Example 10. For $s = 3$ and $s = 4$, their respective trivial solutions generate the complete sequences of s-agonal Pell equation solutions for each. Moreover, as we will see later, in these cases, these sequences of solutions already satisfy the three conditions in Definition 5 to be s-agonal-balancing and s-agonal Lucas-balancing numbers for $s = 3$ and $s = 4$.

Note that triangle-balancing numbers are the ordinary balancing numbers shifted by one index. Similarly, the Lucas-triangle-balancing numbers are the shifted Lucas-balancing numbers.

Each solution (x, y) to the s-agonal Pell equation $y^2 - 8x^2 = s^2 - 8$ may be paired with another solution $(\bar{x}, \bar{y})$ by a reflection. We say that $(\bar{x}, \bar{y})$ is *conjugate* to (x, y).

Theorem 11. *Let $(x, y) \in \mathbb{Z}^2$ be a solution of $y^2 - 8x^2 = s^2 - 8$. Suppose that for some $\alpha, \beta \in \mathbb{R}$, we have*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then, another solution $(\bar{x}, \bar{y}) \in \mathbb{Z}^2$ of the s-agonal Pell equation is given by

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Proof. We have $x = \alpha + \beta \in \mathbb{Z}$ and $y = 4\alpha + 2\beta \in \mathbb{Z}$. It follows that $2\alpha = y - 2x \in \mathbb{Z}$ and $2\beta = -y + 4x \in \mathbb{Z}$. From these, we see that $2\alpha - 2\beta = 2y - 6x$, which implies that $\bar{x} = \alpha - \beta = y - 3x \in \mathbb{Z}$. We also find that $8\alpha - 4\beta = 6y - 16x$, from which it follows that $\bar{y} = 4\alpha - 2\beta = 3y - 8x \in \mathbb{Z}$.

Now, we need only show that $(\bar{y})^2 - 8(\bar{x})^2 = y^2 - 8x^2 = s^2 - 8$:

$$\begin{aligned}(\bar{y})^2 - 8(\bar{x})^2 &= (4\alpha - 2\beta)^2 - 8(\alpha - \beta)^2 \\ &= 16\alpha^2 - 16\alpha\beta + 4\beta^2 - 8(\alpha^2 - 2\alpha\beta + \beta^2) \\ &= 16\alpha^2 + 4\beta^2 - 8(\alpha^2 + \beta^2) \\ &= 16\alpha^2 + 16\alpha\beta + 4\beta^2 - 8(\alpha^2 + 2\alpha\beta + \beta^2) \\ &= (4\alpha + 2\beta)^2 - 8(\alpha + \beta)^2 \\ &= y^2 - 8x^2 \\ &= s^2 - 8.\end{aligned}$$

□

From the proof of Theorem 11, we see that we may obtain the conjugate solution $(\bar{x}, \bar{y})$ from a solution (x, y) using a matrix form:

$$\begin{bmatrix}\bar{x} \\ \bar{y}\end{bmatrix} = \begin{bmatrix}-3 & 1 \\ -8 & 3\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}. \tag{31}$$

Definition 12. We denote the *conjugation transformation* by

$$R\left(\begin{bmatrix}x \\ y\end{bmatrix}\right) = R\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}-3 & 1 \\ -8 & 3\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix},$$

where we denote the matrix performing the transformation by R as well as the function.

We may express the duality between conjugate solutions directly.

Theorem 13. *If (x, y) and $(\bar{x}, \bar{y})$ are conjugate solutions of $y^2 - 8x^2 = s^2 - 8$, then*

$$\bar{y} - 4\bar{x} = -(y - 4x) \text{ and } \bar{y} - 2\bar{x} = y - 2x.$$

Proof. From Equation (31), we see that $\bar{x} = -3x + y$ and $\bar{y} = -8x + 3y$. Upon substitution, we see that

$$\bar{y} - 4\bar{x} = (-8x + 3y) - 4(-3x + y) = 4x - y = -(y - 4x).$$

Similarly, we have $\bar{y} - 2\bar{x} = y - 2x$. □

Theorem 13 shows that, in the real plane, the transformation induced by R is a reflection across the line $y = 4x$, and fixes points on that line. Thus, self-conjugate solutions of the s-agonal Pell equation must lie on that line. It follows that for a self-conjugate solution (x, y) to exist, we must have that $8x^2 = s^2 - 8$. Thus, we can characterize those values of s whose corresponding s-agonal Pell equation has self-conjugate solutions.

Theorem 14. *The s-agonal equation $y^2-8x^2=s^2-8$ has a self-conjugate solution (x,y) if and only if s is of the form $s=4t$, where (x,t) is a solution to the negative Pell equation*

$$x^2-2t^2=-1.$$

Moreover, for such an s, a self-conjugate solution of the s-agonal Pell equation has the form

$$x=\pm\sqrt{\frac{s^2-8}{8}} \text{ and } y=\pm 4\sqrt{\frac{s^2-8}{8}}.$$

Proof. For a self-conjugate solution $(x,4x)$ of the s-agonal Pell equation, we must have that $(4x)^2-8x^2=8x^2=s^2-8$. Hence s^2 must be divisible by 8. Therefore, s must be divisible by 4. By setting $s=4t$ in the equation $8x^2=s^2-8$, we find that x and t satisfy the Pell equation

$$x^2-2t^2=-1.$$

The converse follows immediately. Finally, a self-conjugate solution $(x,4x)$ of the s-agonal Pell equation satsfies $8x^2=s^2-8$. Hence, s^2-8 is divisible by 8, and we have

$$x=\pm\sqrt{\frac{s^2-8}{8}}=\pm\sqrt{2t^2-1} \text{ and } y=\pm 4\sqrt{\frac{s^2-8}{8}}=\pm 4\sqrt{2t^2-1}.$$

□

For s-agonal-balancing numbers, we are interested in solutions on the branch of the hyperbola $y^2-8x^2=s^2-8$ with $y>0$. Thus, in the following we will restrict ourselves to non-negative solutions where $y=4x$.

Among the Pell numbers, A000129, we see that the values of t are the odd indexed Pell numbers P_{2j-1}, for $j\in\mathbb{N}$. These odd indexed Pell numbers appear separately on the OEIS as A001653. The x values are A002315, the odd indexed Lucas-Pell numbers. These are also half the members of the sequence A077444. Table 7 shows the first few positive values of x and t and the corresponding values of $s=4t$. These values of s appear as sequence A077445 in the OEIS.

k	0	1	2	3	4	5	6	7
x_k	1	7	41	239	1393	8119	47321	275807
t_k	1	5	29	169	985	5741	33461	195025
s_k	4	20	116	676	3940	22964	133844	780100

Table 7: Solutions of $x^2-2t^2=-1$, and $s=4t$

Example 15. For $s \geq 4$, the conjugate of the trivial solution $(x_0, y_0) = (1, s)$ is $(\bar{x}_0, \bar{y}_0) = (s-3, 3s-8)$, where we have suppressed the dependence of x_0 and y_0 on s. Notice the relationship between $(\bar{x}_0, \bar{y}_0)$ and the image $(x_1, y_1) = (s+3, 3s+8)$ of the trivial solution under the generator functions $f(x;s)$ and $g(y;s)$. Moreover, the conjugation of (x_1, y_1) is $(\bar{x}_1, \bar{y}_1) = (-1, s)$, which is also the image of $(\bar{x}_0, \bar{y}_0)$ under the inverse generator functions $f^{-1}(x;s)$ and $g^{-1}(y;s)$. These relationships illustrate how the generator matrix F and the conjugation matrix R interact, as in Theorem 16.

The cases when $s = 3$ and $s = 4$ deserve special attention. For the triangle-balancing Pell equation the conjugate of the trivial solution, $(x_0, y_0) = (1, 3)$ is $(\bar{x}_0, \bar{y}_0) = (0, 1)$. This is the only case where $\bar{x}_0 < x_0$. This causes $s = 3$ to be an exceptional case in some later results. For $s = 4$, the conjugate of the trivial solution $(x_0, y_0) = (1, 4)$ is itself. The trivial solution is self-conjugate in this, and only this, case.

Theorem 16. *The generator matrix F and conjugation matrix R satisfy*

$$RFR = F^{-1}.$$

Moreover, as a reflection, R satisfies $R^2 = I$.

Proof. We calculate as follows:

$$RFR = \begin{bmatrix} -3 & 1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -8 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -8 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -8 & 3 \end{bmatrix} = F^{-1}.$$

The fact that $R^2 = I$ follows easily from Equation (31) or Theorem 13. ☐

From the presentation $\langle F, R \colon R^2 = I,\ RFR = F^{-1} \rangle$, we may identify the group of symmetries.

Corollary 5. *The generator matrix F and conjugation matrix R generate an infinite dihedral group $\mathcal{G}$ that acts on the solutions of any s-agonal Pell equation.*

In the proof of Theorem 16 another reflection $S = RF$, which is given explicitly by

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

appears. As usual, we denote the matrix associated with this S transformation by S as well. We may alternatively use the two reflections R and S to generate the group $\mathcal{G}$.

Corollary 6. *The reflections R and S generate the group $\mathcal{G}$ of symmetries of the solutions to any s-agonal Pell equation. That is, $\mathcal{G}$ is an instance of the group with presentation $\langle R, S \colon R^2 = S^2 = I \rangle$.*

Proof. We need only express the generator matrix F in terms of the matrices R and S. To that end, we see that

$$RS = \begin{bmatrix} -3 & 1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} = F.$$

Moreover, we have F^{-1} is given by

$$SR = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -8 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -8 & 3 \end{bmatrix} = F^{-1}.$$

□

Thus, we see that every symmetry in $\mathcal{G}$ may be given as an alternating sequence of R and S transformations, or may be represented by the identity matrix.

As with the generator matrix F, we may express the conjugation R, as well as the simple reflection S, in terms of functions in x or y.

Theorem 17. *For a given $s \geq 3$, the actions of the transformations F, R, and S on the solutions of $y^2 - 8x^2 = s^2 - 8$ may be expressed in functional form. That is, we have*

$$\begin{aligned} F: &\quad x' = f(x;s) = 3x + \sqrt{8x^2 + s^2 - 8},\ y' = g(y;s) = 3y + \sqrt{8y^2 - 8(s^2 - 8)} \\ R: &\quad x' = p(x;s) = -3x + \sqrt{8x^2 + s^2 - 8},\ y' = q(y;s) = 3y - \sqrt{8y^2 - 8(s^2 - 8)} \\ S: &\quad x' = h(x;s) = -x, \quad y' = i(y;s) = y. \end{aligned}$$

Proof. Recall that, for a given $s \geq 3$, we have

$$y = \sqrt{8x^2 + s^2 - 8} \text{ and } 8x = \sqrt{8y^2 - 8(s^2 - 8)}.$$

Substituting these alternately into the matrix forms of F, we see that

$$\begin{aligned} F \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x;s) \\ g(y;s) \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x \\ \sqrt{8x^2 + s^2 - 8} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{8}\sqrt{8y^2 - 8(s^2 - 8)} \\ y \end{bmatrix}, \end{aligned}$$

yields the given functional form for F. The functional form for R follows similarly. And, for S, the functional form is clear. □

5. Fundamental Interval for s-agonal Pell Equation Solutions

The group $\mathcal{G}$, acting on the set of solutions to the s-agonal Pell equation, naturally divides the solutions into orbits. We will represent these orbits by those solutions (x, y) where x lies in a particular interval determined by s which will be called the $\mathcal{G}$-*fundamental interval* for a given s. The group $\mathcal{G}$ is generated by the transformation F and R. We also consider the group generated by F alone.

Consider the function $f(x; s)$ in Theorem 17 as acting on the real line $\mathbb{R}$. We see that $f(x; s) = 3x + \sqrt{8x^2 + s^2 - 8}$ maps a certain partition of $\mathbb{R}$ into intervals to itself.

Lemma 2. *For all $k \in \mathbb{Z}$, we have $f(B_k\sqrt{s^2-8}; s) = B_{k+1}\sqrt{s^2-8}$, where B_k is an ordinary balancing number.*

Proof. For any $k \in \mathbb{Z}$ we have

$$\begin{aligned} f(B_k\sqrt{s^2-8}; s) &= 3B_k\sqrt{s^2-8} + \sqrt{8B_k^2(s^2-8) + s^2 - 8} \\ &= 3B_k\sqrt{s^2-8} + \sqrt{(8B_k^2+1)(s^2-8)} \\ &= 3B_k\sqrt{s^2-8} + \sqrt{C_k^2(s^2-8)}, \text{ since } C_k^2 - 8B_k^2 = 1 \\ &= 3B_k\sqrt{s^2-8} + C_k\sqrt{(s^2-8)}, \text{ since } C_k \geq 0 \\ &= B_{k+1}\sqrt{s^2-8}, \text{ by Equation (5).} \end{aligned}$$

□

Corollary 7. *For all $k \in \mathbb{Z}$, the monotone increasing function $f(x; s) = 3x + \sqrt{8x^2 + s^2 - 8}$ maps the interval $(B_{k-1}\sqrt{s^2-8}, B_k\sqrt{s^2-8}]$ onto the interval given by $(B_k\sqrt{s^2-8}, B_{k+1}\sqrt{s^2-8}]$. Thus, every such interval is the image of $(0, \sqrt{s^2-8}]$ under $f^k(x; s)$ for some $k \in \mathbb{Z}$.*

As a result, we see that all solutions (x, y) of an s-agonal Pell equation are generated by solutions for which the x values are in the interval $(0, \sqrt{s^2-8}]$. That is, the transformation group $\mathcal{F} =< F >$ generated by F naturally partitions the set of solutions into orbits each with a unique representative in the interval $(0, \sqrt{s^2-8}]$. We say a solution (x, y) of the s-agonal Pell equation, with $y > 0$, is a $\mathcal{F}$-*fundamental solution* if $x \in \big(0, \sqrt{s^2-8}\big]$. It is easy to see, for $s \geq 3$, that $s-2 \leq \sqrt{s^2-8}$. In fact, for $s > 3$, the largest possible integer x of a fundamental solution (x, y), relative to $\mathcal{F}$, is $s - 3$, while the smallest is $x = 1$.

Theorem 18. *For $s > 3$, a representative solution (x, y) to the s-agonal Pell equation, with $x \in \big(0, \sqrt{s^2-8}\big]$, satisfies $1 \leq x \leq s - 3$.*

Proof. The solution (x, y) to the s-agonal Pell equation having the smallest positive value of x is $(1, s)$. The conjugate of this trivial solution, is $(\bar{x}, \bar{y}) = (s-3, 3s-8)$. Since $p(x)$ in Theorem 17 is monotone decreasing, for any $x' \in \big(0, \sqrt{s^2-8}\big]$ with $x' > s-3$, its conjugate value satisfies $\bar{x}' = p(x') < 1$. Thus, since $(1, s)$ has the smallest x value of any fundamental solution, there can not be any fundamental solution in $\big(0, \sqrt{s^2-8}\big]$ with $x > s-3$. □

Adjoining the reflection R to the group $\mathcal{F}$ splits the $\mathcal{F}$-fundamental interval in two, creating a new fundamental interval relative to the group $\mathcal{G}$. But, we note that the conjugation R maps $(0, \sqrt{s^2-8}]$ onto $[0, \sqrt{s^2-8})$. Except for $s = 3$ this is of no consequence. Observe that for $s = 3$, the value $x = \sqrt{s^2-8} = 1$ has conjugate $\bar{x} = 0$. For $s > 3$, the conjugate of any solution (x, y) with $x \in (0, \sqrt{s^2-8}]$ again lies in that same half-open interval.

Theorem 19. *Suppose $s > 3$ and let (x, y) be a solution of the s-agonal Pell equation with $x \in (0, \sqrt{s^2-8}]$. Then, the corresponding conjugate solution $(\bar{x}, \bar{y})$ also satisfies $\bar{x} \in (0, \sqrt{s^2-8}]$. Moreover, the x value of the self-conjugate point*

$$(x, y) = (\sqrt{(s^2-8)/8}, 4\sqrt{(s^2-8)/8})$$

clearly lies in the interval $(0, \sqrt{s^2-8})$.

Proof. From Theorem 17, we see that $\bar{x} = p(x; s) = -3x + \sqrt{8x^2+s^2-8}$. Since $p(x; s)$ is monotone decreasing and satisfies $p(p(x; s); s) = x$, we need only note that $p(0; s) = \sqrt{s^2-8}$ and $p\left(\sqrt{s^2-8}; s\right) = 0$. □

Relative to the group $\mathcal{G}$ generated by both F and R, at least for $s > 3$, Theorem 19 implies that we need only consider fundamental solutions (x, y) with $1 \le x \le \sqrt{(s^2-8)/8}$ and $y > 0$.

Corollary 8. *For $s > 3$, the individual orbits of solutions of $y^2 - 8x^2 = s^2 - 8$ with $y > 0$, under the symmetry group $\mathcal{G}$ may be represented by the solutions (x, y) where x is found in the $\mathcal{G}$-fundamental interval $(0, \sqrt{(s^2-8)/8}]$.*

Thus, to determine the solutions to any particular s-agonal Pell equation, we need only search for solutions (x, y) for which x lies in $(0, \sqrt{(s^2-8)/8}]$ and $y > 0$. But, Theorem 14 on self-conjugate solutions shows that solutions for different values of s may be related. In fact, all non-trivial fundamental solutions for any s are directly related to trivial solutions of an s-agonal Pell equation for smaller values of s, as we shall now see.

6. Interrelated Solutions in the Polygonal-Balancing Family of Pell Equations

For any $s > 3$, the trivial solution $(1, s)$ lies in the $\mathcal{G}$-fundamental interval given by $(0, \sqrt{(s^2-8)/8}]$. We are interested in the form of non-trivial fundamental solutions. To that end, We note that for $s = 3$ and $s = 4$, the only solution (x, y) with x in the $\mathcal{G}$-fundamental interval is the trivial solution $(1, s)$. But, for $s > 4$, any non-trival $\mathcal{G}$-fundamental solution (x, y) may be found from the trivial solution $(1, s_1)$ for some $s_1 < s$. The following result generalizes Theorem 14, which found the values of s for which self-conjugate solutions exist.

Theorem 20. *Consider the s_2-agonal Pell equation $y^2 - 8x^2 = s_2^2 - 8$, with a non-trivial $\mathcal{G}$-fundamental solution (x_2, y_2) where x_2 lies in the interval $(0, \sqrt{(s_2^2-8)/8}]$. We set d, y_1, x_1, s_1 and u as follows:*

$$d = x_2 - 1, \quad y_1 = y_2 - 4d = y_2 - 4x_2 + 4, \quad x_1 = x_2 - d = 1$$
$$s_1 = y_1, \quad u = 2d + (s_1 - 2) = y_2 - 2x_2.$$

Then, $(x_1, y_1) = (1, s_1)$ is the trivial $\mathcal{G}$-fundamental solution to $y^2 - 8x^2 = s_1^2 - 8$ in the interval $(0, \sqrt{(s_1^2-8)/8}]$. Moreover, s_1 and s_2 satisfy

$$2u^2 + 8 = (s_1 - 4)^2 + s_2^2. \tag{32}$$

Proof. We observe that $(x_1, y_1) = (1, s_1)$ is the trivial solution to $y^2 - 8x^2 = s_1^2 - 8$. Now, since $s_1 - 4 = y_2 - 4x_2$, we have that

$$\begin{aligned}(s_1 - 4)^2 + s_2^2 &= (y_2 - 4x_2)^2 + y_2^2 - 8x_2^2 + 8\\ &= 2y_2^2 - 8y_2x_2 + 8x_2^2 + 8\\ &= 2(y_2 - 2x_2)^2 + 8\\ &= 2u^2 + 8.\end{aligned}$$

□

Conversely, we can determine when a non-trivial fundamental solution (x_2, y_2) to the equation $y^2 - 8x^2 = s_2^2 - 8$ may be generated by the trivial solution $(1, s_1)$ to $y^2 - 8x^2 = s_1^2 - 8$.

Theorem 21. *Suppose for some integers $u > 0$ and $3 < s_1 < s_2$, we have that*

$$2u^2 + 8 = (s_1 - 4)^2 + s_2^2 \text{ and } u \geq (s_1 - 2).$$

Set $d = [u - (s_1 - 2)]/2$. Then we have $d \in \mathbb{N}$, and $(x_2, y_2) = (1 + d, s_1 + 4d)$ is a $\mathcal{G}$-fundamental solution of $y^2 - 8x^2 = s_2^2 - 8$.

Proof. Since the left-hand side of $2u^2+8=(s_1-4)^2+s_2^2$ is even, (s_1-4) and s_2 have the same parity. Since the right-hand side is congruent to 0 or 2 modulo 4, it follows that u has the same parity as (s_1-4) and s_2. Therefore, we know that $u-(s_1-2)$ is divisible by 2, and hence, d is an integer.

Now, we see that

$$\begin{aligned}
y_2^2-8x_2^2 &= [s_1+4d]^2-8\,[1+d]^2\\
&= \left[s_1+4\left(\frac{u-(s_1-2)}{2}\right)\right]^2-8\left[1+\left(\frac{u-(s_1-2)}{2}\right)\right]^2\\
&= [s_1+2u-2(s_1-2)]^2-2\,[2+u-(s_1-2)]^2\\
&= [2u-(s_1-4)]^2-2\,[u-(s_1-4)]^2\\
&= 4u^2-4u(s_1-4)+(s_1-4)^2-2\left[u^2-2u(s_1-4)+(s_1-4)^2\right]\\
&= 2u^2-(s_1-4)^2\\
&= s_2^2-8.
\end{aligned}$$

Finally, we need only show that $0<x_2\le\sqrt{(s_2^2-8)/8}$. Since $u\ge(s_1-2)$, we see that $d\ge0$ and $x_2=1+d>0$. Moreover, we have

$$\begin{aligned}
x_2^2=(1+d)^2=\left(\frac{u-(s_1-4)}{2}\right)^2 &= \frac{u^2-2u(s_1-4)+(s_1-4)^2}{4}\\
=\frac{2u^2-4u(s_1-4)+2(s_1-4)^2}{8} &= \frac{3(s_1-4)^2-4v(s_1-4)+s_2^2-8}{8}\\
\le\frac{-(s_1-4)^2+s_2^2-8}{8} &\le \frac{s_2^2-8}{8}.
\end{aligned}$$

□

We have shown that non-trivial $\mathcal{G}$-fundamental solutions may be given in terms of trivial solutions for smaller values of s. We now see that this is true for non-trivial $\mathcal{F}$-fundamental solutions. That is, the solutions for $y^2-8x^2=s_2^2-8$ generated from a non-trivial $\mathcal{F}$-fundamental solution (x_2,y_2) are related to the solutions for $y^2-8x^2=s_1^2-8$ generated from the associated trivial solution $(x_1,y_1)=(1,s_1)$.

Theorem 22. *Suppose we have* $2u^2+8=(s_1-4)^2+s_2^2$ *with* $3<s_1<s_2$*. Set*

$$x_2=1+d,\quad y_2=s_1+4d,\quad x_1=1,\quad y_1=s_1.$$

Then, we have

$$\begin{aligned}
(x_2)_k &= f^k(x_2;s_2)=f^k(x_1;s_1)+dB_k^{[4]}=(x_1)_k+dB_k^{[4]}\\
(y_2)_k &= g^k(y_2;s_2)=g^k(x_1;s_1)+dC_k^{[4]}=(y_1)_k+dC_k^{[4]}
\end{aligned}$$

Proof. From Theorem 21, (x_2, y_2) is a solution to $y - 8x^2 = s_2^2 - 8$. Using Corollary 2, the members of the orbit under the group $\mathcal{F}$ of solutions generated by the representative (x_2, y_2), to $y^2 - 8x^2 = s_2^2 - 8$, are given by

$$(x_2)_k = f^k(x_2; s_2) = C_k x_2 + B_k y_2 \text{ and } (y_2)_k = g^k(y_2; s_2) = C_k y_2 + 8B_k x_2,$$

where C_k and B_k are the ordinary Lucas-balancing and balancing numbers, respectively.

We then have

$$\begin{aligned}(x_2)_k &= C_k x_2 + B_k y_2 = C_k(1+d) + B_k(s_1 + 4d) \\ &= (C_k \cdot 1 + B_k \cdot s_1) + d(C_k \cdot 1 + B_k \cdot 4) \\ &= f^k(x_1; s_1) + d\left(C_k \cdot B_1^{[4]} + B_k \cdot C_1^{[4]}\right) \\ &= f^k(x_1; s_1) + dB_k^{[4]} \\ &= (x_1)_k + dB_k^{[4]}.\end{aligned}$$

Similarly, we see that

$$(y_2)_k = g^k(y_1; s_1) + dC_k^{[4]} = (y_1)_k + dC_k^{[4]}.$$

□

By selecting a particular value for s_1 we can find a subfamily of related s_2-agonal Pell equations by solving the Pell Equation (32) for s_2 and u. For example, when s_1 is either 3 or 5 we find the sequence $s_2 = 3, 5, 13, 27, 75, \ldots$ of values for the Pell equation $s_2^2 - 2u^2 = 7$ that provide interrelated solutions according to Theorem 21. Geometrically, Theorem 21 shows $(1, s_1)$ is related to the non-trivial fundamental solution $(x_2, y_2) = (1 + d, s_1 + 4d)$ of the s_2-agonal Pell equation if and only if the points lie on a line of slope 4. Figure 1 shows some such interrelated solutions of the subfamily $y^2 - 8x^2 = s^2 - 8$ where $s = 3, 5, 13, 27, \ldots$. We note that these s values are the first few terms of the OEIS sequence A077443. And, the corresponding values $u = 1, 3, 9, 19, \ldots$ is sequence A077442.

7. Examples of Fundamental Solutions

Finally, we present several examples of finding fundamental representatives of the $\mathcal{F}$ orbits of solutions to an s-agonal-balancing Pell equation.

Example 23. For the pentagonal-balancing equation we have $\sqrt{(s^2 - 8)/8} \approx 1.46$, where $s = 5$. Thus, only the trivial solution $x_0 = 1$ occurs in the $\mathcal{G}$-fundamental interval $(0, \sqrt{(s^2 - 8)/8}]$. We adjoin its conjugate $x_1 = 2$ to find the solutions to

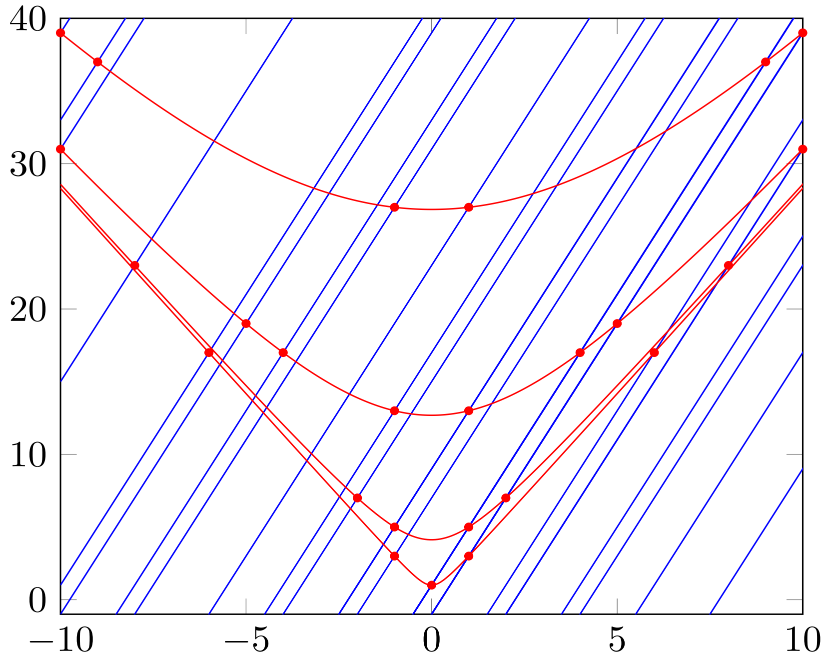

Figure 1: Interrelated solutions of $y^2 - 8x^2 = s^2 - 8$ for $s = 3, 5, 13, 27$.

the pentagonal Pell equation $y^2 - 8x^2 = 17$ with x values in the $\mathcal{F}$-fundamental interval $(0, \sqrt{s^2-8}]$. The corresponding solutions to the pentagonal Pell equation are $(x_0, y_0) = (1, 5)$ and $(x_1, y_1) = (2, 7)$. Table 8 displays initial values of these orbits. We note that the two orbits, shown in Table 8, of the pentagonal-balancing equation, $y^2 - 8x^2 = 17$, yield the two classes of 3-gap balancing numbers appearing in Rout and Panda [12]. Moreover, these solutions of the pentagonal-balancing equation are two less than the upper 3-gap balancing numbers introduced in [1].

k	0	1	2	3	4	5	6	7
$(x_0)_k$	1	8	47	274	1597	9308	54251	316198
$(y_0)_k$	5	23	133	775	4517	26327	153445	894343
$(x_1)_k$	2	13	76	443	2582	15049	87712	511223
$(y_1)_k$	7	37	215	1253	7303	42565	248087	1445957

Table 8: Orbits of the pentagonal-balancing Pell equation

Example 24. For the hexagonal Pell equation, $y^2 - 8x^2 = 28$. We have, for $s = 6$, $\sqrt{(s^2-8)/8} \approx 1.87$, where $s = 6$. Hence, only the trivial solution, $(x_0, y_0) = (1, 6)$

has its x value in the $\mathcal{G}$-fundamental interval $(0, \sqrt{3.5}]$. We adjoin the conjugate of the trivial solution, namely $(x_1, y_1) = (3, 10)$, to find the $\mathcal{F}$-fundamental solutions: $\{(1, 6), (3, 10)\}$, with some orbital values appearing in Table 9.

k	0	1	2	3	4	5	6	7
$(x_0)_k$	1	9	53	309	1801	10497	61181	356589
$(y_0)_k$	6	26	150	874	5094	29690	173046	1008586
$(x_1)_k$	3	19	111	647	3771	21979	128103	746639
$(y_1)_k$	10	54	314	1830	10666	62166	362330	2111814

Table 9: Orbits of the hexagonal-balancing Pell equation

Example 25. For the triskaidecagonal-balancing Pell equation, where $s = 13$, we see that $\sqrt{(s^2 - 8)/8} \approx 4.86$. Thus, the possible $\mathcal{G}$-fundamental solutions to the triskaidecagonal Pell equation, $y^2 - 8x^2 = 161$, satisfy $1 \le x \le 4$. The only such solutions (x, y), with x in the $\mathcal{G}$ fundamental interval $(0, \sqrt{20.125}]$, are $(1, 13)$ and $(4, 17)$. Note that the solution $(4, 17)$ may be obtained from the pentagonal-balancing trivial solution $(1, 5)$ as in Theorem 20, with $d = 3$. Adjoining the conjugates, we obtain the set of $\mathcal{F}$-fundamental 13-agonal Pell equation solutions: $\{(1, 13), (4, 17), (5, 19), (10, 31)\}$. Several values in each orbit appear in Table 10.

k	0	1	2	3	4	5	6	7
$(x_0)_k$	1	16	95	554	3229	18820	109691	639326
$(y_0)_k$	13	47	269	1567	9133	53231	310253	1808287
$(x_1)_k$	4	29	170	991	5776	33665	196214	1143619
$(y_1)_k$	17	83	481	2803	16337	95219	554977	3234643
$(x_2)_k$	5	34	199	1160	6761	39406	229675	1338644
$(y_2)_k$	19	97	563	3281	19123	111457	649619	3786257
$(x_3)_k$	10	61	356	2075	12094	70489	410840	2394551
$(y_3)_k$	31	173	1007	5869	34207	199373	1162031	6772813

Table 10: Orbits of the triskaidecagonal-balancing Pell equation

Example 26. Finally, consider the icosagonal Pell equation $y^2 - 8x^2 = 392$. The set of $\mathcal{G}$-fundamental solutions, with x in the interval $(0, \sqrt{(20^2 - 8)/8}]$, are $\{(1, 20), (7, 28)\}$. The non-trivial, self-conjugate solution $(7, 28)$ may be obtained from the trivial square-balancing solution $(1, 4)$ with $d = 6$, as in Theorem 20. Adjoining the conjugate of $(1, 20)$, we obtain the set of $\mathcal{F}$-fundamental solutions, with x in the interval $(0, \sqrt{20^2 - 8}]$: $\{(1, 20), (7, 28), (17, 52)\}$. Several orbital pairs are displayed in Table 11.

k	0	1	2	3	4	5	6	7
$(x_0)_k$	1	23	137	799	4657	27143	158201	922063
$(y_0)_k$	20	68	388	2260	13172	76772	447460	2607988
$(x_1)_k$	7	49	287	1673	9751	56833	331247	1930649
$(y_1)_k$	28	140	812	4732	27580	1607548	936908	5460700
$(x_2)_k$	17	103	601	3503	20417	1118999	693577	4042463
$(y_2)_k$	52	292	1700	9908	57748	1336580	1961732	11433812

Table 11: Orbits of the icosagonal-balancing Pell equation

8. Conclusion and Open Problems

We have reduced finding the solutions to the s-agonal-balancing Diophantine equation $y^2-8x^2 = s^2-8$ to finding representative solutions where x lies in a fundamental interval. Numerical evidence suggests that the number of $\mathcal{F}$ fundamental solutions depends on the divisors of $s^2 - 8$ for all s. Moreover, we have demonstrated that all solutions for the entire family of s-agonal-balancing equations are interrelated. In part II, we will apply the congruence conditions to the s-agonal-balancing equation solutions, determining which produce s-agonal-balancing triples of s-agonal-balancing numbers, s-agonal Lucas-balancing and s-agonal-counterbalancing numbers.

From data generated, the problem of explicitly counting the number of distinct orbits under the $\mathcal{G}$ generated by the maps F and R appears to have a satisfactory solution. Additionally, applying the congruence conditions produces a modular period of s-agonal-balancing triples. Data indicate that the periods for different values of s fall into natural categories. Explicitly specifying those categories is an open problem.

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