#A56 INTEGERS 22 (2022)

HYPERGEOMETRIC BERNOULLI POLYNOMIALS AND r-ASSOCIATED STIRLING NUMBERS OF THE SECOND KIND

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Received: 7/27/21, Revised: 4/1/22, Accepted: 6/6/22, Published: 6/21/22

Abstract

We define a class of hypergeometric Bernoulli polynomials that generalize previous results recently presented in the literature. We study their connections to the corresponding r-associated Stirling numbers of the second kind and define fractional values for these units. Finally, we make a conjecture about some special values of a generalized hypergeometric form of the Riemann zeta function.

1. Introduction

As D. E. Smith noted [23], in the field of polynomial functions there are few mathematical entities as important as Bernoulli polynomials, which intervene in very different areas, from series expansions to the Riemann zeta function to the last theorem of Fermat. The literature on this subject is very extensive, as one can read in [7].

Numerous extensions of these mathematical entities have been proposed in the literature. See the articles [2, 4, 8, 18, 19, 20, 21], where the most important generalizations are recalled. Among these, the articles by F. T. Howard [12], B. Kurt [14, 15], and M. Miloud and M. Tiachachat [16, 17], the latter with respect to the r-associated Stirling numbers [3, 6, 22], have received special attention. All these expansions are obtained by replacing the denominator $e^x - 1$ by $e^x - T_r(x)$, where $T_r(x)$ (a polynomial of degree $r-1$) is the partial sum of order r of the MacLaurin expansion of the exponential.

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Moreover, A. Hassen and H. D. Nguyen [11] extended the definition of generalized Bernoulli polynomials to non-integer values of the parameter r (see also [1, 10, 13]). A connection of these new entities with a fractional hypergeometric version of the Riemann zeta function was made in [9].

More general extensions can be considered by raising the denominator $e^x - T_r(x)$ to the integer power k, thus defining a set of polynomials, denoted $B_n^{[r-1,\,k]}(t)$, and numbers that are their values at the origin. These polynomials are naturally associated with the r-associated Stirling numbers of the second kind $S(n,k;r)$, so that the possibility of using the $B_n^{[r-1,\,k]}(t)$ polynomials also for non-integer values of r can be extended to this set of Stirling numbers.

This topic is explored in this article, along with its connection to a more general extension of the Riemann zeta function. A conjecture in this framework will be made in Section 4. Some tables of introduced numbers for rational values of r, derived by the second author using the computer algebra program Mathematica©, can be found in the appendix at the end of this article.

2. Basic Definitions

When dealing with hypergeometric functions in the following, for typographical convenience we use the Pochhammer symbol for the increasing factorial according to the notation:

$$(x)_n = \begin{cases} x(x+1)\cdots(x+n-1) = \dfrac{\Gamma(x+n)}{\Gamma(x)}\,, & n \geq 1\,, \\ 1\,, & n = 0\,. \end{cases} \tag{1}$$

Since we never use the falling factorial in this article, this will not be misleading.

Stirling numbers of the second kind are defined by

$$S(n,k) = \frac{1}{k!}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}m^n\,. \tag{2}$$

For brevity, we write:

$$T_r(x) := \sum_{\ell=0}^{r-1}\frac{x^\ell}{\ell!}\,, \tag{3}$$

i.e., the r-th partial sum of the exponential series, which is a polynomial of degree $r-1$. Then the r-associated Stirling numbers of the second kind $S(n,k;r)$ are defined by

$$\left(\sum_{\ell=r}^{\infty}\frac{x^\ell}{\ell!}\right)^k = (e^x - T_r(x))^k = k!\sum_{n=rk}^{\infty}S(n,k;r)\,\frac{x^n}{n!}\,. \tag{4}$$

Of course, $S(n,k;1) = S(n,k)$. Note that

$$e^x - T_r(x) = \frac{x^r}{r!}\ {}_1F_1(1, r+1, x) = \sum_{n=0}^{\infty} \frac{1}{(r+1)_n}\, x^n\,. \tag{5}$$

2.1. Hypergeometric Bernoulli Polynomials of Order k

We introduce the hypergeometric Bernoulli polynomials of order k starting from the equation

$$\frac{\left(\dfrac{x^r}{r!}\right)^k e^{tx}}{[e^x - T_r(x)]^k} = \frac{e^{tx}}{[{}_1F_1(1, r+1, x)]^k} = \sum_{n=0}^{\infty} B_n^{[r-1,\,k]}(t)\, \frac{x^n}{n!}\,. \tag{6}$$

Remark 1. Note that if the polynomials introduced in [18] are denoted by $\mathcal{B}_n^{[r-1]}(t)$, (in our notation the variables x and t are interchanged), they are related to those in Equation (6) by means of the equation

$$r!\, B_n^{[r-1,\,1]}(t) = \mathcal{B}_n^{[r-1]}(t)$$

and similarly for the generalized Bernoulli polynomials considered in [14], denoted here by $\mathcal{B}_n^{[r-1,\,k]}(t)$, the following relation holds:

$$(r!)^k\, B_n^{[r-1,\,k]}(t) = \mathcal{B}_n^{[r-1,\,k]}(t)\,.$$

From the fact that

$$[{}_1F_1(1, r+1, x)]^k = \sum_{n=0}^{\infty} \sum_{\substack{0 \le h_i \le n \\ h_1 + h_2 + \cdots + h_k = n}} \frac{x^n}{(r+1)_{h_1}(r+1)_{h_2}\cdots(r+1)_{h_k}}\ , \tag{7}$$

we have the following theorem.

Theorem 1. *The exponential generating function of the hypergeometric Bernoulli polynomials $B_n^{[r-1,\,k]}(t)$ is given by*

$$\frac{e^{tx}}{\displaystyle\sum_{n=0}^{\infty} \sum_{\substack{0 \le h_i \le n \\ h_1 + h_2 + \cdots + h_k = n}} \frac{x^n}{(r+1)_{h_1}(r+1)_{h_2}\cdots(r+1)_{h_k}}} = \sum_{n=0}^{\infty} B_n^{[r-1,\,k]}(t)\, \frac{x^n}{n!}\,. \tag{8}$$

Introducing the hypergeometric Bernoulli numbers $B_n^{[r-1,\,k]} := B_n^{[r-1,\,k]}(0)$, we find the relevant generating function [24].

$$\frac{1}{\displaystyle\sum_{n=0}^{\infty} \sum_{\substack{0\le h_i\le n\\ h_1+h_2+\cdots+h_k=n}} \frac{[\Gamma(r+1)]^k}{\Gamma(r+h_1+1)\Gamma(r+h_2+1)\cdots\Gamma(r+h_k+1)}\, x^n} = \sum_{n=0}^{\infty} B_n^{[r-1,\,k]} \frac{x^n}{n!}\,. \tag{9}$$

Then we can prove the following result.

Theorem 2. *A representation formula for the hypergeometric r-associated Stirling numbers of the second kind $S(n+kr,k;r)$ is expressed by*

$$S(n+kr,k;r) = \frac{\Gamma(n+kr+1)}{k!} \sum_{\substack{0\le h_i\le n\\ h_1+h_2+\ldots+h_k=n}} \frac{1}{\Gamma(r+h_1+1)\Gamma(r+h_2+1)\cdots\Gamma(r+h_k+1)}\,. \tag{10}$$

Proof. The exponential generating function considered in [21], according to our notation, becomes

$$G^{[r-1,k]}(x) = \frac{1}{\displaystyle k!\,(r!)^k \sum_{n=0}^{\infty} \frac{n!}{(n+kr)!}\, S(n+kr,k;r)\, \frac{x^n}{n!}} = \sum_{n=0}^{\infty} B_n^{[r-1,k]} \frac{x^n}{n!}\,. \tag{11}$$

□

Comparing Equations (9) and (11), we find the equation

$$\frac{1}{\displaystyle k!\,(r!)^k \sum_{n=0}^{\infty} \frac{n!}{(n+kr)!}\, S(n+kr,k;r)\, \frac{x^n}{n!}} = \frac{1}{\displaystyle \sum_{\substack{0\le h_i\le n\\ h_1+h_2+\cdots+h_k=n}} \frac{[\Gamma(r+1)]^k}{\Gamma(r+h_1+1)\Gamma(r+h_2+1)\cdots\Gamma(r+h_k+1)}\, x^n}\,, \tag{12}$$

that is,

$$\frac{k!}{\Gamma(n+kr+1)}\, S(n+kr,k;r) = \sum_{\substack{0\le h_i\le n\\ h_1+h_2+\cdots+h_k=n}} \frac{1}{\Gamma(r+h_1+1)\Gamma(r+h_2+1)\cdots\Gamma(r+h_k+1)}\,.$$

Then we find, in particular, the following theorem.

Theorem 3. *The Stirling numbers of the second kind $S(n+k,k)$ can be computed by means of the equation*

$$S(n+k,k) = \frac{(n+k)!}{k!} \sum_{\substack{0\le h_i\le n \\ h_1+h_2+\cdots+h_k=n}} \frac{1}{(h_1+1)!(h_2+1)!\cdots(h_k+1)!} \,. \tag{13}$$

Proof. Putting $r=1$ in Equation (10), we find

$$\begin{aligned} S(n+k,k;1) &= S(n+k,k) \\ &= \frac{\Gamma(n+k+1)}{k!} \sum_{\substack{0\le h_i\le n \\ h_1+h_2+\cdots+h_k=n}} \frac{1}{\Gamma(h_1+2)\Gamma(h_2+2)\cdots\Gamma(h_k+2)} \,, \end{aligned} \tag{14}$$

that is, Equation (13). □

3. The Particular Case $k = 2$

In the particular case $k=2$, putting

$$\frac{\left(\dfrac{x^r}{r!}\right)^2 e^{tx}}{[e^x - T_r(x)]^2} = \frac{e^{tx}}{[{}_1F_1(1,r+1,x)]^2} = \sum_{n=0}^{\infty} B_n^{[r-1,\,2]}(t)\,\frac{x^n}{n!} \,, \tag{15}$$

and recalling the equation

$${}_1F_1(a,b,x)\ {}_1F_1(\alpha,\beta,x) = \sum_{n=0}^{\infty}\sum_{h=0}^{n} \binom{n}{k} \frac{(a)_h(\alpha)_{n-h}}{(b)_h(\beta)_{n-h}}\,\frac{x^n}{n!} \,, \tag{16}$$

we find

$$\begin{aligned} [{}_1F_1(1,r+1,x)]^2 &= \sum_{n=0}^{\infty}\sum_{h=0}^{n} \binom{n}{h} \frac{(1)_h(1)_{n-h}}{(r+1)_h(r+1)_{n-h}}\,\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{h=0}^{n} \frac{x^n}{(r+1)_h(r+1)_{n-h}} \,, \end{aligned} \tag{17}$$

so that

$$\frac{e^{tx}}{\displaystyle\sum_{n=0}^{\infty}\sum_{h=0}^{n} \frac{x^n}{(r+1)_h(r+1)_{n-h}}} = \sum_{n=0}^{\infty} B_n^{[r-1,\,2]}(t)\,\frac{x^n}{n!} \,. \tag{18}$$

Introducing the hypergeometric Bernoulli numbers $B_n^{[r-1,\,2]} := B_n^{[r-1,\,2]}(0)$, we find the exponential generating function:

$$\frac{1}{\displaystyle\sum_{n=0}^{\infty}\sum_{h=0}^{n}\frac{x^n}{(r+1)_h(r+1)_{n-h}}} = \sum_{n=0}^{\infty} B_n^{[r-1,\,2]}\,\frac{x^n}{n!}\,, \tag{19}$$

which, recalling Equation (1), gives the generating function of the hypergeometric Bernoulli numbers $B_n^{[r-1,\,2]}$:

$$\frac{1}{\displaystyle\sum_{n=0}^{\infty}\sum_{h=0}^{n}\frac{[\Gamma(r+1)]^2}{\Gamma(r+h+1)\Gamma(r+n-h+1)}\,x^n} = \sum_{n=0}^{\infty} B_n^{[r-1,\,2]}\,\frac{x^n}{n!}\,. \tag{20}$$

4. Connection with Generalization of the Riemann Zeta Function

Many extensions of the Riemann zeta function have been considered in the past, most notably in the form $\zeta(z) = \displaystyle\sum_{n=1}^{\infty}\frac{f(n)}{n^z}$, where $f(n)$ is a function of n, in connection with certain versions of the Riemann hypothesis. The number of these generalizations was so large that Atle Selberg, the 1950 Fields Medalist, jokingly proposed a "nonproliferation pact of ζ-functions" in the second half of the last century.

An interesting extension was recently introduced by A. Hassen and H. D. Nguyen [10], who replace $e^x - 1$ in the denominator of the integral representation by $e^x - T_r(x)$, where $T_r(x)$ is the partial sum of order r of the Maclaurin expansion of the exponential. H. L. Geleta and A. Hassen [9] studied a continuous version of the hypergeometric zeta function depending on a positive rational number a, including the value $a = r$ as a special case. The corresponding analytic continuation and the connection with the hypergeometric Bernoulli polynomials were also studied.

Here we set up the equation

$$\begin{aligned} \zeta^{[r]}(n) &= \frac{1}{\Gamma(r+n-1)}\int_0^{\infty}\frac{x^{r+n-2}}{e^x - T_r(x)}\,dx \\ &= \frac{\Gamma(r+1)}{\Gamma(r+n-1)}\int_0^{\infty}\frac{x^{n-2}}{{}_1F_1(1,r+1,x)}\,dx\,, \end{aligned} \tag{21}$$

which for $r = 1$, according to Equation (3), returns the classical Riemann formula:

$$\zeta(n) = \frac{1}{\Gamma(n)}\int_0^{\infty}\frac{x^{n-1}}{e^x - 1}\,dx\,.$$

A connection between the Bernoulli numbers and the Riemann zeta function, which was conjectured by Euler in 1750, was proved by Riemann as

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}\,(2\pi)^{2n}}{2\,(2n)!}\,. \tag{22}$$

Since the Bernoulli numbers B_{2n} change their signs, the above equation writes

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}\,|B_{2n}|}{(2n)!}\,. \tag{23}$$

A possible extension of Equation (21) could be to introduce the function:

$$\begin{aligned}\zeta^{[r,k]}(n) &= \frac{1}{\Gamma(r+n-1)} \int_0^\infty \frac{x^{kr+n-2}}{(r!)^k\,[e^x - T_r(x)]^k}\;dx \\ &= \frac{1}{\Gamma(r+n-1)} \int_0^\infty \frac{x^{n-1}}{[{}_1F_1(1,r+1,x)]^k}\;dx\,,\end{aligned} \tag{24}$$

so that, for $r = k = 1$, the Riemann zeta function (22) is recovered.

We would like to conclude this article with the following conjecture.

Conjecture. The generalized Bernoulli polynomials $B_n^{[r-1,k]}$ are related to the generalized zeta function $\zeta^{[r,k]}(n)$ by the equation

$$\zeta^{[r,k]}(2n) = \frac{2^{2n-1}\pi^{2n}\,|B_{2n}^{[r-1,k]}|}{(2n)!}\,, \tag{25}$$

which is an extension of Equation (23).

5. Conclusion

Based on some recent generalizations of the Bernoulli polynomials, we have introduced generalized hypergeometric versions of these polynomials and the corresponding r-associated Stirling numbers.

It is possible to define the values of these mathematical units also for non-integers r, thus extending the original definitions. In this way, the combinatorial meaning is lost [5], but as in the case of the gamma function, a significant extension to the complex case is achieved. The tables in the appendix show the initial values of the generalized hypergeometric Bernoulli polynomials and the corresponding r-associated Stirling numbers for rational r. The not-so-simple proof of the conjecture proposed in Section 4 could be the subject of further investigation.

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Appendix

Examples of fractional r-associated Stirling numbers of the second kind are reported in the following tables.

	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	$S\left(\frac{1}{2},1;\frac{1}{2}\right)=1$	$S\left(1,2;\frac{1}{2}\right)=\frac{2}{\pi}$	$S\left(\frac{3}{2},3;\frac{1}{2}\right)=\frac{1}{\pi}$	$S\left(2,4;\frac{1}{2}\right)=\frac{4}{3\pi^2}$
$n=1$	$S\left(\frac{3}{2},1;\frac{1}{2}\right)=1$	$S\left(2,2;\frac{1}{2}\right)=\frac{16}{3\pi}$	$S\left(\frac{5}{2},3;\frac{1}{2}\right)=\frac{5}{\pi}$	$S\left(3,4;\frac{1}{2}\right)=\frac{32}{3\pi^2}$
$n=2$	$S\left(\frac{5}{2},1;\frac{1}{2}\right)=1$	$S\left(3,2;\frac{1}{2}\right)=\frac{176}{15\pi}$	$S\left(\frac{7}{2},3;\frac{1}{2}\right)=\frac{56}{3\pi}$	$S\left(4,4;\frac{1}{2}\right)=\frac{896}{15\pi^2}$
$n=3$	$S\left(\frac{7}{2},1;\frac{1}{2}\right)=1$	$S\left(4,2;\frac{1}{2}\right)=\frac{512}{21\pi}$	$S\left(\frac{9}{2},3;\frac{1}{2}\right)=\frac{188}{3\pi}$	$S\left(5,4;\frac{1}{2}\right)=\frac{54784}{189\pi^2}$
$n=4$	$S\left(\frac{9}{2},1;\frac{1}{2}\right)=1$	$S\left(5,2;\frac{1}{2}\right)=\frac{15616}{315\pi}$	$S\left(\frac{11}{2},3;\frac{1}{2}\right)=\frac{1001}{5\pi}$	$S\left(6,4;\frac{1}{2}\right)=\frac{1235456}{945\pi^2}$
$n=5$	$S\left(\frac{11}{2},1;\frac{1}{2}\right)=1$	$S\left(6,2;\frac{1}{2}\right)=\frac{346112}{3465\pi}$	$S\left(\frac{13}{2},3;\frac{1}{2}\right)=\frac{9347}{15\pi}$	$S\left(7,4;\frac{1}{2}\right)=\frac{1679360}{297\pi^2}$

Table 1: $S(n+\frac{k}{2},k;\frac{1}{2}),\ k=1,2,3,4;\ n=0,1,2,3,4,5.$

	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	$S\left(\frac{3}{2},1;\frac{3}{2}\right)=1$	$S\left(3,2;\frac{3}{2}\right)=\frac{16}{3\pi}$	$S\left(\frac{9}{2},3;\frac{3}{2}\right)=\frac{35}{3\pi}$	$S\left(6,4;\frac{3}{2}\right)=\frac{2560}{27\pi^2}$
$n=1$	$S\left(\frac{5}{2},1;\frac{3}{2}\right)=1$	$S\left(4,2;\frac{3}{2}\right)=\frac{256}{15\pi}$	$S\left(\frac{11}{2},3;\frac{3}{2}\right)=\frac{77}{\pi}$	$S\left(7,4;\frac{3}{2}\right)=\frac{28672}{27\pi^2}$
$n=2$	$S\left(\frac{7}{2},1;\frac{3}{2}\right)=1$	$S\left(5,2;\frac{3}{2}\right)=\frac{4352}{105\pi}$	$S\left(\frac{13}{2},3;\frac{3}{2}\right)=\frac{1716}{5\pi}$	$S\left(8,4;\frac{3}{2}\right)=\frac{1015808}{135\pi^2}$
$n=3$	$S\left(\frac{9}{2},1;\frac{3}{2}\right)=1$	$S\left(6,2;\frac{3}{2}\right)=\frac{4096}{45\pi}$	$S\left(\frac{15}{2},3;\frac{3}{2}\right)=\frac{19448}{15\pi}$	$S\left(9,4;\frac{3}{2}\right)=\frac{29229056}{675\pi^2}$
$n=4$	$S\left(\frac{11}{2},1;\frac{3}{2}\right)=1$	$S\left(7,2;\frac{3}{2}\right)=\frac{661504}{3465\pi}$	$S\left(\frac{17}{2},3;\frac{3}{2}\right)=\frac{157131}{35\pi}$	$S\left(10,4;\frac{3}{2}\right)=\frac{11518738432}{51975\pi^2}$
$n=5$	$S\left(\frac{13}{2},1;\frac{3}{2}\right)=1$	$S\left(8,2;\frac{3}{2}\right)=\frac{17629184}{45045\pi}$	$S\left(\frac{19}{2},3;\frac{3}{2}\right)=\frac{517123}{35\pi}$	$S\left(11,4;\frac{3}{2}\right)=\frac{9250537472}{8775\pi^2}$

Table 2: $S(n+\frac{3k}{2},k;\frac{3}{2}),\ k=1,2,3,4;\ n=0,1,2,3,4,5.$

	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	$S\left(\frac{5}{2},1;\frac{5}{2}\right)=1$	$S\left(5,2;\frac{5}{2}\right)=\frac{256}{15\pi}$	$S\left(\frac{15}{2},3;\frac{5}{2}\right)=\frac{1001}{5\pi}$	$S\left(10,4;\frac{5}{2}\right)=\frac{917504}{75\pi^2}$
$n=1$	$S\left(\frac{7}{2},1;\frac{5}{2}\right)=1$	$S\left(6,2;\frac{5}{2}\right)=\frac{2048}{35\pi}$	$S\left(\frac{17}{2},3;\frac{5}{2}\right)=\frac{7293}{5\pi}$	$S\left(11,4;\frac{5}{2}\right)=\frac{11534336}{75\pi^2}$
$n=2$	$S\left(\frac{9}{2},1;\frac{5}{2}\right)=1$	$S\left(7,2;\frac{5}{2}\right)=\frac{47104}{315\pi}$	$S\left(\frac{19}{2},3;\frac{5}{2}\right)=\frac{739024}{105\pi}$	$S\left(12,4;\frac{5}{2}\right)=\frac{1891631104}{1575\pi^2}$
$n=3$	$S\left(\frac{11}{2},1;\frac{5}{2}\right)=1$	$S\left(8,2;\frac{5}{2}\right)=\frac{131072}{385\pi}$	$S\left(\frac{21}{2},3;\frac{5}{2}\right)=\frac{990964}{35\pi}$	$S\left(13,4;\frac{5}{2}\right)=\frac{82661343232}{11025\pi^2}$
$n=4$	$S\left(\frac{13}{2},1;\frac{5}{2}\right)=1$	$S\left(9,2;\frac{5}{2}\right)=\frac{32964608}{45045\pi}$	$S\left(\frac{23}{2},3;\frac{5}{2}\right)=\frac{32442443}{315\pi}$	$S\left(14,4;\frac{5}{2}\right)=\frac{1358266630144}{33075\pi^2}$
$n=5$	$S\left(\frac{15}{2},1;\frac{5}{2}\right)=1$	$S\left(10,2;\frac{5}{2}\right)=\frac{68681728}{45045\pi}$	$S\left(\frac{25}{2},3;\frac{5}{2}\right)=\frac{22190423}{63\pi}$	$S\left(15,4;\frac{5}{2}\right)=\frac{227848585984}{11025\pi^2}$

Table 3: $S(n+\frac{5k}{2},k;\frac{5}{2}),\ k=1,2,3,4;\ n=0,1,2,3,4,5.$

Examples of fractional hypergeometric Bernoulli numbers of the second kind are reported in the following tables.

	$k=1$	$k=2$	$k=3$
$n=0$	1	1	1
$n=1$	- 2/3	- 4/3	- 2
$n=2$	16/45	8/5	56/15
$n=3$	- 32/315	- 512/315	- 400/63
$n=4$	- 256/4725	5632/315	14848/1575
$n=5$	512/6237	- 2048/10395	- 115712/10395
$n=6$	47104/14189175	- 14495744/14189175	21204992/2837835
$n=7$	- 94208/868725	753664/552825	827392/135135
$n=8$	10289152/140970375	13238272/20675655	- 906100736/34459425
$n=9$	12720668672/68746552875	- 61313908736/13749310575	332049154048/13749310575
$n=10$	- 846506491904/2268636244875	591383232512/174510480375	44381353541632/756212081625

Table 4: $B_n^{[-\frac{1}{2},k]},\ k=1,2,3;\ n=0,1,2,\ldots,10.$

	$k=1$	$k=2$
$n=0$	1	1
$n=1$	- 2/5	- 4/5
$n=2$	16/175	88/175
$n=3$	32/2625	- 512/2625
$n=4$	- 12032/1010625	- 512/40425
$n=5$	- 91648/13138125	735232/13138125
$n=6$	6649856/1379503125	2060288/153278125
$n=7$	17502208/2393015625	- 91783168/1861234375
$n=8$	- 334267875328/122534365078125	- 370138021888/13614929453125
$n=9$	- 423312621568/36039519140625	9201921818624/122534365078125
$n=10$	- 463452541288448/1282322130542578125	6061106293374976/75430713561328125

Table 5: $B_n^{[\frac{1}{2},k]},\ k=1,2;\ n=0,1,2,\ldots,10.$

	$k=3$
$n=0$	1
$n=1$	- 6/5
$n=2$	216/175
$n=3$	- 176/175
$n=4$	35328/67375
$n=5$	3072/625625
$n=6$	- 109113344/459834375
$n=7$	5627904/1861234375
$n=8$	839532806144/2722985890625
$n=9$	143964504064/11139487734375
$n=10$	- 6002797926416384/8723279799609375

Table 6: $B_n^{[\frac{1}{2},k]},\ k=3;\ n=0,1,2,\ldots,10.$

	$k = 1$	$k = 2$
$n = 0$	1	1
$n = 1$	- 2/7	- 4/7
$n = 2$	16/441	104/441
$n = 3$	32/3773	- 512/11319
$n = 4$	- 7424/9270261	- 40448/3090087
$n = 5$	- 147968/64891827	83968/21630609
$n = 6$	- 218281984/254830204629	34697216/5200616421
$n = 7$	3723309056/4841773887951	4910645248/484177887951
$n = 8$	4641498791936/3965412814231869	- 1174667395072/233259577307757
$n = 9$	10506370678784/86700569061785679	- 207875695181824/45900301268004183
$n = 10$	- 12538170178011136/18907393330012492305	2425498645823488/677124279036260055

Table 7: $B_n^{[\frac{3}{2},k]},\ k = 1, 2;\ n = 0, 1, 2, \ldots, 10.$

	$k = 3$
$n = 0$	1
$n = 1$	- 6/7
$n = 2$	88/147
$n = 3$	- 1136/3773
$n = 4$	215552/3090087
$n = 5$	80896/3090087
$n = 6$	- 387739648/36404314947
$n = 7$	- 10268106752/537974876439
$n = 8$	- 26915504128/120164024673693
$n = 9$	332811646271488/15300100422668061
$n = 10$	1190747127453581312/81932037763387466655

Table 8: $B_n^{[\frac{3}{2},k]},\ k = 3;\ n = 0, 1, 2, \ldots, 10.$