#A58 INTEGERS 22 (2022)

ON THE INDEX OF ODD PERFECT NUMBERS

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Received: 11/8/21, Revised: 2/15/22, Accepted: 6/14/22, Published: 6/21/22

Abstract

We prove the non-existence of odd perfect numbers for which the square-free part of their index satisfies some mild conditions.

1. Introduction

A perfect number is a positive integer n such that the sum of all its divisors including 1 and n, say $\sigma(n)$, equals $2n$. We know (see remark below) that all perfect numbers n, even or odd, have the following form:

$$n = P^k Q^2, \tag{1}$$

where P is a prime number, and k, Q are positive integers such that $P \nmid Q$. More precisely, if n is even, then $k = 1$, $P = 2^p - 1$, with p an odd prime number, and $Q = 2^{\frac{p-1}{2}}$. While if n is odd, then $k \equiv 1 \pmod 4$ and also $P \equiv 1 \pmod 4$.

Remark 1. We often find in books about elementary number theory a proof that an even perfect number n satisfies Equation (1). However, we were unable to find a reference that the same holds for odd perfect numbers. Moreover, Brauer [2] and Guy [14, Section B1] refers to Euler's original proof [9]. For completeness, we give a short proof of this fact in Lemma 2.

The multiplicative property of σ implies the following:

$$2P^k Q^2 = \sigma(P^k)\sigma(Q^2).$$

One sees that Equation (1) can be written as

$$\frac{\sigma(Q^2)}{Q^2} = \frac{P^k}{\sigma(P^k)/2}, \tag{2}$$

and that the right-hand side fraction in Equation (2) is in the lowest terms, i.e., $\gcd(P^k, \sigma(P^k)/2) = 1$. Therefore, there exists a positive integer d such that

$$\sigma(Q^2) = d \cdot P^k, \tag{3}$$

and

$$Q^2 = d \cdot \sigma(P^k)/2. \tag{4}$$

It is then natural to define the index d of n as

$$d := \gcd(Q^2, \sigma(Q^2)). \tag{5}$$

When n is even, we have $\sigma(Q^2) = \sigma(2^{p-1}) = 2^p - 1 = P$, and $Q^2 = 2^{p-1}$. Thus,

$$\sigma(Q^2) = 2Q^2 - 1. \tag{6}$$

Therefore, we obtain the value of d as follows:

$$d = 1. \tag{7}$$

However, when n is odd, we cannot easily determine d.

As usual, for integers a, b, c, and a prime number q, we let $o_a(b)$ denote the order of b modulo a. We let $v_q(c)$ denote the maximal integer $m \geq 0$ such that $q^m \mid c$. If $q^j \mid c$ while $q^{j+1} \nmid c$ (i.e., we have $j = v_q(c)$) then we write $q^j || c$. We let $\omega(c)$ denote the number of distinct prime numbers that divide c. We let $\Phi_m(x)$ denote the m-th cyclotomic polynomial. Likewise, we let $\operatorname{rad}(n)$ (the *radical* of n) denote the product of all distinct prime numbers that divide n. Finally, we let $\sharp S$ denote the cardinal of S. A few new results follow.

Our first result is the simplest possible when n is odd, and generalizes directly (see Equation (6) and Equation (7)) the case n even considered above.

Proposition 1. *Let $n := P^k Q^2$ be an odd perfect number, with P an odd prime congruent to $1 \pmod 4$, $k > 1$ an integer congruent to $1 \pmod 4$, and with Q a positive integer such that $\gcd(P, Q) = 1$. Let $d := \gcd(Q^2, \sigma(Q^2))$. Let d_1 and d_2 be two positive divisors of d, such that d_2 is square-free and $d = d_1^2 d_2$. Put $\ell := (k+1)/2$, $R := \frac{P^\ell - 1}{P-1}$ and $S := \frac{P^\ell + 1}{2}$. Then, it is impossible to have $d = 1$ and $\sigma(Q^2) = 2Q^2 - 1$.*

We now focus on d_2, the square-free part of d.

Proposition 2. *With the same notation and hypotheses as Proposition 1 we have the following. It is impossible that d be a square (i.e., that $d_2 = 1$).*

Proposition 3. *With the same notation and hypotheses as Proposition 1 we have the following. It is impossible that d_2 divides S.*

Proposition 4. *With the same notation and hypotheses as Proposition 1 we have the following. Assume that $\gcd(d_2, S) = 1$. Then $P \equiv 1 \pmod 8$. Moreover, it is impossible to have $P \leq 100$ when $d_2 < P$.*

Proposition 5. *With the same notation and hypotheses as Proposition 1 we have the following. It is impossible that $\gcd(d_2, S) > 1$ and $\omega(d_2) = 2$, say $d_2 = p_1 p_2$, with primes $p_1 \neq p_2$, such that $p_2 \mid S$.*

Proposition 6. *With the same notation and hypotheses as Proposition 1 we have the following. Assume that* $\omega(d_2) = 1$ *so that* $d_2 = p_1$*, a prime divisor of* Q*. Then* $p_1 \mid R$*,* $P \equiv 1 \pmod 8$*, and we have*

$$v_{p_1}(R) \geq \omega(\ell). \tag{8}$$

Moreover, let h_1 *be the order of* P *modulo* p_1 *(i.e.,* $h_1 = o_{p_1}(P)$*). Then, either* $h_1 = 1$*, so that* $P \equiv 1 \pmod{p_1}$*, or* h_1 *is a power of a prime. Furthermore, if* $a = 0$ *(e.g., if* $p < p_1$ *), and* ℓ *is square-free, then* $q \neq p_1$*. The prime* q *is a divisor of* ℓ *such that* $p_1 \mid \Phi_{q^m}(P)$*, for some positive integer* m*.*

Remark 2. Slowak [22] gave another proof of the special case of Proposition 2 when $d = 1$. Broughan et al. [3] proved Proposition 2 by similar methods as ours. Moreover, they also proved that $d \geq 135$, improving on older estimates (see their references).

Recent work on Goormaghtigh's equation (Equation (17)) by Bennett et al. [1] leads to the following numerical result.

Proposition 7. *With the same notation and hypotheses as Proposition 1 we have the following. Assume that*

$$Q^2 = d \cdot (1 + d_2 + \cdots + d_2^{m-1}) \cdot S \tag{9}$$

for some integer m *such that* $m > \ell$*, and that* $\gcd(d_2, S) = 1$*. Then the following hold.*

(a) *If* $1 < d_2 < P$ *then* $P > 10^5$*.*

(b) *The following statements are false.*

  (1) $m = \ell + 1$ *and* $\ell \leq 17$*.*

  (2) $\gcd(m-1, \ell-1) > 1$ *and* $m \leq 50$*.*

  (3) $(\ell, m) \in \{(3, 6), (3, 8)\}$*.*

There are many papers about odd perfect numbers. Brauer [2] address cases in which the possible odd perfect number has a special factorization. Gallardo and Rahavandrainy [10, 11] considered special cases like reducing the problem modulo some prime number. Gallardo [12] showed the existence of a family of sums of two cubes, say $s(t)$, such that $s(t)$ is odd and $\sigma(s(t)) \equiv 2s(t) \pmod 4$.

The present paper relies on existence results and numerical results by specialists [1, 4, 5, 6, 17, 19, 20] about some classical Diophantine equations. Namely, Nagell-Ljunggren's equation (Equation (14)), and Goormaghtigh's equation (Equation (17)). More precisely, Lemmas 3, 4, 5, and 6, are key in order to prove our main results. It is also worth mentioning that Yuan [24] proved that $(x, y, m) \in$

$\{(5,2,5),(90,2,13)\}$ are the only solutions of the special case of Goormaghtigh's equation in which $n = 3$, and m is odd.

Since we discuss the square-free part (namely, d_2) of a special divisor d of an odd perfect number $n = P^k Q^2$ (see Lemma 1), it is worth mentioning some previous work in this area. First, Luca and Pomerance [18] proved that $\mathrm{rad}(m) < 2m^{17/26}$ for any perfect number m. Later, Klurman [16] improved this (for big enough m) to $\mathrm{rad}(m) \ll n^{9/14}$. Moreover, he proved that for any polynomial $P(x)$ with integer coefficients, with simple roots, and of degree at least 3, the set of positive integers k such that $P(k)$ is perfect, is a finite set. Ellia [8] proved that one has $\mathrm{rad}(n) \leq \sqrt{n}$ provided that $3 \nmid Q$ and $P \leq 148207$ (resp. provided that $3 \mid n$ and $P \leq 223$). Ochem and Rao [21] improved on the result of Ellia by proving that $\mathrm{rad}(n) > \sqrt{n}$ implies that $P > 10^{60}$.

Let us describe where our paper sits in the literature about the index d of an odd perfect number n (see Equation (5)). It is worth mentioning that Broughan et al. [3] (see also Remark 2) proved that $d \geq 315$ as a byproduct of proving the impossibility that d be a product of some special prime powers. Later, Chen and Chen [7] improved on this, and proved that the index d cannot be of the form p^{2u}, where p is a prime and u is a positive integer such that $2u+1$ is composite.

Remark 3. We took $k > 1$ in all our results for the following reason. Consider the idea that one might be able to prove the case $k = 1$. Namely, that there do not exist odd perfect numbers n of the form $n = PQ^2$, with P, Q as in Lemma 1. This is itself a long-standing open problem, dating back to Descartes and Frenicle. Indeed, to some extent (we hope) this entire paper can be though of as advancing towards that goal.

The core of the results in the present paper is due to the solvers of the difficult Diophantine equations discussed above.

2. Tools

Euler showed that odd perfect numbers have a special form.

Lemma 1 ([9])**.** *Let n be an odd perfect number. Then $n = P^k Q^2$, where P is a prime divisor of n such that $P \equiv 1 \pmod 4$, k is a positive integer such that $k \equiv 1 \pmod 4$, and Q is a positive integer.*

Lemma 1 follows from the next result, since $\sigma(n) = 2n$ implies that $\sigma(n) \equiv 2 \pmod 4$ for an odd number n.

Lemma 2. *Let $n > 0$ be an odd integer such that*

$$\sigma(n) \equiv 2 \pmod 4. \tag{10}$$

Then

$$n = P^k Q^2, \tag{11}$$

where P is a prime divisor of n such that $P \equiv 1 \pmod 4$, k is a positive integer such that $k \equiv 1 \pmod 4$, and Q is a positive integer.

Proof. Let p_1 be a prime divisor of n, and let k_1 be a positive integer such that $p_1^{k_1} || n$ and $\sigma(p_1^{k_1})$ is even. We have that $0 \equiv \sigma(p_1^{k_1}) \equiv k_1 + 1 \pmod 2$ since p_1 is odd. In other words, k_1 is odd. Thus, for any other prime divisor p_2 of n we have the following. If $p_2^{k_2} || n$, for some positive integer k_2, then k_2 is even. Thus, n divided by $p_1^{k_1}$ is a square, say Q^2, with Q a positive integer. Put $P := p_1$, and $k := k_1$. Therefore, Equation (11) holds. Assume that $P \equiv -1 \pmod 4$. Then

$$\sigma(P^k) \equiv (1 + P) + \cdots + (P^{k-1} + P^k) \equiv 0 + \cdots + 0 \equiv 0 \pmod 4. \tag{12}$$

One sees that Equation (12) contradicts Equation (10). Thus, P is congruent to 1 modulo 4. Then

$$\sigma(P^k) \equiv 1 + P + \cdots + P^k \equiv k + 1 \pmod 4. \tag{13}$$

Hence, since k is odd, Equation (10) and Equation (13) imply that $k \equiv 1 \pmod 4$. This proves the lemma. □

Bougeaud and Mihailescu proved the following lemma about Nagell-Ljunggren's equation (Equation (14)) that goes back to the papers of Nagell [19, 20] and Ljunggren [17].

Lemma 3 ([5]). *Besides the solutions*

$$\frac{3^5 - 1}{3 - 1} = 11^2, \frac{7^4 - 1}{7 - 1} = 20^2, \frac{18^3 - 1}{18 - 1} = 7^3,$$

the equation

$$\frac{x^n - 1}{x - 1} = y^q \tag{14}$$

has no other solution in integers (x, y, n, q), with

$$x > 1, y > 1, n > 2, q \geq 2, \tag{15}$$

if any one of the following conditions hold,

(a) $q = 2$, (b) 3 *divides* n, (c) 4 *divides* n, (d) $q = 3$ *and* $n \not\equiv 5 \pmod 6$.

Bougeaud and Mignotte proved the following result.

Lemma 4 ([4])**.** *The unique solutions (a, x, y, n, q), with*

$$n > 2, x > 1, y > 1, 1 \leq a < x, \text{and } q > 1 \tag{16}$$

of the equation

$$a \cdot \frac{x^n - 1}{x - 1} = y^q,$$

with $x \leq 100$ or $x = 1000$, are

$$\begin{aligned}(a, x, y, n, q) \in \{&(1, 3, 11, 5, 2), (1, 7, 20, 4, 2), (4, 7, 40, 4, 2), (1, 18, 7, 3, 3)\}\\ &\bigcup \{(7, 18, 49, 3, 2), (7, 18, 7, 3, 4), (8, 18, 14, 3, 3), (3, 22, 39, 3, 2)\}\\ &\bigcup \{(12, 22, 78, 3, 2), (19, 30, 133, 3, 2), (21, 41, 1218, 4, 2)\}\\ &\bigcup \{(13, 68, 247, 3, 2), (52, 68, 494, 3, 2), (58, 99, 7540, 4, 2)\}.\end{aligned}$$

Bennett et al. proved the following two results.

Lemma 5 ([1])**.** *If (x, y, m, n) is a solution to*

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}, \tag{17}$$

with

$$m > n > 2, \quad y > x \geq 2, \tag{18}$$

and with $2 \leq x < y \leq 10^5$, then

$$(x, y, m, n) \in \{(2, 5, 5, 3), (2, 90, 13, 3)\}.$$

Lemma 6. *The only solutions to Equation (17) together with Equation (18), and*

$$m = n + 1 \;\; \text{and} \;\; 3 \leq n \leq 17, \tag{19}$$

or

$$\gcd(m - 1, n - 1) > 1 \;\; \text{and} \;\; m \leq 50, \tag{20}$$

or

$$(n, m) \in \{(3, 6), (3, 8)\}, \tag{21}$$

are given by

$$(x, y, m, n) \in \{(2, 5, 5, 3), (2, 90, 13, 3)\}.$$

Iannucci proved the following.

Lemma 7 ([15])**.** *For primes p and q, $q \mid \Phi_m(p)$ if and only if $m = h \cdot q^\gamma$, where $h = o_q(p)$ and $\gamma \geq 0$. If $\gamma > 0$, then $q || \Phi_m(p)$. Moreover, if $v_q(\sigma(p^a)) \neq 0$ then either $h = 1$ and $v_q(\sigma(p^a)) = v_q(a + 1)$, or $h \mid a + 1$, $h > 1$, and $v_q(\sigma(p^a)) = v_q(a + 1) + v_q(\Phi_h(p))$.*

3. Proofs of Propositions 1, 2, 3, 4, 5, and 6

Observe that

$$\gcd(R, S) = 1, \tag{22}$$

since a common prime divisor q of R and S must be an odd prime number such that $q \mid P^\ell - 1$ and $q \mid P^\ell + 1$. Therefore, q divides the difference $2 = (P^\ell + 1) - (P^\ell - 1)$. We write Equation (4) as

$$Q^2 = d \cdot R \cdot S. \tag{23}$$

Proof of Proposition 1. Assume, contrary to what we want to prove, that $d = 1$ and $\sigma(Q^2) = 2Q^2 - 1$. Observe that we have $2Q^2 = P^k + 1$ from Equation (3), and $2Q^2 = \sigma(P^k)$ from Equation (4). Thus,

$$P\left(\frac{P^{k-1} - 1}{P - 1}\right) = \sigma(P^k) - (P^k + 1) = 0. \tag{24}$$

But Equation (24) is impossible, since $k > 1$. This proves the proposition. □

Proof of Proposition 2. Assume that, contrary to what we want to prove, we have $d_2 = 1$ (i.e., we assume that d equals the square of d_1). From Equation (23) we have

$$\left(\frac{Q}{d_1}\right)^2 = R \cdot S. \tag{25}$$

By Equation (22), we have $\gcd(R, S) = 1$. In particular, Equation (25) implies that R is a square, say $R = R_1^2$, with $R_1 > 1$, since $\ell \geq 3$. More precisely, Equation (25) implies that R and S are both squares. Let R_1 be the positive integer such that $R = R_1^2$. Observe that $\ell \geq 3$, and that $P \geq 5$, since $P \equiv 1 \pmod 4$. Thus,

$$R_1^2 = R = \frac{P^\ell - 1}{P - 1} \geq \frac{P^3 - 1}{P - 1} = P^2 + P + 1 \geq 31. \tag{26}$$

Therefore, $x = P, y = R_1, n = \ell$, and $q = 2$ satisfy Equation (15). Lemma 3 implies then that either $P = 3$ or $P = 7$. This is impossible since $P \equiv 1 \pmod 4$. This proves the proposition. □

Proof of Proposition 3. We assume, contrary to what we want to prove, that d_2 divides S. Let T be the integer such that $S = Td_2$. From Equation (23) it follows that $Q^2 = d_1^2 d_2 RS = d_1^2 d_2 RT d_2$. Thus,

$$\left(\frac{Q}{d_1 d_2}\right)^2 = T \cdot R. \tag{27}$$

Since by Equation (22) one has $\gcd(R, S) = 1$, it follows that $\gcd(R, T) = 1$. Thus, Equation (27) implies that R and T are both squares. Let R_1 be the positive integer such that $R = R_1^2$. Observe, as before, that $\ell \geq 3$, and that $P \geq 5$. Thus, as in Equation (26), we obtain that $R \geq 31$. In other words, we have that $R_1 \geq 6$. Therefore, $x = P, y = R_1, n = \ell$, and $q = 2$ satisfy Equation (15). Lemma 3 implies then that either $P = 3$ or $P = 7$. This is impossible since $P \equiv 1 \pmod 4$. This proves the proposition. □

Proof of Proposition 4. We have $\gcd(d_2, S) = 1$. This means, by Equation (23), that $d_2 \mid R$, since d_2 is square-free, and R and S are coprime (see Equation (22)). Write $d_2 := p_1 \cdots p_h$ for some odd primes $p_1 < \ldots < p_h$, with h a positive integer. For positive integers $e_1, \ldots, e_h$ we have $p_1^{2e_1+1} || R, \ldots, p_h^{2e_h+1} || R$. Thus, $R = R_2 \prod_{j=1}^{h} p_j^{2e_j+1}$ for some positive integer R_2 coprime with p_j, for all j.

Therefore, Equation (23) implies that

$$\left(\frac{Q}{\prod_{j=1}^{h} p_j^{e_j}} \right)^2 = R_2 \cdot S. \tag{28}$$

Observe that S is coprime with R_2 since S is coprime with R. From Equation (28) it follows that S is an odd square, say $S = S_2^2$. Thus, by the definition of S, one has

$$P^\ell = 2S_2^2 - 1. \tag{29}$$

Reducing Equation (29) modulo 8 we see that $P^\ell \equiv 1 \pmod 8$. Thus, $P \equiv 1 \pmod 8$, since ℓ is odd. We recall that ℓ is odd since $2\ell = k + 1$, and $k \equiv 1 \pmod 4$. It follows from Lemma 4 that $d_2 < P$ implies that $P > 100$. In more detail, from Equation (23) we have that the equation

$$a \cdot \frac{x^n - 1}{x - 1} = y^2, \tag{30}$$

where $a = d_2, x = P, y = Q/(d_1 \cdot S_2), n = \ell$, and $q = 2$, satisfies Equation (16). Thus, Equation (30) has no solutions (a, x, y, n) with $0 < a < x$, and $P \equiv 1 \pmod 8$, besides $(21, 41, 1218, 4)$. But for this solution, $\ell = 4$ is not odd. This proves the proposition. □

Proof of Proposition 5. For some integer $m \geq 0$ we have $p_2^{2m+1} || S$. One has

$$\left(\frac{Q}{d_1} \right)^2 = p_1 p_2 \cdot SR. \tag{31}$$

By Proposition 3, p_1 cannot divide S. Thus, Equation (31) implies that for some positive integer T_3 one has

$$p_2^{2m+1} T_3^2 = \frac{P^\ell + 1}{2} \cdot R. \tag{32}$$

If $p_2^{2m+1}||(P^\ell+1)/2$, then Equation (32) implies that R is a square. But this is impossible by the same proof as in part (b). Thus, $p_2^{2m+1}||R$. This is impossible since p_2 divides S and S and R are coprime. This proves the proposition. □

Proof of Proposition 6. Observe that Proposition 3 implies that $p_1 \mid R$. Thus,

$$\left(\frac{Q}{d_1}\right)^2 = p_1 \cdot SR. \tag{33}$$

But Equation (33) implies that S is a square, say $S = S_2^2$. This implies as in Equation (29) that $P \equiv 1 \pmod 8$. By Equation (33), we obtain then that there exists an integer $h \geq 0$ such that

$$R = p_1^{2h+1} R_2^2, \tag{34}$$

for some positive integer R_2 coprime with p_1. Put

$$S(\ell) := \{q\colon q \text{ is prime, and } q \mid \ell\}.$$

Clearly, $\sharp S(\ell) = \omega(\ell)$. Observe that we have the following:

$$p_1^{2h+1} R_2^2 = R = \prod_{j>1, j\mid\ell} \Phi_j(P). \tag{35}$$

Assume, contrary to what we want to prove, that

$$2h+1 = v_{p_1}(R) < \omega(\ell) = \sharp S(\ell). \tag{36}$$

In other words, p_1 could appear as a factor in at most $2h+1$ of the terms of the product of all the $\Phi_j(P)$ in Equation (35). It follows from Equation (36) and from Equation (34) that at least one element q of $S(\ell)$ has the following property. For some positive integer m, for which $q^m||\ell$ one has that

$$\Phi_q(P^{q^{m-1}}) = \Phi_{q^m}(P),$$

is the square K^2 of a positive integer K. But this is impossible. Indeed, we apply Lemma 3 as in the proof of Proposition 2 (see Equation (26)) with ℓ replaced by q^{m-1} to the equality

$$\Phi_q(P^{q^{m-1}}) = K^2,$$

to get a contradiction. This proves that $v_{p_1}(R) \geq \omega(\ell)$. Another short proof (of the case in which $m=1$) is the following. Since $R = \sigma(P^{\ell-1})$ and $p_1 \mid R$, one has $v_{p_1}(\sigma(P^{\ell-1}) \neq 0$. The result follows then from Lemma 7.

Since $p_1 \mid R$, we can assume from Equation (35) that $p_1 \mid \Phi_{q^m}(P)$ for some positive integer m for which $q^m||\ell$, with $q \in S(\ell)$. Lemma 7 implies that for some integer $a \geq 0$, and for $h_1 = o_{p_1}(P)$, one has

$$q^m = h_1 \cdot p_1^a. \tag{37}$$

Since q and p_1 are primes, Equation (37) implies that either $p_1 = q$, $h_1 = 1$ and $m = a \geq 1$, or $a = 0$. If $a = 0$, then Equation (37) implies that $h_1 = q^m$. But $q^m \mid \ell$, and since we assume that ℓ is square-free, we obtain that $m = 1$, so that $h_1 = q$. Since $p_1 \mid \Phi_{q^m}(P)$, we get that

$$\sigma(P^{q-1}) = \frac{P^q - 1}{P - 1} \equiv 0 \pmod{p_1}.$$

But $h_1 = o_{p_1}(P) = q > 1$. Thus, Lemma 7 implies that

$$v_{p_1}(\sigma(P^{q-1})) = v_{p_1}(q) + v_{p_1}(\Phi_q(P)). \tag{38}$$

Observe that Equation (38) says that $v_{p_1}(q) = 0$. Thus, $p_1 \neq q$. The last sentence of the proposition is a consequence of Equation (8) and the fact that p_1 divides R. □

4. Proof of Proposition 7

As in the proof of Proposition 4, we have that $S = S_2^2$ for a positive integer S_2. By Equation (9) we have the following equality:

$$\left(\frac{Q}{d_1 \cdot S_2}\right)^2 \cdot \frac{1}{d_2} = \frac{d_2^m - 1}{d_2 - 1}. \tag{39}$$

But Equation (23) implies that

$$\left(\frac{Q}{d_1 \cdot S_2}\right)^2 \cdot \frac{1}{d_2} = \frac{P^\ell - 1}{P - 1}. \tag{40}$$

Equation (39) and Equation (40) implies that Goormaghtigh's equation (Equation (17)) has the solution

$$x = d_2, y = P, m = m, n = \ell.$$

Proof of part (a). Assume that $1 < d_2 < P$ and that $P \leq 10^5$. Thus, Equation (18) holds. This implies by Lemma 5 that

$$P = 5, m = 5, \ell = 3, \text{ and } d_2 = 2. \tag{41}$$

But Equation (41) implies that Equation (9) can be rewritten as follows:

$$Q^2 = (d_1 S_2)^2 \cdot 62. \tag{42}$$

This is impossible since 62 is not a square. Therefore, $P > 10^5$. This proves (a). □

Proof of part (b). Observe that Equation (19), Equation (20), and Equation (21) imply that Equation (18) holds. Thus, Lemma 6 implies that Equation (41) and Equation (42) hold. This implies that Equation (19), Equation (20), and Equation (21) do not hold. □

Acknowledgements. We thank the referee, and the editor, for helpful detailed comments and suggestions. We are especially grateful to the referee for letting us know about the recent results of Bennett et al. [1].

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