PRIMES IN FLOOR FUNCTION SETS

Randell Heyman
School of Mathematics and Statistics, University of New South Wales, Sydney, N.S.W., Australia
randell@unsw.edu.au

Received: 11/23/21, Revised: 4/11/22, Accepted: 6/13/22, Published: 6/21/22

Abstract

Let x be a positive integer. We give an asymptotic formula for the number of primes in the set $\{\lfloor x/n \rfloor : 1 \le n \le x\}$ and give some related results.

1. Introduction

There is an extensive body of research on arithmetic functions with integer parts of real-valued functions, most commonly, with Beatty $\lfloor \alpha n + \beta \rfloor$ sequences; see, for example, [1, 3, 6, 12, 13], and Piatetski–Shapiro $\lfloor n^\gamma \rfloor$ sequences; see, for example, [2, 4, 5, 7, 17, 20], with real α, β and γ.

Recently there has been much research on sums of the form

$$\sum_{n \le x} f\left(\left\lfloor \frac{x}{n} \right\rfloor\right), \tag{1}$$

where throughout x is a positive integer, f is an arithmetic function, and $\lfloor \cdot \rfloor$ is the floor function. In [8] the authors used exponential sums to find asymptotic bounds and formulas for various classes of arithmetic functions. Subsequent papers by various authors have mainly been focused on improvements in exponential sum techniques (see [9, 11, 15, 18, 19, 22, 23, 24, 25, 26, 27]).

It is natural to examine more fundamental questions about the set$\{\lfloor x/n \rfloor : 1 \le n \le x\}$. In [14] an exact formula for the cardinality of this set was given. In this paper, we count primes in this floor function set. Let

$$\mathcal{G}(x) = \left\{ \left\lfloor \frac{x}{n} \right\rfloor : 1 \le n \le x, \left\lfloor \frac{x}{n} \right\rfloor \text{ is prime} \right\},$$

and set $G(x) := |\mathcal{G}(x)|$. Using exponential sums, this quantity can estimated with the following result.

Theorem 1. *We have*

$$G(x) = \frac{4\sqrt{x}}{\log x} + O\left(\frac{\sqrt{x}}{(\log x)^2}\right).$$

The OEIS sequence A068050 attributes to Adams-Watters the statement that for p prime not equal to 3 we have $G(p) = G(p-1) + 1$. We prove the following result.

Theorem 2. *For any prime x not equal to 3 we have $G(x) = G(x-1) + 1$.*

It is possible to link up $G(x)$ and $G(x-1)$ for some other classes of x. An example is the following result.

Theorem 3. *Let $x = pq$ with p, q odd primes, not necessarily distinct. Then*

$$G(x) = G(x-1) + 1.$$

These relationships between $G(x)$ and $G(x-1)$ may generalize, but with considerable difficulties. For example, based on a somewhat limited investigation using Maple, we have the following result.

Conjecture 1. Suppose $x = p_1p_2p_3$ with $2 < p_1 < p_2 < p_3$. Then

$$G(x) = \begin{cases} G(x-1) & \text{if } p_1p_2 > p_3, \\ G(x-1) + 1 & \text{if } p_1p_2 < p_3. \end{cases}$$

Let $\mathbb{N} = \{1, 2, \cdots\}$. We can also examine the cardinality of the set

$$\mathcal{F}(x) := \left\{ n : \left\lfloor \frac{x}{n} \right\rfloor \text{ is prime} \right\}.$$

This might more naturally be thought of as the cardinality of the subsequence $(\mathcal{F}_{n_k})$ created from the sequence $(\mathcal{F}_n)_{\substack{n \le x \\ n \in \mathbb{N}}}$, $\mathcal{F}_n = \lfloor x/n \rfloor$, where you retain n for which $\mathcal{F}_n$ is prime and remove n for which $\mathcal{F}_n$ is not prime. For example, we have

$$\mathcal{F}(10) = \{2, 3, 4, 5\},$$

whereas it is more natural to think of the sequence (for $x = 10$)

$$(\mathcal{F}_{n_k}) = 5, 3, 2, 2.$$

Of course, the cardinalities are the same. The cardinality of $\mathcal{F}(x)$ (or of $(\mathcal{F}_{n_k})$) can be obtained by substituting $f(m) = \mathbf{1}_{\mathbb{P}}(m)$ into Equation (1) and using a recent result from Zhai [25]. As is usual, $\mathbf{1}_{\mathbb{P}}(m) = 1$ if m is prime and 0 otherwise. We obtain the following result.

Theorem 4. *Let $F(x) := |\mathcal{F}(x)|$. Then*

$$F(x) = \mathcal{P}x + O\left(x^{1/2}\right),$$

where

$$\mathcal{P} = \sum_{n=1}^{\infty} \frac{\mathbf{1}_{\mathbb{P}}(n)}{n(n+1)} = \sum_{p} \frac{1}{p(p+1)} \cong 0.330230.$$

We can use an alternate elementary approach, without exponential sums, to arrive at a result with a slightly better lower bound. Specifically, we have the following result.

Theorem 5. *There exists calculable constants A_1 and A_2 such that for all x,*

$$\mathcal{P}x - \frac{A_1\sqrt{x}}{\log x} \leq F(x) \leq \mathcal{P}x + A_2\sqrt{x}.$$

The methodology of Theorem 4 can be utilised for all indicator functions since these functions are all bounded by 1. For example, we state, but do not prove, the following result.

Theorem 6. *We have*

$$\left\{n : \left\lfloor \frac{x}{n} \right\rfloor \text{ is a prime power}\right\} = \mathcal{D}x + O\left(x^{1/2}\right),$$

where

$$\mathcal{D} = \sum_{n=p^k} \frac{1}{n(n+1)} \cong 0.41382.$$

Throughout we use p, with or without subscript, to denote a prime number. The notation $f(x) = O(g(x))$ or $f(x) \ll g(x)$ is equivalent to the assertion that there exists a constant $c > 0$ such that $|f(x)| \leq c|g(x)|$ for all x. As is usual, we denote by Λ the von Mangoldt function.

2. Proof of Theorem 1

We have

$$G(x) = \left|\left\{\text{prime } p = \left\lfloor \frac{x}{n} \right\rfloor \text{ for some } 1 \leq n \leq x\right\}\right|.$$

If $\left\lfloor \frac{x}{n} \right\rfloor = p$ then

$$\frac{x}{p+1} < n \leq \frac{x}{p}$$

and such an n will exist if and only if $\left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p+1} \right\rfloor > 0$. So

$$G(x) = \sum_{p \leq x} \delta\left(\left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p+1} \right\rfloor > 0\right),$$

where

$$\delta_p = \begin{cases} 1 \text{ if } \left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p+1} \right\rfloor > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Let

$$G(x) = G_1(x) + G_2(x) + G_3(x) + G_4(x), \tag{2}$$

where

$$G_1 = \sum_{p<b} \delta_p,\ G_2 = \sum_{b \le p \le \sqrt{x}} \delta_p,\ G_3 = \sum_{\sqrt{x} < p \le x^{34/67}} \delta_p,\ G_4 = \sum_{x^{34/67} < p \le x} \delta_p,$$

and

$$b = \frac{\sqrt{4x+1}-1}{2} = \sqrt{x} + O(1).$$

For $G_1(x)$ the condition is always satisfied, since for $p < b$ we have

$$\left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p+1} \right\rfloor > \frac{x}{p} - \frac{x}{p+1} - 1 = \frac{x}{p(p+1)} - 1 > 0.$$

So

$$G_1(x) = \sum_{p<b} 1 = \pi(\sqrt{x}) + O(1) = \frac{2\sqrt{x}}{\log x} + O\left(\frac{\sqrt{x}}{(\log x)^2}\right). \tag{3}$$

Trivially, we have that

$$G_2(x) = O(1). \tag{4}$$

Next, we estimate $G_4(x)$. If $p > x^{34/67}$ then $p = \lfloor \frac{x}{n} \rfloor$ for some $n \le x^{33/67}$. Since there can be at most $x^{33/67}$ values for n we have

$$G_4(x) = O\left(x^{33/67}\right). \tag{5}$$

For $G_3(x)$ (and $G_4(x)$) p is large enough that $\left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p+1} \right\rfloor$ can equal only 0 or 1. So

$$G_3(x) = \sum_{\sqrt{x} < p \le x^{34/67}} \delta\left(\left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p+1} \right\rfloor > 0\right) = \sum_{\sqrt{x} < p \le x^{34/67}} \left(\left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{p+1} \right\rfloor\right).$$

Then, using $\psi(x) = x - \lfloor x \rfloor - \frac{1}{2}$,

$$G_3(x) = x \sum_{\sqrt{x} < p \le x^{34/67}} \frac{1}{p(p+1)} + \sum_{\sqrt{x} < p \le x^{34/67}} \left(\psi\left(\frac{x}{p+1}\right) - \psi\left(\frac{x}{p}\right)\right). \tag{6}$$

Using Riemann-Stieltjes integration and the Prime Number Theorem we have, for the first sum,

$$\begin{aligned} x \sum_{\sqrt{x} < p \le x^{34/67}} \frac{1}{p(p+1)} &= x \int_{\sqrt{x}}^{x^{34/67}} \frac{1}{n(n+1)} d(\pi(n)) \\ &= \frac{2\sqrt{x}}{\log x} + O\left(\frac{\sqrt{x}}{(\log x)^2}\right). \end{aligned} \tag{7}$$

For the second sum of $G_3(x)$ we will use the following result([10, Theorem 6.25]).

Lemma 1. *Let $\delta \in [0,1], x \geq 1$ be a large real number and R, R_1 be positive integers such that $1 \leq R \leq R_1 \leq 2R \leq x^{2/3}$. Then, for all $\epsilon \in (0, \frac{1}{2}]$,*

$$x^{-\epsilon} \sum_{R \leq n \leq R_1} \Lambda(n)\psi\left(\frac{x}{n+\delta}\right) \ll \left(x^2R^{33}\right)^{1/38} + \left(x^2R^{19}\right)^{1/24}\left(x^3R^2\right)^{1/9} + \left(x^3R^{-1}\right)^{1/6} + R^{5/6}.$$

Returning to the second sum of $G_3(x)$ and using the Lemma we have

$$\sum_{\sqrt{x} < p \leq x^{34/67}} \left(\psi\left(\frac{x}{p+1}\right) - \psi\left(\frac{x}{p}\right)\right) \leq \left|\sum_{\sqrt{x} < p \leq x^{34/67}} \psi\left(\frac{x}{p+1}\right)\right| + \left|\sum_{\sqrt{x} < p \leq x^{34/67}} \psi\left(\frac{x}{p}\right)\right|.$$

We now bound the sum involving $\psi(\frac{x}{p})$. The calculations for the sum involving $\psi(\frac{x}{p+1})$ is virtually identical. Let $m \sim N$ denote the inequalities $N < m \leq 2N$. We have

$$\sum_{\sqrt{x} < p \leq x^{34/67}} \psi\left(\frac{x}{p}\right) \ll \max_{\sqrt{x} < N \leq x^{34/67}} \left|\sum_{p \sim N} \psi\left(\frac{x}{p}\right)\right| \log x.$$

Next, using Abel summation,

$$\begin{aligned}\left|\sum_{p \sim N} \psi\left(\frac{x}{p}\right)\right| &= \left|\sum_{p \sim N}\left(\frac{1}{\log p} \times \psi\left(\frac{x}{p}\log p\right)\right)\right| \\ &\leq \frac{2}{\log N} \max_{N \leq N_2 \leq N_1} \left|\sum_{N < p \leq N_2} \psi\left(\frac{x}{p}\right)\log p + \mathbf{1}_p(N)\psi\left(\frac{x}{N}\log N\right)\right| \\ &\leq \frac{2}{\log N} \max_{N \leq N_2 \leq N_1} \left\{\left|\sum_{N < n \leq N_2} \Lambda(n)\psi\left(\frac{x}{n}\right)\right| + |R(N)| + \log N\right\},\end{aligned}$$

where

$$|R(N)| \leq \left(\left(\sum_{\sqrt{N} < p \leq \sqrt{N_2}} \log p\right)\left(\sum_{2 \leq a \leq \frac{\log N_2}{\log p}} 1\right)\right) < 2\sqrt{N}.$$

Using Lemma 1 with $N_1 = N_2 = x^{34/67}$ and $N = \sqrt{x}$ we obtain

$$\sum_{\sqrt{x} < p \leq x^{34/67}} \psi\left(\frac{x}{p}\right) \ll x^{\frac{1256}{2546}+\epsilon}. \tag{8}$$

Substituting Equations (7) and (8) into Equation (6) and we see that

$$G_3(x) = \frac{2\sqrt{x}}{\log x} + \left(\frac{\sqrt{x}}{(\log x)^2}\right),$$

and substituting this equation and Equations (3), (4) and (5) into Equation (2) completes the proof.

3. Proof of Theorem 2

Fix a prime x not equal to 3. Suppose further that $p \in G(x)$ but $p \neq x$. So for some n we have $\lfloor x/n \rfloor = p$. As x is a prime we have $x = np + u$ where $1 \le u \le n-1$. So $x - 1 = np + u - 1$ from which

$$\left\lfloor \frac{x-1}{n} \right\rfloor = \left\lfloor \frac{np+u-1}{n} \right\rfloor = \left\lfloor p + \frac{u-1}{n} \right\rfloor = p.$$

Thus $p \in G(x-1)$.

Conversely, suppose $p \in G(x-1)$. So $\lfloor (x-1)/n \rfloor = p$ for some n. So $x - 1 = np + u$ where $0 \le u \le n-1$. But $u \neq n-1$, for then $x = np + n = n(p+1)$, which contradicts the supposition that x is prime. Thus $x - 1 = np + u$ with $0 \le u \le n-2$, and then

$$\left\lfloor \frac{x}{n} \right\rfloor = \left\lfloor \frac{np+u+1}{n} \right\rfloor = \left\lfloor p + \frac{u+1}{n} \right\rfloor = p.$$

Therefore $p \in G(x)$.

We conclude that there is a one-to-one correspondence between an $p \neq x \in G(x)$ and $p \neq x \in G(x-1)$. Noting that we have $x \in G(x)$ but $x \notin G(x-1)$ concludes the proof.

4. Proof of Theorem 3

We have $x = pq$ with p, q odd primes, not necessarily distinct. Without loss of generality assume $p \le q$.

Case 1: Suppose that $r \in G(x)$ with $r \neq p, q$. So $x = nr + u$ with $0 \le u \le n-1$. But if $u = 0$ then $x = nr$ which is impossible since $r \neq p, q$. So $1 \le u \le n-1$ and thus

$$\left\lfloor \frac{x-1}{n} \right\rfloor = \left\lfloor \frac{nr+u-1}{n} \right\rfloor = \left\lfloor r + \frac{u-1}{n} \right\rfloor = r.$$

So $r \in G(x)$. Conversely, suppose $r \in G(x-1)$. So $x-1 = nr+u$ with $0 \le u \le n-1$. But if $u = n-1$ then $x = nr + n - 1 + 1 = (n+1)r$ which is again impossible since

$r \neq p, q$. So $0 \leq u \leq n-2$ and then

$$\left\lfloor \frac{x}{n} \right\rfloor = \left\lfloor \frac{nr+u+1}{n} \right\rfloor = \left\lfloor r + \frac{u+1}{n} \right\rfloor = r.$$

So $r \in G(x-1)$.

Case 2: Suppose that $r \in G(x)$ with $r = p = q$. Since $r = \lfloor x/r \rfloor$ we have $r \in G(x)$. But

$$\left\lfloor \frac{x-1}{r} \right\rfloor = \left\lfloor r - \frac{1}{r} \right\rfloor = r - 1$$

and

$$\left\lfloor \frac{x-1}{r-1} \right\rfloor = \left\lfloor \frac{r^2-1}{r-1} \right\rfloor = r + 1.$$

So $r \notin G(x-1)$.

Case 3: Suppose that $r \in G(x)$ with $r = p \neq q$ or $r = q \neq p$. Recall that $p \leq q$. It is clear that $p \in G(x)$ and $q \in G(x)$. Then

$$\left\lfloor \frac{x-1}{q-1} \right\rfloor = \left\lfloor \frac{pq-1}{q-1} \right\rfloor = \left\lfloor p + \frac{p-1}{q-1} \right\rfloor = p,$$

so $p \in G(x-1)$. But

$$\left\lfloor \frac{x-1}{p} \right\rfloor = \left\lfloor \frac{pq-1}{p} \right\rfloor = \left\lfloor q - \frac{1}{p} \right\rfloor = q - 1,$$

and

$$\left\lfloor \frac{x-1}{p-1} \right\rfloor = \left\lfloor \frac{pq-1}{p-1} \right\rfloor = \left\lfloor q + \frac{q-1}{p-1} \right\rfloor > q.$$

So $q \notin G(x-1)$.

Reviewing the three cases, we see that $\mathcal{G}(x) = \mathcal{G}(x-1) + 1$, which proves the theorem.

5. Proof of Theorem 4

We have

$$F(x) = \sum_{n \leq x} \mathbf{1}_{\mathbb{P}} \left(\left\lfloor \frac{x}{n} \right\rfloor \right).$$

We will require the following result ([25, Theorem 1]).

Lemma 2. *Let f be a complex-valued arithmetic function with $f(n) \ll n^{\alpha} (\log n)^{\theta}$ for some $\alpha \in [0, 1)$ and $\theta \geq 0$. Then*

$$\sum_{n \leqslant x} f(\lfloor x/n \rfloor) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O\left(x^{\frac{1}{2}(\alpha+1)} (\log x)^{\theta} \right).$$

Using this lemma with $\alpha = 0$ and $\theta = 0$ we have

$$F(x) = \mathcal{P}x + O\left(x^{1/2}\right),$$

where

$$\mathcal{P} = \sum_{n=1}^{\infty} \frac{\mathbf{1}_{\mathbb{P}}(n)}{n(n+1)} = \sum_{p} \frac{1}{p(p+1)} \cong 0.330230,$$

completing the proof.

6. Proof of Theorem 5

Let $\mathbb{P}$ be the set of (positive) primes and $\overline{\mathbb{P}}$ be the set consisting of 1 and the positive composite numbers. We create upper and lower bounds for $F(x)$ from the set $\mathcal{C}(x) := \{n : 1 \le n \le x\}$. Note that $\mathcal{F}(x) = \{n \in \mathcal{C}(x) : \lfloor x/n \rfloor \text{ is prime}\}$. For the upper bound we truncate the process by removing from $\mathcal{C}(x)$ only those n such that $\lfloor x/n \rfloor$ is a non-prime less than or equal to $\sqrt{x}$. In total we remove

$$\sum_{\substack{c \in \overline{\mathbb{P}} \\ c \le \sqrt{x}}} \left(\left\lfloor \frac{x}{c} \right\rfloor - \left\lfloor \frac{x}{c+1} \right\rfloor \right) > \sum_{\substack{c \in \overline{\mathbb{P}} \\ c \le \sqrt{x}}} \left(\frac{x}{c(c+1)} - 1 \right)$$

values of n.

So

$$\begin{aligned}
F(x) &< x - \sum_{\substack{c \in \overline{\mathbb{P}} \\ c \le \sqrt{x}}} \left(\frac{x}{c(c+1)} - 1 \right) \\
&= x - \sum_{\substack{c \in \overline{\mathbb{P}} \\ c \le \sqrt{x}}} \frac{x}{c(c+1)} + \sum_{\substack{c \in \overline{\mathbb{P}} \\ c \le \sqrt{x}}} 1 \\
&= x - x \left(\sum_{c \le \sqrt{x}} \frac{1}{c(c+1)} - \sum_{\substack{c \in \mathbb{P} \\ c \le \sqrt{x}}} \frac{1}{c(c+1)} \right) + \sqrt{x} - \pi(\sqrt{x}) \\
&= x - \frac{x\sqrt{x}}{\sqrt{x}+1} + x \sum_{\substack{c \in \mathbb{P} \\ c \le \sqrt{x}}} \frac{1}{c(c+1)} + O\left(\sqrt{x}\right) \\
&= \frac{x}{\sqrt{x}+1} + x \sum_{p} \frac{1}{p(p+1)} - x \sum_{p > \sqrt{x}} \frac{1}{p(p+1)} + O\left(\sqrt{x}\right).
\end{aligned}$$

Then

$$\sum_{p > \sqrt{x}} \frac{1}{p(p+1)} \le \sum_{n > \sqrt{x}} \frac{1}{n(n+1)} = O\left(\frac{1}{\sqrt{x}}\right),$$

and so

$$F(x) \leq \mathcal{P}x + O\left(\sqrt{x}\right). \tag{9}$$

For the lower bound we add up the number of n for which $\lfloor x/n \rfloor$ is a prime less than or equal to $\sqrt{x}$. Then, using $\pi(m) = \frac{m}{\log m} + O\left(\frac{m}{(\log m)^2}\right)$ and Riemann-Stieltjes integration,

$$\begin{aligned} F(x) = \sum_{\substack{c \in \mathbb{P} \\ c \leq \sqrt{x}}} \left(\left\lfloor \frac{x}{c} \right\rfloor - \left\lfloor \frac{x}{c+1} \right\rfloor \right) &\geq \sum_{\substack{c \in \mathbb{P} \\ c \leq \sqrt{x}}} \left(\frac{x}{c(c+1)} + 1 \right) \\ &= x \sum_{\substack{c \in \mathbb{P} \\ c \leq \sqrt{x}}} \frac{1}{c(c+1)} + \sum_{\substack{c \in \mathbb{P} \\ c \leq \sqrt{x}}} 1 = x \left(\mathcal{P} - \sum_{p > \sqrt{x}} \frac{1}{p(p+1)} \right) + \sum_{\substack{c \in \mathbb{P} \\ c \leq \sqrt{x}}} 1 \\ &= \mathcal{P}x - x \sum_{p > \sqrt{x}} \frac{1}{p(p+1)} + O\left(\frac{\sqrt{x}}{\log x} \right) = \mathcal{P}x + O\left(\frac{\sqrt{x}}{\log x} \right), \end{aligned}$$

completing the proof.

Acknowledgements. I thank Joshua Stucky who sketched out the proof of the asymptotic formula for $G(x)$. I thank Olivier Bordellès for assisting in the proof of Theorem 1. I thank Igor Shparlinski for some initial discussions.

References

[1] A. G. Abercrombie, W. D. Banks and I. E. Shparlinski, Arithmetic functions on Beatty sequences, *Acta Arith.* **136** (2009), 81–89.

[2] Y. Akbal, Friable values of Piatetski-Shapiro sequences, *Proc. Amer. Math. Soc.* **145** (2017), 4255–4268.

[3] R. C. Baker and W. D. Banks, Character sums with Piatetski–Shapiro sequences, *Quart. J. Math.* **66** (2015), 393–416.

[4] R. C. Baker, W. D. Banks, J. Brüdern, I. E. Shparlinski and A. Weingartner, Piatetski–Shapiro sequences, *Acta Arith.* **157** (2013), 37–68.

[5] R. C. Baker, W. D. Banks, V. Z. Guo and A. M. Yeager, Piatetski–Shapiro primes from almost primes, *Monatsh Math.* **174** (2014), 357–370.

[6] R. C. Baker and L. Zhao, Gaps between primes in Beatty sequences, *Acta Arith.* **172** (2016), 207–242.

[7] W. D. Banks, V. Z. Guo and I. E. Shparlinski, Almost primes of the form $\lfloor p^c \rfloor$, *Indag. Math.* **27** (2016), 423–436.

[8] O. Bordellès, L. Dai, R. Heyman, H. Pan and I. E. Shparlinski, On a sum involving the Euler function, J. Number Theory **202** (2019), 278–297.

[9] O. Bordellès, On certain sums of number theory, *preprint*, Available at arXiv:2009.05751 [math.NT].

[10] O. Bordellès, *Arithmetic tales (2nd ed.)*, Springer Nature, Switzerland, 2020.

[11] S. Chern, Note on sums involving the Euler function (English summary), *Bull. Aust. Math. Soc.* **100** (2019), 194–200.

[12] A. M. Güloğlu and C. W. Nevans, Sums with multiplicative functions over a Beatty sequences, *Bull. Austral. Math. Soc.* **78** (2008), 327–334.

[13] G. Harman, Primes in Beatty sequences in short intervals, *Mathematika* **62** (2016), 572–586.

[14] R. Heyman, Cardinality of a floor function set, Integers **19** (2019), A67.

[15] K. Liu, J. Wu and Z. Yang, On some sums involving the integral part function, *preprint*, Available at arXiv:2109.01382 [math.NT]

[16] K. Liu, J. Wu and Z. Yang, A variant of the Prime Number Theorem, *preprint*, Available at arXiv:2105.10844 [math.NT]

[17] K. Liu, I. E. Shparlinski and T. Zhang, Squares in Piatetski–Shapiro sequences, *Acta Arith.* **181** (2017), 239–252.

[18] J. Ma and J. Wu, On a sum involving the Mangoldt function (English summary), *Period. Math. Hungar.* **83** (2021), 39–48.

[19] J. Ma and H. Sun, On a sum involving certain arithmetic functions and the integral part function, *preprint*, Available at arXiv:2109.02924 [math.NT]

[20] J. F. Morgenbesser, The sum of digits of $\lfloor n^c \rfloor$, *Acta Arith.* **148** (2011), 367–393.

[21] J. Rosser, J. Barkley and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962) 64–94.

[22] J. Stucky, The fractional sum of small arithmetic functions, *preprint*, Available at arXiv:2106.14142 [Math.NT].

[23] J. Wu, On a sum involving the Euler totient function, *Indag. Math. (N.S.)* **30** (2019), 536–541.

[24] J. Wu, Note on a paper by Bordellès, Dai, Heyman, Pan and Shparlinski, *Period. Math. Hungar.* **80** (2020), 95–102.

[25] W. Zhai, On a sum involving the Euler function, *J. Number Theory* **211** (2020), 199–219.

[26] W. Zhai, Corrigendum to "On a sum involving the Euler function", *J. Number Theory* **222** (2021), 423–425.

[27] F. Zhao and J. Wu, On a Sum Involving the Sum-of-Divisors Function, *J. Math.* **2021**.