ON HYPER \(k\)-PELL, HYPER \(k\)-PELL-LUCAS AND HYPER MODIFIED \(k\)-PELL SEQUENCES

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Abstract
In this paper, the hyper \(k\)-Pell, hyper \(k\)-Pell-Lucas and hyper modified \(k\)-Pell sequences are introduced as well as some properties of each. Furthermore, properties about the relationships between them are presented. In addition, the convexity, concavity, log-concavity and log-convexity properties for these sequences are established.

1. Introduction
The Fibonacci numbers are defined by the recurrence relation \(F_{n+1} = F_n + F_{n-1}\), \(n \geq 1\) with \(F_0 = 0\) and \(F_1 = 1\) as initial conditions. Related to this sequence we can not fail to speak of the Lucas sequence \(\{L_n\}_{n=0}^{\infty}\), as well as several generalizations of both sequences. The Lucas sequence is defined by the recurrence relation \(L_{n+1} = L_n + L_{n-1}\), \(n \geq 1\) with \(L_0 = 2\) and \(L_1 = 1\).

Also, the Pell sequence \(\{P_n\}_{n=0}^{\infty}\) defined by \(P_{n+1} = 2P_n + P_{n-1}\), \(n \geq 1\) with initial conditions \(P_0 = 0\) and \(P_1 = 1\) has been studied by several researchers (see for instance, [10] and [13]). The same has happened for the Pell-Lucas sequence \(\{Q_n\}_{n=0}^{\infty}\) and modified Pell sequence \(\{q_n\}_{n=0}^{\infty}\) defined by a similar recurrence rela-
tion, but with different initial conditions, being \( Q_0 = Q_1 = 2 \) and \( q_0 = q_1 = 1 \), accordingly.

One of the generalizations of Pell sequences was introduced and studied by Catarino in [6] and [7], and Catarino and Vasco in [8] and [9]. Such sequences are denoted by \( \{P_{k,n}\}_{n=0}^\infty \), \( \{Q_{k,n}\}_{n=0}^\infty \) and \( \{q_{k,n}\}_{n=0}^\infty \), and they are known as the \( k \)-Pell, \( k \)-Pell-Lucas and modified \( k \)-Pell sequences, accordingly. For non-negative integers \( k \), these sequences are defined by the following recurrence relations and initial conditions:

\[
P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \quad P_{k,0} = 0, \quad P_{k,1} = 1, \quad n \geq 1, \tag{1.1}
\]

\[
Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}, \quad Q_{k,0} = 2, \quad Q_{k,1} = Q_{k,1} = 2, \quad n \geq 1, \tag{1.2}
\]

and

\[
q_{k,n+1} = 2q_{k,n} + kq_{k,n-1}, \quad q_{k,0} = q_{k,1} = 1, \quad n \geq 1, \tag{1.3}
\]

accordingly.

The corresponding characteristic equation of (1.1), (1.2) and (1.3) is

\[ x^2 - 2x - k = 0 \]

and its roots are \( r_1 = 1 + \sqrt{1 + k} \) and \( r_2 = 1 - \sqrt{1 + k} \). These roots verify the identities \( r_1 + r_2 = 2 \), \( r_1 - r_2 = 2\sqrt{1 + k} \) and \( r_1 r_2 = -k \), and the Binet formulae of these sequences are well known and given by

\[ P_{k,n} = \frac{(r_1)^n - (r_2)^n}{r_1 - r_2}, \]

\[ Q_{k,n} = (r_1)^n + (r_2)^n, \]

and

\[ q_{k,n} = \frac{(r_1)^n + (r_2)^n}{2}. \]

Using the Fibonacci and Lucas numbers, Zheng and Liu in [18] established the properties of the hyper-Fibonacci numbers \( F_n^{(r)} \) and the hyper-Lucas numbers \( L_n^{(r)} \) defined for non-negative integers \( r \) as

\[ F_n^{(r)} = \sum_{i=0}^n F_i^{(r-1)}, \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1 \]

and

\[ L_n^{(r)} = \sum_{i=0}^n L_i^{(r-1)}, \quad L_n^{(0)} = L_n, \quad L_0^{(r)} = 2, \quad L_1^{(r)} = 1. \]

For more details about these "hyper" sequences, see for instance, the works in [3], [4], [5], [12] and [14].

Our aim is to introduce and investigate the "hyper" approach for the \( k \)-Pell, \( k \)-Pell-Lucas and modified \( k \)-Pell sequences. More specifically, in Section 2, the definitions of hyper \( k \)-Pell, hyper \( k \)-Pell-Lucas and hyper modified \( k \)-Pell numbers
are presented and some of their properties are established. Finally, in Section 3, we discuss the concavity, convexity, log-concavity and log-convexity properties for these sequences.

2. The Hyper $k$-Pell, Hyper $k$-Pell-Lucas and Hyper Modified $k$-Pell Numbers

In this section, we consider the hyper $k$-Pell, hyper $k$-Pell-Lucas and hyper modified $k$-Pell numbers and some of their properties are investigated. We start with the definitions of these types of sequences.

For non-negative integers $r$, $n$ and $k$, the $n$th term of the hyper $k$-Pell, hyper $k$-Pell-Lucas and hyper modified $k$-Pell sequences are defined by

\[
P^{(r)}_{k,n} = \sum_{i=0}^{n} P^{(r-1)}_{k,i}, \quad P^{(0)}_{k,n} = P_{k,n}, \quad P^{(r)}_{k,0} = 0, \quad P^{(r)}_{k,1} = 1, \tag{2.1}
\]

\[
Q^{(r)}_{k,n} = \sum_{i=0}^{n} Q^{(r-1)}_{k,i}, \quad Q^{(0)}_{k,n} = Q_{k,n}, \quad Q^{(r)}_{k,0} = 2, \quad Q^{(r)}_{k,1} = 2(r + 1), \tag{2.2}
\]

and

\[
q^{(r)}_{k,n} = \sum_{i=0}^{n} q^{(r-1)}_{k,i}, \quad q^{(0)}_{k,n} = q_{k,n}, \quad q^{(r)}_{k,0} = 1, \quad q^{(r)}_{k,1} = r + 1, \tag{2.3}
\]

accordingly.

In the following tables we present a few hyper $k$-Pell, hyper $k$-Pell-Lucas and hyper modified $k$-Pell sequences.

We start with the hyper $k$-Pell sequences.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r = 0$</th>
<th>$r = 1$</th>
<th>$r = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P^{(0)}<em>{k,n} = P</em>{k,n}$</td>
<td>$P^{(1)}_{k,n}$</td>
<td>$P^{(2)}_{k,n}$</td>
</tr>
<tr>
<td>$n = 0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>$4 + k$</td>
<td>$7 + k$</td>
<td>$11 + k$</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>$8 + 4k$</td>
<td>$15 + 5k$</td>
<td>$26 + 6k$</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>$16 + 12k + k^2$</td>
<td>$31 + 17k + k^2$</td>
<td>$57 + 23k + k^2$</td>
</tr>
</tbody>
</table>

Table 1: The hyper $k$-Pell sequences for $r = 0, 1, 2$ and $0 \leq n \leq 5$.

In Table 2 we have the hyper $k$-Pell-Lucas sequences and in Table 3 we present the hyper modified $k$-Pell sequences.
Table 2: The hyper $k$-Pell-Lucas sequences for $r = 0, 1, 2$ and $0 \leq n \leq 5$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r = 0$</th>
<th>$r = 1$</th>
<th>$r = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q^{(0)}_{k,n}$</td>
<td>$Q^{(1)}_{k,n}$</td>
<td>$Q^{(2)}_{k,n}$</td>
<td></td>
</tr>
<tr>
<td>$n = 0$</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>$4 + 2k$</td>
<td>$8 + 2k$</td>
<td>$14 + 2k$</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>$8 + 6k$</td>
<td>$16 + 8k$</td>
<td>$30 + 10k$</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>$16 + 16k + 2k^2$</td>
<td>$32 + 24k + 2k^2$</td>
<td>$62 + 34k + 2k^2$</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>$32 + 40k + 10k^2$</td>
<td>$64 + 64k + 12k^2$</td>
<td>$126 + 98k + 14k^2$</td>
</tr>
</tbody>
</table>

Table 3: The hyper modified $k$-Pell sequences for $r = 0, 1, 2$ and $0 \leq n \leq 5$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r = 0$</th>
<th>$r = 1$</th>
<th>$r = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^{(0)}_{k,n}$</td>
<td>$q^{(1)}_{k,n}$</td>
<td>$q^{(2)}_{k,n}$</td>
<td></td>
</tr>
<tr>
<td>$n = 0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>$2 + k$</td>
<td>$4 + k$</td>
<td>$7 + k$</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>$4 + 3k$</td>
<td>$8 + 4k$</td>
<td>$15 + 5k$</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>$8 + 8k + k^2$</td>
<td>$16 + 12k + k^2$</td>
<td>$31 + 17k + k^2$</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>$16 + 20k + 5k^2$</td>
<td>$32 + 32k + 6k^2$</td>
<td>$63 + 49k + 7k^2$</td>
</tr>
</tbody>
</table>

The following proposition shows some useful identities in which some relations between the terms of these sequences are given.

**Proposition 1.** For non-negative integers $k$ and natural numbers $r$ and $n$, the hyper $k$-Pell, hyper $k$-Pell-Lucas and hyper modified $k$-Pell sequences satisfy the following identities:

1. $P^{(r)}_{k,n} = P^{(r-1)}_{k,n} + P^{(r)}_{k,n-1}$;
2. $Q^{(r)}_{k,n} = Q^{(r-1)}_{k,n} + Q^{(r)}_{k,n-1}$;
3. $q^{(r)}_{k,n} = q^{(r-1)}_{k,n} + q^{(r)}_{k,n-1}$.

**Proof.** For item (1) we use (2.1) and we obtain

$$P^{(r)}_{k,n} := \sum_{i=0}^{n} P^{(r-1)}_{k,i} = \sum_{i=0}^{n-1} P^{(r-1)}_{k,i} + P^{(r-1)}_{k,n} = P^{(r)}_{k,n-1} + P^{(r-1)}_{k,n}.$$
as required. Items (2) and (3) can be proved in a similar way using (2.2) and (2.3) accordingly.

Taking into account the previous proposition and the fact that $P_{k,n}^{(0)} = P_{k,n}$, $Q_{k,n}^{(0)} = Q_{k,n}$ and $q_{k,n}^{(0)} = q_{k,n}$, we have the following corollary for the particular case of $r = 1$.

**Corollary 1.** For natural numbers $n$, $r = 1$ and non-negative integers $k$, the hyper $k$-Pell, hyper $k$-Pell-Lucas and hyper modified $k$-Pell sequences satisfy the following identities:

1. $P_{k,n}^{(1)} = P_{k,n} + P_{k,n-1}$;
2. $Q_{k,n}^{(1)} = Q_{k,n} + Q_{k,n-1}$;
3. $q_{k,n}^{(1)} = q_{k,n} + q_{k,n-1}$.

For each of the three identities of Corollary 1, a different expression can be given which refers to the sum of the first $n$ terms of the sequences of $k$-Pell, $k$-Pell-Lucas and modified $k$-Pell numbers. Such expressions are shown in the following result.

**Proposition 2.** For non-negative integers $k$ and natural numbers $n$, the hyper $k$-Pell, hyper $k$-Pell-Lucas and hyper modified $k$-Pell sequences satisfy the following identities:

1. $P_{k,n}^{(1)} = \frac{1}{k+1} (-1 + q_{k,n})$;
2. $Q_{k,n}^{(1)} = \frac{1}{k+1} (Q_{k,n+1} + kQ_{k,n})$;
3. $q_{k,n}^{(1)} = \frac{1}{2(k+1)} (Q_{k,n+1} + kQ_{k,n})$.

**Proof.** For the first identity, by successively applying item (1) of Corollary 1, we obtain

$$P_{k,n}^{(1)} = P_{k,n} + P_{k,n-1} = \cdots = (P_{k,n} + P_{k,n-1} + \cdots + P_{k,1}) + P_{k,0}.$$

Now taking into account Proposition 3 of [7] and the fact that $Q_{k,j} = 2q_{k,j}$, the
result is obtained as follows:

\[
(P_{k,n} + P_{k,n-1} + \cdots + P_{k,1}) + P_{k,0}^{(1)} = \frac{1}{k+1} \left( -1 + kP_{k,n} + (k+2)P_{k,n+1} \right) - P_{k,n+1}
\]

\[
= \frac{1}{k+1} \left( -1 + kP_{k,n} + \left( \frac{k+2}{k+1} - 1 \right) P_{k,n+1} \right)
\]

\[
= \frac{1}{k+1} \left( -1 + kP_{k,n} + \frac{1}{k+1} P_{k,n+1} \right)
\]

\[
= \frac{1}{k+1} \left( -1 + kP_{k,n} + P_{k,n+1} \right)
\]

\[
= \frac{1}{k+1} \left( -1 + (P_{k,n+2} - 2P_{k,n+1}) + P_{k,n+1} \right)
\]

\[
= \frac{1}{k+1} \left( -1 + (P_{k,n+2} - P_{k,n+1}) \right)
\]

\[
= \frac{1}{k+1} \left( -1 + \frac{1}{2} Q_{k,n+1} \right)
\]

\[
= \frac{1}{k+1} \left( -1 + q_{k,n+1} \right).
\]

For item (2) we use (2.2), the fact that \(Q^{(0)}_{k,n} = Q_{k,n}\), the initial condition \(Q_{k,0} = 2\) and item (1) of Proposition 3 of [15] and then

\[
Q_{k,n}^{(1)} = \sum_{i=0}^{n} Q^{(0)}_{k,i} = Q_{k,0} + Q_{k,1} + \cdots + Q_{k,n-1} + Q_{k,n}
\]

\[
= 2 + \sum_{j=1}^{n} Q_{k,j} = 2 + \frac{1}{k+1} (Q_{k,n+1} + kQ_{k,n}) - Q_{k,0}
\]

\[
= \frac{1}{k+1} (Q_{k,n+1} + kQ_{k,n})
\]

as required.

As for the last item of this proposition, using the fact that \(Q_{k,n} = 2q_{k,n}\) and item (2) of Proposition 3 of [15], it is easy to show the last identity.

The next result shows other different expressions for \(q_{k,n}^{(1)}\) and \(Q_{k,n}^{(1)}\) in terms of \(P_{k,n+1}^{(1)}\).

**Lemma 1.** For non-negative integers \(k\) and natural numbers \(n\), the hyper \(k\)-Pell, hyper \(k\)-Pell-Lucas and hyper modified \(k\)-Pell sequences satisfy the following identities:

1. \(Q_{k,n}^{(1)} = 2q_{k,n}^{(1)}\).
2. \( Q_{k,n}^{(1)} = 2P_{k,n+1} \);
3. \( q_{k,n}^{(1)} = P_{k,n+1} \).

**Proof.** To prove item (1) it is sufficient to use items (2) and (3) of Proposition 2 and obviously the statement holds.

For the proof of item (2), using item (2) of Proposition 2, Proposition 3 of [7], the recurrence relation (1.1) and finally item (1) of Proposition 2 we have

\[
Q_{k,n}^{(1)} = \frac{1}{k+1} (Q_{k,n+1} + kQ_{k,n}) \\
= \frac{1}{k+1} (2P_{k,n+2} - 2P_{k,n+1} + k(2P_{k,n+1} - 2P_{k,n})) \\
= \frac{2}{k+1} (P_{k,n+2} - P_{k,n+1} + kP_{k,n+1} - kP_{k,n}) \\
= \frac{2}{k+1}((-1 + kP_{k,n+1} + P_{k,n+2}) - (-1 + kP_{k,n} + P_{k,n+1})) + 1 - 1 \\
= 2(P_{k,n+1}^{(1)} - P_{k,n}^{(1)}) \\
= 2P_{k,n+1},
\]

as required.

Finally, item (3) is easily proved taking into account the previous items of this lemma.

Now we generalize the identities of Lemma 1 involving sequences in the following proposition.

**Proposition 3.** For non-negative integers \( k \) and natural numbers \( n \) and \( r \), the hyper \( k \)-Pell, hyper \( k \)-Pell-Lucas and hyper modified \( k \)-Pell sequences satisfy the following identities:

1. \( Q_{k,n}^{(r)} = 2q_{k,n}^{(r)} \);
2. \( Q_{k,n}^{(r)} = 2P_{k,n+1}^{(r-1)} \);
3. \( q_{k,n}^{(r)} = P_{k,n+1}^{(r-1)} \).

**Proof.** For the first identity we use induction on \( r \). For \( r = 1 \), taking into account item (1) of Lemma 1, the statement holds. Suppose that the statement is true for \( r \), we also show that it is true for \( r + 1 \). We have

\[
Q_{k,n}^{(r+1)} = \sum_{i=0}^{n} Q_{k,i}^{(r)} = \sum_{i=0}^{n} 2q_{k,i}^{(r)} = 2 \sum_{i=0}^{n} q_{k,i}^{(r)} = 2q_{k,n}^{(r+1)},
\]
which leads us to conclude the proof of item (1).

For the proof of the second identity we use a similar method as before. For \( r = 1 \), using item (2) of Lemma 1, the result follows. Now suppose that the statement is true for \( r \) and we show that it is also true for \( r + 1 \). Therefore, using \( P^{(r-1)}_{k,0} = 0 \), we get

\[
Q^{(r+1)}_{k,n} = \sum_{i=0}^{n} Q_{k,i}^{(r)} = 2 \sum_{i=0}^{n} P_{k,i+1}^{(r-1)} = 2 \sum_{i=0}^{n+1} P_{k,i}^{(r-1)} = 2P^{(r)}_{k,n+1}.
\]

This completes the proof of item (2). Finally, by the use of items (1) and (2) the last identity immediately follows.

Note that the identity of item (1) of the previous proposition is also true for \( r = 0 \). In fact, given that \( Q_{k,n} = 2q_{k,n} \) we get

\[
Q^{(0)}_{k,n} = Q_{k,n} = 2q_{k,n} = 2q^{(0)}_{k,n}
\]

and the statement is true in this case as well.

The next result gives other relationship between \( q_{k,n} \) and \( P_{k,n} \), which will be useful later in this paper.

**Lemma 2.** For non-negative integers \( k \) and natural numbers \( n \), the following identity holds:

\[
\left(P^{(1)}_{k,n}\right)^2 - P^{(1)}_{k,n-1}P^{(1)}_{k,n+1} = \frac{1}{k+1} (q_{k,n} - (-k)^n).
\]

**Proof.** Using item (1) of Corollary 1, (2.1) and item (1) of Proposition 2, we obtain

\[
\left(P^{(1)}_{k,n}\right)^2 - P^{(1)}_{k,n-1}P^{(1)}_{k,n+1} = (P_{k,n})^2 + 2P_{k,n}P^{(1)}_{k,n-1} + P^{(1)}_{k,n-1} (-3P_{k,n} - kP_{k,n-1})
\]

\[
= (P_{k,n})^2 - P^{(1)}_{k,n-1} (P_{k,n} + kP_{k,n-1})
\]

\[
= (P_{k,n})^2 - \frac{1}{k+1} (-1 + kP_{k,n-1} + P_{k,n}) (kP_{k,n-1} + P_{k,n})
\]

\[
= (P_{k,n})^2 - \frac{1}{k+1} \left((P_{k,n+1} - P_{k,n})^2 - kP_{k,n-1} - P_{k,n}\right).
\]

Using item \((ii)\) of Proposition 2.1 of [16] and after some calculations, the result finally follows.

3. The Concavity, Convexity, Log-Concavity and Log-Convexity Properties

In this section, considering various works, such as [1], [2], [11] and [17], we discuss the concavity, convexity, log-concavity and log-convexity properties for hyper \( k \)-Pell, hyper \( k \)-Pell-Lucas and hyper modified \( k \)-Pell sequences.
A given sequence of positive real numbers \( \{a_n\}_{n \geq 0} \) is said to be log-concave (respectively, log-convex) if \( a_n^2 \geq a_{n-1}a_{n+1} \) (respectively, \( a_n^2 \leq a_{n-1}a_{n+1} \)) for all \( n \geq 1 \). Also, we say that the sequence \( \{a_n\}_{n \geq 0} \) is convex (respectively, concave) if for \( n \geq 1 \), \( a_{n-1} + a_{n+1} \geq 2a_n \) (respectively, \( a_{n-1} + a_{n+1} \leq 2a_n \)).

As for the concavity property, if we consider for example \( r = 0 \) and \( n = 2 \), we get
\[
P^{(0)}_{k,1} + P^{(0)}_{k,3} = P_{k,1} + P_{k,3} = 5 + k > 4 = 2P_{k,2} = 2P^{(0)}_{k,2}
\]
for all non-negative integers \( k \), thus we conclude that the hyper \( k \)-Pell sequences are not concave, for all non-negative integers \( k \) and \( r \), and natural numbers \( n \). Similarly, we can conclude that hyper \( k \)-Pell-Lucas and hyper modified \( k \)-Pell sequences are not concave. About the convexity property, we have the following proposition.

**Proposition 4.** For non-negative integers \( k, r \) and natural numbers \( n \), the hyper \( k \)-Pell, hyper \( k \)-Pell-Lucas and hyper modified \( k \)-Pell sequences are convex.

**Proof.** For non-negative integers \( k \), natural numbers \( n \), and \( r = 0 \), by (2.1) and (1.1), we have
\[
P^{(0)}_{k,n-1} + P^{(0)}_{k,n+1} = P_{k,n-1} + P_{k,n+1}
= P_{k,n-1} + (2P_{k,n} + kP_{k,n-1})
= 2P_{k,n} + (k + 1)P_{k,n-1}
\geq 2P_{k,n}
= 2P^{(0)}_{k,n}.
\]
For non-negative integers \( k \), natural numbers \( n \) and \( r = 1 \), by the use of item (1) of Corollary 1 and (1.1), we obtain
\[
P^{(1)}_{k,n-1} + P^{(1)}_{k,n+1} = P^{(1)}_{k,n-1} + (P_{k,n+1} + P^{(1)}_{k,n})
= P^{(1)}_{k,n-1} + (2P_{k,n} + kP_{k,n-1}) + P^{(1)}_{k,n}
= 2P_{k,n} + P^{(1)}_{k,n-1} + kP_{k,n-1} + (P_{k,n} + P^{(1)}_{k,n-1})
= 3P_{k,n} + 2P^{(1)}_{k,n-1} + kP_{k,n-1}
\geq 2P_{k,n} + 2P^{(1)}_{k,n-1}
= 2P^{(1)}_{k,n-1}.
\]
We now consider \( r \geq 2 \) and the hyper \( k \)-Pell sequence. By the use of (2.1) and
Proposition 1 we have

\[ P_{k,n-1}^{(r)} + P_{k,n+1}^{(r)} = \sum_{i=0}^{n-1} P_{k,i}^{(r-1)} + \sum_{i=0}^{n+1} P_{k,i}^{(r-1)} \]

\[ = 2 \sum_{i=0}^{n-1} P_{k,i}^{(r-1)} + P_{k,n}^{(r-1)} + P_{k,n+1}^{(r-1)} \]

\[ = 2P_{k,n}^{(r)} - \left( P_{k,n}^{(r-1)} - P_{k,n+1}^{(r-1)} \right) \]

\[ = 2P_{k,n}^{(r)} - (-P_{k,n+1}^{(r-2)}) \]

\[ = 2P_{k,n}^{(r)} - P_{k,n+1}^{(r-2)} \geq 2P_{k,n}^{(r)}, \]

as required. For the other sequences the proofs are similar and for this reason we have chosen not to present them.

In the context of log-concavity and log-convexity, we consider the hyper $k$-Pell, hyper $k$-Pell-Lucas and hyper modified $k$-Pell sequences for the case of $r = 0$. First of all, we establish the following lemma which will be necessary in some proofs.

Lemma 3. For non-negative integers $j$ and $k$, the following identity

\[ P_{k,j} P_{k,j+2} = \frac{1}{4(k+1)} \left( Q_{k,2j+2} - 2(-k)^j(2 + k) \right) \]

holds.

Proof. We use the Binet formulas for the sequences of $k$-Pell and $k$-Pell-Lucas and also the sum, product and difference between the roots $r_1$ and $r_2$ mentioned in Section 1. We have

\[ P_{k,j} P_{k,j+2} = \frac{(r_1)^j - (r_2)^j}{r_1 - r_2} \frac{(r_1)^{j+2} - (r_2)^{j+2}}{r_1 - r_2} \]

\[ = \frac{(r_1)^{2j+2} - (r_1)^j(r_2)^{j+2} - (r_1)^j(r_2)^{j+2} + (r_2)^{2j+2}}{r_1 - r_2} \]

\[ = \frac{((r_1)^{2j+2} + (r_2)^{2j+2}) - 2(-k)^j(2 + k)}{4(k+1)} \]

\[ = \frac{1}{4(k+1)} \left( Q_{k,2j+2} - 2(-k)^j(2 + k) \right), \]

and this completes the proof.

For the particular case of $r = 0$ we give the first result.
Proposition 5. For non-negative integers \( k \) and \( r \), and natural numbers \( n \), the hyper \( k \)-Pell, hyper \( k \)-Pell-Lucas and hyper modified \( k \)-Pell sequences are log-convex if \( n \) is even and they are log-concave if \( n \) is odd.

Proof. First we prove this for the hyper \( k \)-Pell sequence. By (2.1), Lemma 3 and item (iv) of Proposition 2.1 in [16] we have

\[
\left( P_{k,n}^{(0)} \right)^2 - P_{k,n-1}^{(0)} P_{k,n+1}^{(0)} = \left( P_{k,n} \right)^2 - P_{k,n-1} P_{k,n+1} = \frac{1}{4(k+1)} \left( (Q_{k,2n} - 2 (-k)^{n}) - (Q_{k,2n} - 2 (-k)^{n-1} (2 + k)) \right) = (-1)^{n+1} (k)^{n-1},
\]

and the result follows. For the hyper \( k \)-Pell-Lucas sequence, using (2.2) and the respective Binet formula we obtain

\[
\left( Q_{k,n}^{(0)} \right)^2 - Q_{k,n-1}^{(0)} Q_{k,n+1}^{(0)} = 4 (-k)^{n-1} (k + 1),
\]

and therefore it is easy to prove that for \( n \) even, \( \left( Q_{k,n}^{(0)} \right)^2 - Q_{k,n-1}^{(0)} Q_{k,n+1}^{(0)} \leq 0 \), and for \( n \) odd we have the reverse inequality, as required. For the proof of this result for the hyper modified \( k \)-Pell sequence, it suffices to recall item (1) of Proposition 3.

Taking into consideration Lemma 1 and Lemma 3, we have

\[
\left( Q_{k,n}^{(1)} \right)^2 - Q_{k,n-1}^{(1)} Q_{k,n+1}^{(1)} = 4 \left( (P_{k,n+1})^2 - \frac{1}{4(k+1)} (Q_{k,2n+2} - 2 (-k)^{n} (2 + k)) \right) = 4 (-k)^{n}.
\]

When \( n = 1 \), the last identity gives

\[
\left( Q_{k,1}^{(1)} \right)^2 - Q_{k,0}^{(1)} Q_{k,2}^{(1)} = -4k \leq 0
\]

for all non-negative integers \( k \). However, when \( n = 2 \) the respective result is

\[
\left( Q_{k,2}^{(1)} \right)^2 - Q_{k,1}^{(1)} Q_{k,3}^{(1)} = 4k^2 \geq 0
\]

for all non-negative integers \( k \), which shows that the sequence \( \{ Q_{k,n}^{(1)} \}_{n \geq 1} \) is not log-concave. Moreover, as \( 2Q_{k,n}^{(1)} = q_{k,n}^{(1)} \), the sequence \( \{ q_{k,n}^{(1)} \}_{n \geq 1} \) is also not log-concave.

Corollary 2. For non-negative integers \( k \) and natural numbers \( n \), the sequences \( \{ \sum_{i=0}^{n} \binom{n}{k} P_{k,n}^{(0)} \}_{n \geq 1}, \{ \sum_{i=0}^{n} \binom{n}{k} Q_{k,n}^{(0)} \}_{n \geq 1} \) and \( \{ \sum_{i=0}^{n} \binom{n}{k} q_{k,n}^{(0)} \}_{n \geq 1} \) are log-convex if \( n \) is even and log-concave if \( n \) is odd.
Proof. When \( n \) is even, by [1, Lemma 3.3] and [11] the result follows. Analogously, by [1, Lemma 3.2] and [17] we conclude the result when \( n \) is odd.

The next result continues the discussion of the log-concavity of hyper \( k \)-Pell, hyper \( k \)-Pell-Lucas and hyper modified \( k \)-Pell sequences.

**Proposition 6.** For non-negative integers \( k \) and natural numbers \( n \), the sequence \( \{P_{k,n}^{(r)}\}_{n \geq 1} \) is log-concave for \( r \geq 1 \) and \( n \) odd.

**Proof.** Taking into account Lemma 2 and the fact that \( n \) is odd, it appears that

\[
\left( P_{k,n}^{(1)} \right)^2 - P_{k,n-1}^{(1)} P_{k,n+1}^{(1)} \geq 0 \quad (3.1)
\]

from which it is concluded that for \( r = 1 \) the sequence \( \{P_{k,n}^{(r)}\}_{n \geq 1} \) is log-concave for \( n \) odd.

By [1, Lemma 3.1] and [17], we derive the result when \( n \) is odd and \( r \geq 1 \).

As a consequence of this result we have the following corollary.

**Corollary 3.** For non-negative integers \( k \) and natural numbers \( n \), the sequence \( \{\sum_{i=0}^{n} \binom{n}{i} P_{k,n}^{(r)}\}_{n \geq 1} \) is log-concave for \( r \geq 1 \) and \( n \) odd.

**Proof.** Using Lemma 3.2 of [1], the result follows.

Note that for \( n \) even, the inequality (3.1) does not hold for some non-negative integers \( k \). For example, if we consider \( n = 2 \) in (3.1), we obtain that

\[
\left( P_{k,2}^{(1)} \right)^2 - P_{k,1}^{(1)} P_{k,3}^{(1)} = \frac{1}{k+1} (q_{k,2} - k^2) = \frac{-1}{k+1} (k + 1) (k - 2) = 2 - k
\]

and then, for \( k \geq 3 \) this is negative.

The next two propositions and the corresponding corollaries give us not only the log-concavity property of the hyper \( k \)-Pell-Lucas and hyper modified \( k \)-Pell sequences, but also the log-concavity property of other sequences related to them.

**Proposition 7.** For non-negative integers \( k \) and natural numbers \( n \), the sequence \( \{Q_{k,n}^{(r)}\}_{n \geq 1} \) is log-concave for \( r \geq 2 \).

**Proof.** This proof can be done by induction on \( r \) and taking into account Propositions 3 and 6.

**Corollary 4.** For non-negative integers \( k \) and natural numbers \( n \), the sequence \( \{\sum_{i=0}^{n} \binom{n}{i} Q_{k,n}^{(r)}\}_{n \geq 1} \) is log-concave for \( r \geq 2 \).

**Proof.** Using Lemma 3.2 from [1], the result is easy to prove.
Proposition 8. For non-negative integers $k$ and natural numbers $n$, the sequence\[ \{q^{(r)}_{k,n}\}_{n \geq 1} \text{ is log-concave for } r \geq 2. \]

Proof. This statement is obvious by Propositions 3 and 6.

Corollary 5. For non-negative integers $k$ and natural numbers $n$, the sequence\[ \{\sum_{i=0}^{n} \binom{n}{i} q^{(r)}_{k,n}\}_{n \geq 1} \text{ is log-concave for } r \geq 2. \]

Proof. Using Lemma 3.2 of [1], the result follows.

Taking into consideration Propositions 6, 7 and 8, and Lemma 3.1 of [1], it is easy to prove the following proposition, and hence we omit the proof.

Proposition 9. For non-negative integers $k$, natural numbers $n$ and $r \geq 2$, the sequences\[ \{\sum_{i=0}^{n} \binom{n}{k} P^{(r)}_{k,n} Q^{(r)}_{k,n}\}_{n \geq 1} , \{\sum_{i=0}^{n} \binom{n}{k} P^{(r)}_{k,n} q^{(r)}_{k,n}\}_{n \geq 1} \text{ and } \{\sum_{i=0}^{n} \binom{n}{k} Q^{(r)}_{k,n} q^{(r)}_{k,n}\}_{n \geq 1} \text{ are log-concave.} \]

From Proposition 9 we immediately have the following corollary.

Corollary 6. For non-negative integers $k$, natural numbers $n$ and $r \geq 2$, the sequences\[ \{\sum_{i=0}^{n} \binom{n}{k} P^{(r)}_{k,n} Q^{(r)}_{k,n}\}_{n \geq 1} , \{\sum_{i=0}^{n} \binom{n}{k} P^{(r)}_{k,n} q^{(r)}_{k,n}\}_{n \geq 1} \text{ and } \{\sum_{i=0}^{n} \binom{n}{k} Q^{(r)}_{k,n} q^{(r)}_{k,n}\}_{n \geq 1} \text{ are log-concave.} \]

Proof. Using [17] and [1, Lemma 3.3] the result follows.

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References


