ON HYPER k-PELL, HYPER k-PELL-LUCAS AND HYPER MODIFIED k-PELL SEQUENCES

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Abstract

In this paper, the hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences are introduced as well as some properties of each. Furthermore, properties about the relationships between them are presented. In addition, the convexity, concavity, log-concavity and log-convexity properties for these sequences are established.

1. Introduction

The Fibonacci numbers are defined by the recurrence relation $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$ with $F_0 = 0$ and $F_1 = 1$ as initial conditions. Related to this sequence we can not fail to speak of the Lucas sequence $\{L_n\}_{n=0}^{\infty}$, as well as several generalizations of both sequences. The Lucas sequence is defined by the recurrence relation $L_{n+1} = L_n + L_{n-1},\ n \geq 1$ with $L_0 = 2$ and $L_1 = 1$.

Also, the Pell sequence $\{P_n\}_{n=0}^{\infty}$ defined by $P_{n+1} = 2P_n + P_{n-1},\ n \geq 1$ with initial conditions $P_0 = 0$ and $P_1 = 1$ has been studied by several researchers (see for instance, [10] and [13]). The same has happened for the Pell-Lucas sequence $\{Q_n\}_{n=0}^{\infty}$ and modified Pell sequence $\{q_n\}_{n=0}^{\infty}$ defined by a similar recurrence relation, but with different initial conditions, being $Q_0 = Q_1 = 2$ and $q_0 = q_1 = 1$, accordingly.

One of the generalizations of Pell sequences was introduced and studied by Catarino in [6] and [7], and Catarino and Vasco in [8] and [9]. Such sequences are denoted by $\{P_{k,n}\}_{n=0}^{\infty}$, $\{Q_{k,n}\}_{n=0}^{\infty}$ and $\{q_{k,n}\}_{n=0}^{\infty}$, and they are known as the k-Pell, k-Pell-Lucas and modified k-Pell sequences, accordingly. For non-negative integers k, these sequences are defined by the following recurrence relations and initial conditions:

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \quad P_{k,0} = 0, \quad P_{k,1} = 1, \quad n \geq 1, \tag{1.1}$$

$$Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}, \quad Q_{k,0} = Q_{k,1} = 2, \quad n \geq 1, \tag{1.2}$$

and

$$q_{k,n+1} = 2q_{k,n} + kq_{k,n-1}, \quad q_{k,0} = q_{k,1} = 1, \quad n \geq 1, \tag{1.3}$$

accordingly.

The corresponding characteristic equation of (1.1), (1.2) and (1.3) is $x^2 - 2x - k = 0$ and its roots are $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$. These roots verify the identities $r_1 + r_2 = 2$, $r_1 - r_2 = 2\sqrt{1+k}$ and $r_1 r_2 = -k$, and the Binet formulae of these sequences are well known and given by

$$P_{k,n} = \frac{(r_1)^n - (r_2)^n}{r_1 - r_2},$$
$$Q_{k,n} = (r_1)^n + (r_2)^n,$$

and

$$q_{k,n} = \frac{(r_1)^n + (r_2)^n}{2}.$$

Using the Fibonacci and Lucas numbers, Zheng and Liu in [18] established the properties of the hyper-Fibonacci numbers $F_n^{(r)}$ and the hyper-Lucas numbers $L_n^{(r)}$ defined for non-negative integers r as

$$F_n^{(r)} = \sum_{i=0}^{n} F_i^{(r-1)}, \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \; F_1^{(r)} = 1$$

and

$$L_n^{(r)} = \sum_{i=0}^{n} L_i^{(r-1)}, \quad L_n^{(0)} = L_n, \quad L_0^{(r)} = 2, \; L_1^{(r)} = 1.$$

For more details about these "hyper" sequences, see for instance, the works in [3], [4], [5], [12] and [14].

Our aim is to introduce and investigate the "hyper" approach for the k-Pell, k-Pell-Lucas and modified k-Pell sequences. More specifically, in Section 2, the definitions of hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell numbers are presented and some of their properties are established. Finally, in Section 3, we discuss the concavity, convexity, log-concavity and log-convexity properties for these sequences.

2. The Hyper k-Pell, Hyper k-Pell-Lucas and Hyper Modified k-Pell Numbers

In this section, we consider the hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell numbers and some of their properties are investigated. We start with the definitions of these types of sequences.

For non-negative integers r, n and k, the nth term of the hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences are defined by

$$P_{k,n}^{(r)} = \sum_{i=0}^{n} P_{k,i}^{(r-1)},\ P_{k,n}^{(0)} = P_{k,n},\ P_{k,0}^{(r)} = 0,\ P_{k,1}^{(r)} = 1, \tag{2.1}$$

$$Q_{k,n}^{(r)} = \sum_{i=0}^{n} Q_{k,i}^{(r-1)},\ Q_{k,n}^{(0)} = Q_{k,n},\ Q_{k,0}^{(r)} = 2,\ Q_{k,1}^{(r)} = 2(r+1), \tag{2.2}$$

and

$$q_{k,n}^{(r)} = \sum_{i=0}^{n} q_{k,i}^{(r-1)},\ q_{k,n}^{(0)} = q_{k,n},\ q_{k,0}^{(r)} = 1,\ q_{k,1}^{(r)} = r+1, \tag{2.3}$$

accordingly.

In the following tables we present a few hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences.

We start with the hyper k-Pell sequences.

$n \backslash r$	$r=0$	$r=1$	$r=2$
	$P_{k,n}^{(0)} = P_{k,n}$	$P_{k,n}^{(1)}$	$P_{k,n}^{(2)}$
$n=0$	0	0	0
$n=1$	1	1	1
$n=2$	2	3	4
$n=3$	$4+k$	$7+k$	$11+k$
$n=4$	$8+4k$	$15+5k$	$26+6k$
$n=5$	$16+12k+k^2$	$31+17k+k^2$	$57+23k+k^2$

Table 1: The hyper k-Pell sequences for $r=0,1,2$ and $0 \le n \le 5$.

In Table 2 we have the hyper k-Pell-Lucas sequences and in Table 3 we present the hyper modified k-Pell sequences.

n \ r	$r=0$	$r=1$	$r=2$
	$Q_{k,n}^{(0)} = Q_{k,n}$	$Q_{k,n}^{(1)}$	$Q_{k,n}^{(2)}$
$n=0$	2	2	2
$n=1$	2	4	6
$n=2$	$4+2k$	$8+2k$	$14+2k$
$n=3$	$8+6k$	$16+8k$	$30+10k$
$n=4$	$16+16k+2k^2$	$32+24k+2k^2$	$62+34k+2k^2$
$n=5$	$32+40k+10k^2$	$64+64k+12k^2$	$126+98k+14k^2$

Table 2: The hyper k-Pell-Lucas sequences for $r = 0, 1, 2$ and $0 \leq n \leq 5$.

n \ r	$r=0$	$r=1$	$r=2$
	$q_{k,n}^{(0)} = q_{k,n}$	$q_{k,n}^{(1)}$	$q_{k,n}^{(2)}$
$n=0$	1	1	1
$n=1$	1	2	3
$n=2$	$2+k$	$4+k$	$7+k$
$n=3$	$4+3k$	$8+4k$	$15+5k$
$n=4$	$8+8k+k^2$	$16+12k+k^2$	$31+17k+k^2$
$n=5$	$16+20k+5k^2$	$32+32k+6k^2$	$63+49k+7k^2$

Table 3: The hyper modified k-Pell sequences for $r = 0, 1, 2$ and $0 \leq n \leq 5$.

The following proposition shows some useful identities in which some relations between the terms of these sequences are given.

Proposition 1. *For non-negative integers k and natural numbers r and n, the hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences satisfy the following identities:*

1. $P_{k,n}^{(r)} = P_{k,n}^{(r-1)} + P_{k,n-1}^{(r)}$;
2. $Q_{k,n}^{(r)} = Q_{k,n}^{(r-1)} + Q_{k,n-1}^{(r)}$;
3. $q_{k,n}^{(r)} = q_{k,n}^{(r-1)} + q_{k,n-1}^{(r)}$.

Proof. For item (1) we use (2.1) and we obtain

$$P_{k,n}^{(r)} := \sum_{i=0}^{n} P_{k,i}^{(r-1)} = \sum_{i=0}^{n-1} P_{k,i}^{(r-1)} + P_{k,n}^{(r-1)} = P_{k,n-1}^{(r)} + P_{k,n}^{(r-1)},$$

as required. Items (2) and (3) can be proved in a similar way using (2.2) and (2.3) accordingly. □

Taking into account the previous proposition and the fact that $P^{(0)}_{k,n} = P_{k,n}$, $Q^{(0)}_{k,n} = Q_{k,n}$ and $q^{(0)}_{k,n} = q_{k,n}$, we have the following corollary for the particular case of $r = 1$.

Corollary 1. *For natural numbers n, $r = 1$ and non-negative integers k, the hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences satisfy the following identities:*

1. $P^{(1)}_{k,n} = P_{k,n} + P^{(1)}_{k,n-1}$;
2. $Q^{(1)}_{k,n} = Q_{k,n} + Q^{(1)}_{k,n-1}$;
3. $q^{(1)}_{k,n} = q_{k,n} + q^{(1)}_{k,n-1}$.

For each of the three identities of Corollary 1, a different expression can be given which refers to the sum of the first n terms of the sequences of k-Pell, k-Pell-Lucas and modified k-Pell numbers. Such expressions are shown in the following result.

Proposition 2. *For non-negative integers k and natural numbers n, the hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences satisfy the following identities:*

1. $P^{(1)}_{k,n} = \frac{1}{k+1}\left(-1 + q_{k,n}\right)$;
2. $Q^{(1)}_{k,n} = \frac{1}{k+1}\left(Q_{k,n+1} + kQ_{k,n}\right)$;
3. $q^{(1)}_{k,n} = \frac{1}{2(k+1)}\left(Q_{k,n+1} + kQ_{k,n}\right)$.

Proof. For the first identity, by successively applying item (1) of Corollary 1, we obtain

$$P^{(1)}_{k,n} = P_{k,n} + P^{(1)}_{k,n-1} = \cdots = (P_{k,n} + P_{k,n-1} + \ldots + P_{k,1}) + P^{(1)}_{k,0}.$$

Now taking into account Proposition 3 of [7] and the fact that $Q_{k,j} = 2q_{k,j}$, the result is obtained as follows:

$$\begin{aligned}
(P_{k,n} + P_{k,n-1} + \cdots + P_{k,1}) + P_{k,0}^{(1)} &= \frac{1}{k+1}\left(-1 + kP_{k,n} + (k+2)\,P_{k,n+1}\right) - P_{k,n+1} \\
&= \frac{1}{k+1}\left(-1 + kP_{k,n}\right) + \left(\frac{k+2}{k+1} - 1\right) P_{k,n+1} \\
&= \frac{1}{k+1}\left(-1 + kP_{k,n}\right) + \left(\frac{1}{k+1}\right) P_{k,n+1} \\
&= \frac{1}{k+1}\left(-1 + kP_{k,n} + P_{k,n+1}\right) \\
&= \frac{1}{k+1}\left(-1 + (P_{k,n+2} - 2P_{k,n+1}) + P_{k,n+1}\right) \\
&= \frac{1}{k+1}\left(-1 + (P_{k,n+2} - P_{k,n+1})\right) \\
&= \frac{1}{k+1}\left(-1 + \frac{1}{2}Q_{k,n+1}\right) \\
&= \frac{1}{k+1}\left(-1 + q_{k,n+1}\right).
\end{aligned}$$

For item (2) we use (2.2), the fact that $Q_{k,n}^{(0)} = Q_{k,n}$, the initial condition $Q_{k,0} = 2$ and item (1) of Proposition 3 of [15] and then

$$\begin{aligned}
Q_{k,n}^{(1)} &= \sum_{i=0}^{n} Q_{k,i}^{(0)} = Q_{k,0} + Q_{k,1} + \cdots + Q_{k,n-1} + Q_{k,n} \\
&= 2 + \sum_{j=1}^{n} Q_{k,j} = 2 + \frac{1}{k+1}\left(Q_{k,n+1} + kQ_{k,n}\right) - Q_{k,0} \\
&= \frac{1}{k+1}\left(Q_{k,n+1} + kQ_{k,n}\right),
\end{aligned}$$

as required.

As for the last item of this proposition, using the fact that $Q_{k,n} = 2q_{k,n}$ and item (2) of Proposition 3 of [15], it is easy to show the last identity. □

The next result shows other different expressions for $q_{k,n}^{(1)}$ and $Q_{k,n}^{(1)}$ in terms of $P_{k,n+1}^{(1)}$.

Lemma 1. *For non-negative integers k and natural numbers n, the hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences satisfy the following identities:*

1. $Q_{k,n}^{(1)} = 2q_{k,n}^{(1)}$;

2. $Q^{(1)}_{k,n} = 2P_{k,n+1}$;

3. $q^{(1)}_{k,n} = P_{k,n+1}$.

Proof. To prove item (1) it is sufficient to use items (2) and (3) of Proposition 2 and obviously the statement holds.

For the proof of item (2), using item (2) of Proposition 2, Proposition 3 of [7], the recurrence relation (1.1) and finally item (1) of Proposition 2 we have

$$\begin{aligned}
Q^{(1)}_{k,n} &= \frac{1}{k+1}\left(Q_{k,n+1} + kQ_{k,n}\right)\\
&= \frac{1}{k+1}\left(2P_{k,n+2} - 2P_{k,n+1} + k\left(2P_{k,n+1} - 2P_{k,n}\right)\right)\\
&= \frac{2}{k+1}\left(P_{k,n+2} - P_{k,n+1} + kP_{k,n+1} - kP_{k,n}\right)\\
&= \frac{2}{k+1}\left(\left(-1 + kP_{k,n+1} + P_{k,n+2}\right) - \left(-1 + kP_{k,n} + P_{k,n+1}\right)\right) + 1 - 1\\
&= 2\left(P^{(1)}_{k,n+1} - P^{(1)}_{k,n}\right)\\
&= 2P_{k,n+1},
\end{aligned}$$

as required.

Finally, item (3) is easily proved taking into account the previous items of this lemma. □

Now we generalize the identities of Lemma 1 involving sequences in the following proposition.

Proposition 3. *For non-negative integers k and natural numbers n and r, the hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences satisfy the following identities:*

1. $Q^{(r)}_{k,n} = 2q^{(r)}_{k,n}$;

2. $Q^{(r)}_{k,n} = 2P^{(r-1)}_{k,n+1}$;

3. $q^{(r)}_{k,n} = P^{(r-1)}_{k,n+1}$.

Proof. For the first identity we use induction on r. For $r = 1$, taking into account item (1) of Lemma 1, the statement holds. Suppose that the statement is true for r, we also show that it is true for $r + 1$. We have

$$Q^{(r+1)}_{k,n} = \sum_{i=0}^{n} Q^{(r)}_{k,i} = \sum_{i=0}^{n} 2q^{(r)}_{k,i} = 2\sum_{i=0}^{n} q^{(r)}_{k,i} = 2q^{(r+1)}_{k,n},$$

which leads us to conclude the proof of item (1).

For the proof of the second identity we use a similar method as before. For $r = 1$, using item (2) of Lemma 1, the result follows. Now suppose that the statement is true for r and we show that it is also true for $r + 1$. Therefore, using $P_{k,0}^{(r-1)} = 0$, we get

$$Q_{k,n}^{(r+1)} = \sum_{i=0}^{n} Q_{k,i}^{(r)} = 2\sum_{i=0}^{n} P_{k,i+1}^{(r-1)} = 2\sum_{i=0}^{n+1} P_{k,i}^{(r-1)} = 2P_{k,n+1}^{(r)}.$$

This completes the proof of item (2). Finally, by the use of items (1) and (2) the last identity immediately follows. □

Note that the identity of item (1) of the previous proposition is also true for $r = 0$. In fact, given that $Q_{k,n} = 2q_{k,n}$ we get

$$Q_{k,n}^{(0)} = Q_{k,n} = 2q_{k,n} = 2q_{k,n}^{(0)}$$

and the statement is true in this case as well.

The next result gives other relationship between $q_{k,n}$ and $P_{k,n}$, which will be useful later in this paper.

Lemma 2. *For non-negative integers k and natural numbers n, the following identity holds:*

$$\left(P_{k,n}^{(1)}\right)^2 - P_{k,n-1}^{(1)}P_{k,n+1}^{(1)} = \frac{1}{k+1}\left(q_{k,n} - (-k)^n\right).$$

Proof. Using item (1) of Corollary 1, (2.1) and item (1) of Proposition 2, we obtain

$$\begin{aligned}\left(P_{k,n}^{(1)}\right)^2 - P_{k,n-1}^{(1)}P_{k,n+1}^{(1)} &= \left(P_{k,n}\right)^2 + 2P_{k,n}P_{k,n-1}^{(1)} + P_{k,n-1}^{(1)}\left(-3P_{k,n} - kP_{k,n-1}\right)\\ &= \left(P_{k,n}\right)^2 - P_{k,n-1}^{(1)}\left(P_{k,n} + kP_{k,n-1}\right)\\ &= \left(P_{k,n}\right)^2 - \frac{1}{k+1}\left(-1 + kP_{k,n-1} + P_{k,n}\right)\left(kP_{k,n-1} + P_{k,n}\right)\\ &= \left(P_{k,n}\right)^2 - \frac{1}{k+1}\left(\left(P_{k,n+1} - P_{k,n}\right)^2 - kP_{k,n-1} - P_{k,n}\right).\end{aligned}$$

Using item (ii) of Proposition 2.1 of [16] and after some calculations, the result finally follows. □

3. The Concavity, Convexity, Log-Concavity and Log-Convexity Properties

In this section, considering various works, such as [1], [2], [11] and [17], we discuss the concavity, convexity, log-concavity and log-convexity properties for hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences.

A given sequence of positive real numbers $\{a_n\}_{n\geq 0}$ is said to be *log-concave* (respectively, *log-convex*) if $a_n^2 \geq a_{n-1}a_{n+1}$ (respectively, $a_n^2 \leq a_{n-1}a_{n+1}$) for all $n \geq 1$. Also, we say that the sequence $\{a_n\}_{n\geq 0}$ is *convex* (respectively, *concave*) if for $n \geq 1$, $a_{n-1} + a_{n+1} \geq 2a_n$ (respectively, $a_{n-1} + a_{n+1} \leq 2a_n$).

As for the concavity property, if we consider for example $r = 0$ and $n = 2$, we get

$$P_{k,1}^{(0)} + P_{k,3}^{(0)} = P_{k,1} + P_{k,3} = 5 + k > 4 = 2P_{k,2} = 2P_{k,2}^{(0)}$$

for all non-negative integers k, thus we conclude that the hyper k-Pell sequences are not concave, for all non-negative integers k and r, and natural numbers n. Similarly, we can conclude that hyper k-Pell-Lucas and hyper modified k-Pell sequences are not concave. About the convexity property, we have the following proposition.

Proposition 4. *For non-negative integers k, r and natural numbers n, the hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences are convex.*

Proof. For non-negative integers k, natural numbers n, and $r = 0$, by (2.1) and (1.1), we have

$$\begin{aligned} P_{k,n-1}^{(0)} + P_{k,n+1}^{(0)} &= P_{k,n-1} + P_{k,n+1} \\ &= P_{k,n-1} + (2P_{k,n} + kP_{k,n-1}) \\ &= 2P_{k,n} + (k+1)\,P_{k,n-1} \\ &\geq 2P_{k,n} \\ &= 2P_{k,n}^{(0)}. \end{aligned}$$

For non-negative integers k, natural numbers n and $r = 1$, by the use of item (1) of Corollary 1 and (1.1), we obtain

$$\begin{aligned} P_{k,n-1}^{(1)} + P_{k,n+1}^{(1)} &= P_{k,n-1}^{(1)} + \left(P_{k,n+1} + P_{k,n}^{(1)}\right) \\ &= P_{k,n-1}^{(1)} + (2P_{k,n} + kP_{k,n-1}) + P_{k,n}^{(1)} \\ &= 2P_{k,n} + P_{k,n-1}^{(1)} + kP_{k,n-1} + \left(P_{k,n} + P_{k,n-1}^{(1)}\right) \\ &= 3P_{k,n} + 2P_{k,n-1}^{(1)} + kP_{k,n-1} \\ &\geq 2P_{k,n} + 2P_{k,n-1}^{(1)} \\ &= 2P_{k,n}^{(1)}. \end{aligned}$$

We now consider $r \geq 2$ and the hyper k-Pell sequence. By the use of (2.1) and

Proposition 1 we have

$$\begin{aligned}
P^{(r)}_{k,n-1}+P^{(r)}_{k,n+1} &= \sum_{i=0}^{n-1} P^{(r-1)}_{k,i}+\sum_{i=0}^{n+1} P^{(r-1)}_{k,i} \\
&= 2\sum_{i=0}^{n-1} P^{(r-1)}_{k,i}+P^{(r-1)}_{k,n}+P^{(r-1)}_{k,n+1} \\
&= 2P^{(r)}_{k,n}-\left(P^{(r-1)}_{k,n}-P^{(r-1)}_{k,n+1}\right) \\
&= 2P^{(r)}_{k,n}-\left(-P^{(r-2)}_{k,n+1}\right) \\
&= 2P^{(r)}_{k,n}+P^{(r-2)}_{k,n+1}\geq 2P^{(r)}_{k,n},
\end{aligned}$$

as required. For the other sequences the proofs are similar and for this reason we have chosen not to present them. □

In the context of log-concavity and log-convexity, we consider the hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences for the case of $r = 0$. First of all, we establish the following lemma which will be necessary in some proofs.

Lemma 3. *For non-negative integers j and k, the following identity*

$$P_{k,j}P_{k,j+2}=\frac{1}{4(k+1)}\left(Q_{k,2j+2}-2(-k)^j(2+k)\right)$$

holds.

Proof. We use the Binet formulas for the sequences of k-Pell and k-Pell-Lucas and also the sum, product and difference between the roots r_1 and r_2 mentioned in Section 1. We have

$$\begin{aligned}
P_{k,j}P_{k,j+2} &= \left(\frac{(r_1)^j-(r_2)^j}{r_1-r_2}\right)\left(\frac{(r_1)^{j+2}-(r_2)^{j+2}}{r_1-r_2}\right) \\
&= \frac{(r_1)^{2j+2}-(r_1)^j(r_2)^{j+2}-(r_1)^{j+2}(r_2)^j+(r_2)^{2j+2}}{(r_1-r_2)^2} \\
&= \frac{((r_1)^{2j+2}+(r_2)^{2j+2})-(r_1r_2)^j((r_1)^2+(r_2)^2)}{(r_1-r_2)^2} \\
&= \frac{((r_1)^{2j+2}+(r_2)^{2j+2})-2(-k)^j(2+k)}{4(k+1)} \\
&= \frac{1}{4(k+1)}\left(Q_{k,2j+2}-2(-k)^j(2+k)\right),
\end{aligned}$$

and this completes the proof. □

For the particular case of $r = 0$ we give the first result.

Proposition 5. *For non-negative integers k and r, and natural numbers n, the hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences are log-convex if n is even and they are log-concave if n is odd.*

Proof. First we prove this for the hyper k-Pell sequence. By (2.1), Lemma 3 and item (iv) of Proposition 2.1 in [16] we have

$$\begin{aligned}
&\left(P_{k,n}^{(0)}\right)^2 - P_{k,n-1}^{(0)} P_{k,n+1}^{(0)} \\
&\qquad = \left(P_{k,n}\right)^2 - P_{k,n-1} P_{k,n+1} \\
&\qquad = \frac{1}{4\,(k+1)} \left(\left(Q_{k,2n} - 2\,(-k)^n\right) - \left(Q_{k,2n} - 2\,(-k)^{n-1}\,(2+k)\right) \right) \\
&\qquad = (-1)^{n+1}\,(k)^{n-1},
\end{aligned}$$

and the result follows. For the hyper k-Pell-Lucas sequence, using (2.2) and the respective Binet formula we obtain

$$\left(Q_{k,n}^{(0)}\right)^2 - Q_{k,n-1}^{(0)} Q_{k,n+1}^{(0)} = 4\,(-k)^{n-1}\,(k+1)\,,$$

and therefore it is easy to prove that for n even, $\left(Q_{k,n}^{(0)}\right)^2 - Q_{k,n-1}^{(0)} Q_{k,n+1}^{(0)} \leq 0$, and for n odd we have the reverse inequality, as required. For the proof of this result for the hyper modified k-Pell sequence, it suffices to recall item (1) of Proposition 3. □

Taking into consideration Lemma 1 and Lemma 3, we have

$$\begin{aligned}
\left(Q_{k,n}^{(1)}\right)^2 - Q_{k,n-1}^{(1)} Q_{k,n+1}^{(1)} &= 4\left(\left(P_{k,n+1}\right)^2 - \frac{1}{4\,(k+1)} \left(Q_{k,2n+2} - 2\,(-k)^n\,(2+k)\right) \right) \\
&= 4\,(-k)^n\,.
\end{aligned}$$

When $n=1$, the last identity gives

$$\left(Q_{k,1}^{(1)}\right)^2 - Q_{k,0}^{(1)} Q_{k,2}^{(1)} = -4k \leq 0$$

for all non-negative integers k. However, when $n=2$ the respective result is

$$\left(Q_{k,2}^{(1)}\right)^2 - Q_{k,1}^{(1)} Q_{k,3}^{(1)} = 4k^2 \geq 0$$

for all non-negative integers k, which shows that the sequence $\{Q_{k,n}^{(1)}\}_{n\geq 1}$ is not log-concave. Moreover, as $2Q_{k,n}^{(1)} = q_{k,n}^{(1)}$, the sequence $\{q_{k,n}^{(1)}\}_{n\geq 1}$ is also not log-concave.

Corollary 2. *For non-negative integers k and natural numbers n, the sequences $\{\sum_{i=0}^{n} \binom{n}{k} P_{k,n}^{(0)}\}_{n\geq 1}$, $\{\sum_{i=0}^{n} \binom{n}{k} Q_{k,n}^{(0)}\}_{n\geq 1}$ and $\{\sum_{i=0}^{n} \binom{n}{k} q_{k,n}^{(0)}\}_{n\geq 1}$ are log-convex if n is even and log-concave if n is odd.*

Proof. When n is even, by [1, Lemma 3.3] and [11] the result follows. Analogously, by [1, Lemma 3.2] and [17] we conclude the result when n is odd. $\square$

The next result continues the discussion of the log-concavity of hyper k-Pell, hyper k-Pell-Lucas and hyper modified k-Pell sequences.

Proposition 6. *For non-negative integers k and natural numbers n, the sequence $\{P_{k,n}^{(r)}\}_{n\geq 1}$ is log-concave for $r \geq 1$ and n odd.*

Proof. Taking into account Lemma 2 and the fact that n is odd, it appears that

$$\left(P_{k,n}^{(1)}\right)^2 - P_{k,n-1}^{(1)} P_{k,n+1}^{(1)} \geq 0 \tag{3.1}$$

from which it is concluded that for $r = 1$ the sequence $\{P_{k,n}^{(r)}\}_{n\geq 1}$ is log-concave for n odd.

By [1, Lemma 3.1] and [17], we derive the result when n is odd and $r \geq 1$. $\square$

As a consequence of this result we have the following corollary.

Corollary 3. *For non-negative integers k and natural numbers n, the sequence $\{\sum_{i=0}^{n} \binom{n}{k} P_{k,n}^{(r)}\}_{n\geq 1}$ is log-concave for $r \geq 1$ and n odd.*

Proof. Using Lemma 3.2 of [1], the result follows. $\square$

Note that for n even, the inequality (3.1) does not hold for some non-negative integers k. For example, if we consider $n = 2$ in (3.1), we obtain that

$$\left(P_{k,2}^{(1)}\right)^2 - P_{k,1}^{(1)} P_{k,3}^{(1)} = \frac{1}{k+1}\left(q_{k,2} - k^2\right) = \frac{-1}{k+1}(k+1)(k-2) = 2 - k$$

and then, for $k \geq 3$ this is negative.

The next two propositions and the corresponding corollaries give us not only the log-concavity property of the hyper k-Pell-Lucas and hyper modified k-Pell sequences, but also the log-concavity property of other sequences related to them.

Proposition 7. *For non-negative integers k and natural numbers n, the sequence $\{Q_{k,n}^{(r)}\}_{n\geq 1}$ is log-concave for $r \geq 2$.*

Proof. This proof can be done by induction on r and taking into account Propositions 3 and 6. $\square$

Corollary 4. *For non-negative integers k and natural numbers n, the sequence $\{\sum_{i=0}^{n} \binom{n}{k} Q_{k,n}^{(r)}\}_{n\geq 1}$ is log-concave for $r \geq 2$.*

Proof. Using Lemma 3.2 from [1], the result is easy to prove. $\square$

Proposition 8. *For non-negative integers k and natural numbers n, the sequence $\{q_{k,n}^{(r)}\}_{n\geq 1}$ is log-concave for $r \geq 2$.*

Proof. This statement is obvious by Propositions 3 and 6. □

Corollary 5. *For non-negative integers k and natural numbers n, the sequence $\{\sum_{i=0}^{n}\binom{n}{k}q_{k,n}^{(r)}\}_{n\geq 1}$ is log-concave for $r \geq 2$.*

Proof. Using Lemma 3.2 of [1], the result follows. □

Taking into consideration Propositions 6, 7 and 8, and Lemma 3.1 of [1], it is easy to prove the following proposition, and hence we omit the proof.

Proposition 9. *For non-negative integers k, natural numbers n and $r \geq 2$, the sequences $\{\sum_{i=0}^{n} P_{k,n}^{(r)}Q_{k,n}^{(r)}\}_{n\geq 1}$, $\{\sum_{i=0}^{n} P_{k,n}^{(r)}q_{k,n}^{(r)}\}_{n\geq 1}$ and $\{\sum_{i=0}^{n} Q_{k,n}^{(r)}q_{k,n}^{(r)}\}_{n\geq 1}$ are log-concave.*

From Proposition 9 we immediately have the following corollary.

Corollary 6. *For non-negative integers k, natural numbers n and $r \geq 2$, the sequences $\{\sum_{i=0}^{n}\binom{n}{k}P_{k,n}^{(r)}Q_{k,n}^{(r)}\}_{n\geq 1}$, $\{\sum_{i=0}^{n}\binom{n}{k}P_{k,n}^{(r)}q_{k,n}^{(r)}\}_{n\geq 1}$ and $\{\sum_{i=0}^{n}\binom{n}{k}Q_{k,n}^{(r)}q_{k,n}^{(r)}\}_{n\geq 1}$ are log-concave.*

Proof. Using [17] and [1, Lemma 3.3] the result follows. □

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References

[1] M. Ahmia, H. Belbachir and A. Belkhir, The log-concavity and log-convexity properties associated to hyperpell and hyperpell-Lucas sequences, *Ann. Math. Inform.* **43** (2014), 3–12.

[2] N. Asai, I. Kubo and H. Kuo, Bell numbers, log-concavity, and log-convexity, *Acta Appl. Math.* **63** (2000), 79–87.

[3] M. Bahşi and S. Solak, A symmetric algorithm for golden ratio in hyper-Horadam numbers, *Chinese J. Math.*(2016) Article ID 4361582, 1–4.

[4] M. Bahşi, I. Mezö and S. Solak, A symmetric algorithm for hyper-Fibonacci and hyper-Lucas numbers, *Ann. Math. Inform.* **43** (2014), 19–27.

[5] N. -N. Cao and F. -Z. Zhao, Some properties of hyperfibonacci and hyperlucas numbers, *J. Integer Seq.* **13** (2010), Article 10.8.8.

[6] P. Catarino, On some identities and generating functions for k-Pell numbers, *Int. Journal of Math. Analysis* **7** (2013), no. 38, 1877–1884.

[7] P. Catarino, A note involving two-by-two matrices of the k-Pell and k-Pell-Lucas sequences, *Int. Math. Forum* **8** (2013), no. 32, 1561–1568.

[8] P. Catarino and P. Vasco, On some identities and generating functions for k-Pell-Lucas sequence, *Appl. Math. Sci. (Ruse)* **7** (2013), no. 98, 4867–4873.

[9] P. Catarino and P. Vasco, Some basic properties and a two-by-two matrix involving the k-Pell numbers, *Int. Journal of Math. Analysis* **7** (2013), no. 45, 2209–2215.

[10] A. Dasdemir, On the Pell, Pell-Lucas and modified Pell numbers by matrix method, *Appl. Math. Sci. (Ruse)* **5** (2011), no. 64, 3173–3181.

[11] H. Davenport and G. Pólya, On the product of two power series, *Canad. J. Math.* **1** (1949), 1–5.

[12] A. Dil and I. Mezo, A symmetric algorithm hyperharmonic and Fibonacci numbers, *Appl. Math. Comput.* **206** (2008), 942–951.

[13] S. Halici, Some sums formulae for products of terms of Pell, Pell-Lucas and modified Pell sequences, *SAÜ. Fen Bilimleri Dergisi* **15** (2011), Cilt, 2, Say1, 151–155.

[14] T. Komatsu and L. Szalay, A new formula for hyper-Fibonacci numbers, and the number of occurrences, *Turkish J. Math.* **42** (2018), 993–1004.

[15] P. Vasco and P. Catarino, Sums and products involving terms of k-Pell, k-Pell-Lucas and modified k-Pell sequences, *JP Journal of Algebra, Number Theory and Applications* **32** (2014), no. 2, 87–98.

[16] P. Vasco, P. Catarino, H. Campos, A. P. Aires and A. Borges, k-Pell, k-Pell-Lucas and modified k-Pell numbers: some identities and norms of Hankel matrices, *Int. Journal of Math. Analysis* **9** (2015), no. 1, 31–37.

[17] Y. Wang and Y. Yeh, Log-concavity and LC-positivity, *J. Combin. Theory Ser. A* **114** (2007), 195–210.

[18] L. N. Zheng and R. Liu, On the log-concavity of the hyperfibonacci numbers and the hyperlucas numbers, *J. Integer Seq.* **17** (2014), Article 14.1.4.