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COMBINATORIAL FORMULAS FOR ARITHMETIC DENSITY

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Abstract

Let d_S denote the arithmetic density of a subset $S \subseteq \mathbb{N}$. We derive a power series in $q \in \mathbb{C}$, $|q| < 1$, with coëfficients related to integer partitions and integer compositions, that yields $1/d_S$ in the limit as $q \to 1$ radially.

1. Introduction and Statement of Results

In recent works [10, 11], Ono, Wagner, and the first author use methods from partition theory to prove q-series formulas for the arithmetic density d_S of a subset S of natural numbers $\mathbb{N}$:

$$d_S := \lim_{N\to\infty} \frac{\#\{n \in S : n \leq N\}}{N}. \tag{1}$$

In this paper, we prove a combinatorial limiting formula for the reciprocal value $1/d_S$, using ideas related to both partitions and integer compositions.

Let $\mathcal{P}$ denote the set of *integer partitions*, unordered finite sums of natural numbers (see, e.g., [1]), including the empty partition $\emptyset \in \mathcal{P}$. Let $S \subseteq \mathbb{N}$, and let $\mathcal{P}_S$ denote the set of partitions whose parts lie in the subset S. For $\lambda \in \mathcal{P}$, let $|\lambda|$ denote the *size* of the partition (sum of parts), let $\ell(\lambda)$ denote the *length* (number of parts), and let $m_i = m_i(\lambda)$ denote the *multiplicity* (frequency) of i as a part of partition λ. Note that $|\emptyset| = \ell(\emptyset) = 0$.

Define the sum $C_S(n)$ over partitions $\lambda \in \mathcal{P}_S$ with size at most n, noting $m_i(\lambda) = 0$ if $i \notin S$, by

$$C_S(n) := \sum_{\substack{\lambda\in\mathcal{P}_S \\ 0\leq|\lambda|\leq n}} \frac{(-1)^{\ell(\lambda)}\ell(\lambda)!}{m_1!m_2!m_3!\cdots m_n!}. \tag{2}$$

Sums over partitions have a history dating back to work of MacMahon [9, p. 61ff.] and Fine [4, §22]. They are important in modern number theory; the q-bracket of Bloch–Okounkov [3, 14] is an operator from statistical physics that induces modularity in partition-theoretic q-series.

Equation (2) has a natural combinatorial interpretation in terms of *integer compositions*, which are ordered finite sums of natural numbers (see, e.g., [8, Section IV]). Let $\mathcal{C}$ denote the set of all integer compositions. We will extend the partition terms and notations defined above to compositions, with the same meanings. Let $\mathcal{C}_S$ denote the compositions whose parts lie in $S \subseteq \mathbb{N}$. For each partition λ in (2), $\ell(\lambda)!/m_1!m_2!m_3!\cdots$ counts the multiset permutations of the parts of λ, i.e., all compositions $\gamma \in \mathcal{C}_S$ having the same parts as λ.

Proposition 1. *$C_S(n)$ counts the number of compositions γ in $\mathcal{C}_S$ of even length, minus those of odd length, having sizes between 0 and n inclusive:*

$$C_S(n) = \sum_{\substack{\gamma \in \mathcal{C}_S \\ 0 \le |\gamma| \le n}} (-1)^{\ell(\gamma)}.$$

Let $F_S(q) := \sum_{n\geq 0} C_S(n)q^n$ with domain of convergence depending on $S \subseteq \mathbb{N}$; and define the auxiliary series $f_S(q) := 1 + \sum_{n\in S} q^n, |q| < 1$. Our focus is on the behavior of $F_S(q)$ as $q \to 1$.

Theorem 1. *Let $S \subseteq \mathbb{N}$ such that $d_S > 0$, and such that $f_S(q) = 1 + \sum_{n\in S} q^n$ is analytic and has no zeros on $\{q \in \mathbb{C} : |q| < 1\}$. Then as $q \to 1$ radially, we have*

$$\lim_{q\to 1} F_S(q) \;=\; \lim_{q\to 1} \sum_{n=0}^{\infty} C_S(n)q^n \;=\; \frac{1}{d_S}.$$

Theorem 1 has something of an analytic converse.

Theorem 2. *Let $S \subseteq \mathbb{N}$ such that $F_S(q) = \sum_{n=0}^{\infty} C_S(n)q^n$ is analytic and has no zeros on $\{q \in \mathbb{C} : |q| < 1\}$, and such that $\lim_{q\to 1} F_S(q) = L$ exists as $q \to 1$ radially. If the limit L is infinite, then $d_S = 0$. If $L \geq 1$ is finite, then $d_S = 1/L$. The case $L < 1$ cannot occur.*

We postpone proofs until Section 2. Here is an example that uses Theorem 1.

Example 1. For $t, r \in \mathbb{N}$, with $r \leq t$, let $S_{r,t}$ denote the set of positive integers congruent to r modulo t. Then as $q \to 1$ radially, we have

$$\lim_{q\to 1} \sum_{n=0}^{\infty} C_{S_{r,t}}(n)q^n \;= t.$$

We confirm that $d_{S_{r,t}} = 1/t > 0$. Moreover, since $f_{S_{r,t}}(q) = 1 + \sum_{n\in S_{r,t}} q^n = 1 + \sum_{n\geq 0} q^{r+nt} = (1 + q^r - q^t)/(1 - q^t)$ is analytic on $|q| < 1$ with no zeros in the unit disk, then the series satisfies the analytic conditions in Theorem 1. The right-hand side gives $1/d_{S_{r,t}} = t$. Note that this limit can be seen directly, as $\lim_{q\to 1}(1-q)(1+q^r-q^t)/(1-q^t) = 1/t$ by L'Hôpital's rule.

2. Proofs of Theorem 1 and Theorem 2

We prove the theorems using the multinomial theorem, geometric series and the Cauchy product formula for power series, together with a theorem of Frobenius [5]. Our central lemma expresses the coëfficients of the reciprocal of a power series as a sum over partitions.

Lemma 1. *For $a_i \in \mathbb{C}, a_0 \neq 0$, let $f(q) := \sum_{n\geq 0} a_n q^n$ be analytic on $\{q \in \mathbb{C} : |q| < 1\}$. Then on the domain of analyticity of $\phi(q) := 1/f(q)$ we have*

$$\phi(q) = \frac{1}{f(q)} \;=\; \sum_{n=0}^{\infty} c_n q^n, \quad \textit{where } c_n \;=\; \sum_{\substack{\lambda\in\mathcal{P}\\ |\lambda|=n}} \frac{(-1)^{\ell(\lambda)}\ell(\lambda)!\; a_1^{m_1} a_2^{m_2} a_3^{m_3}\cdots a_n^{m_n}}{a_0^{\ell(\lambda)+1}\; m_1!m_2!m_3!\cdots m_n!}.$$

Proof. The result is equivalent to [12, Thm. 3.1], which is proved using the Maclaurin expansion $\phi(q) = 1/f(q) = \sum_{n\geq 0} \phi^{(n)}(0)q^n/n! = \sum_{n\geq 0} c_n q^n$. For completeness, we give a self-contained proof of the identity for c_n. Begin with the *multinomial theorem* (see, e.g., [7]), written as a sum over partitions λ with *largest part* $\mathrm{lg}(\lambda)$ at most k, and length exactly r:

$$(x_1 + x_2 + x_3 + \cdots + x_k)^r = r! \sum_{\substack{0\leq \mathrm{lg}(\lambda)\leq k\\ \ell(\lambda)=r}} \frac{x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots x_k^{m_k}}{m_1!\; m_2!\; m_3!\; \cdots\; m_k!}. \tag{3}$$

Make the substitution $x_i = \frac{a_i}{a_0} q^i$ with a_i as in Lemma 1. We now let k tend to infinity; if $g(q) := \frac{a_1}{a_0}q + \frac{a_2}{a_0}q^2 + \frac{a_3}{a_0}q^3 + \cdots = a_0^{-1}f(q) - 1$ converges absolutely on $|q| < 1$, then (3) becomes

$$(g(q))^r = \frac{r!}{a_0^r} \sum_{\ell(\lambda)=r} q^{|\lambda|} \frac{a_1^{m_1} a_2^{m_2} a_3^{m_3} \cdots}{m_1!\; m_2!\; m_3!\; \cdots}. \tag{4}$$

For $q \in \mathbb{C}$ such that $|g(q)| < 1$, multiplying both sides of (4) by $(-1)^r$ and summing over $r \geq 0$ gives an infinite geometric series on the left, and a sum over all partitions on the right:

$$\frac{1}{1+g(q)} \;=\; \frac{a_0}{f(q)} \;=\; \sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \frac{(-1)^{\ell(\lambda)}\ell(\lambda)!\; a_1^{m_1} a_2^{m_2} a_3^{m_3}\cdots}{a_0^{\ell(\lambda)}\; m_1!\; m_2!\; m_3!\; \cdots}. \tag{5}$$

Dividing through by a_0, then collecting coëfficients of q^n on the right-hand side to write $1/f(q) = \sum_{n\geq 0} c_n q^n$, proves the identity for c_n. While the geometric series representation of $a_0/f(q)$ is only valid when $|g(q)| < 1$, the formula for $c_n = \phi^{(n)}(0)/n!$ holds on the domain of analyticity of $\phi(q) = 1/f(q)$ by uniqueness of the Maclaurin series representation of the analytic function, noting $f(0) \neq 0$ so $1/f(q)$ can be expressed as a power series centered at $q = 0$. □

Lemma 2. *Let $a_i \in \mathbb{C}, a_0 \neq 0$, be such that $f(q) = \sum_{n\geq 0} a_n q^n$ is analytic and has no zeros on $\{q \in \mathbb{C} : |q| < 1\}$. Define c_n as in Lemma 1. Define $A(n) := \sum_{i=0}^{n} a_i$, $C(n) := \sum_{i=0}^{n} c_i$, and let $A := \lim_{n\to\infty} A(n)/n$ if the limit exists. Then if $A \neq 0$, as $q \to 1$ radially we have that*

$$\lim_{q\to 1} \sum_{n=0}^{\infty} C(n)q^n \;=\; \frac{1}{A}.$$

Proof. This lemma can be obtained as a special case of the first asymptotic formula in [6]; for completeness, we prove it directly. For the claimed limit to exist as $q \to 1$, we need the Maclaurin series for $\phi(q) = 1/f(q)$ convergent on the unit disk $|q| < 1$. Take $\varepsilon > 0$ and suppose that $f(q_0) = 0$ for some q_0 with $|q_0| < 1 - \varepsilon$. Then $\phi(q_0) = 1/f(q_0)$ represents a pole, so the series $\phi(q)$ has radius of convergence $\leq 1 - \varepsilon$ and the limit as $q \to 1$ does not exist. Hence the conditions on the domain and zeros are necessary. That this limit is equal to $1/A$ follows from [5], which proves when $q \to 1$ radially from within the unit disk[1] that

$$\lim_{q\to 1}(1 - q)f(q) \;=\; A. \tag{6}$$

Noting that $1/(1 - q)$, $\phi(q)$, and thus $(1 - q)^{-1}\phi(q)$ are analytic on $|q| < 1$, then Lemma 1 gives

$$\frac{1}{A} \;=\; \lim_{q\to 1} \frac{1}{1-q} \sum_{n\geq 0} c_n q^n \;=\; \lim_{q\to 1} \left(\sum_{n\geq 0} q^n\right)\left(\sum_{n\geq 0} c_n q^n\right) \;=\; \lim_{q\to 1} \sum_{n\geq 0} C(n)q^n, \tag{7}$$

with $C(n) = \sum_{0\leq i\leq n} c_i = \sum_{0\leq i\leq n} 1 \cdot c_i$ due to the Cauchy product formula for power series. □

Proof of Theorem 1. Set $a_0 = 1$ and for $n \geq 1$, let a_n be the indicator function of $S \subseteq \mathbb{N}$, i.e., $a_n = 1$ if $n \in S$, and $a_n = 0$ if $n \notin S$. Since $1 + \sum_{n\in S} q^n = \sum_{n\geq 0} a_n q^n$ is analytic on $|q| < 1$ by comparison with geometric series, then with the stipulation it has no zeros, $f_S(q)$ satisfies the analytic conditions of Lemmas 1 and 2. Thus, $a_1^{m_1} a_2^{m_2} \cdots a_r^{m_r} = 1$ if $\lambda \in \mathcal{P}_S$ and $= 0$ otherwise in the expression for c_n in Lemma 1, which yields $C(n) = C_S(n)$ in Lemma 2. Observing that $A = \lim_{n\to\infty} A(n)/n = \lim_{n\to\infty} \#\{i \in S : i \leq n\}/n = d_S$ in Lemma 2 gives the theorem. □

[1] In fact, this limiting result holds if $q \to 1$ through any path in a Stolz sector of the unit disk (see, e.g., [13]), a region with vertex at $q = 1$ such that $\frac{|1-q|}{1-|q|} \leq M$ for some $M > 0$.

Proof of Theorem 2. Since $F_S(q)$ is analytic with no zeros for $|q| < 1$, we have $\Phi_S(q) := 1/F_S(q)$ is also analytic with no zeros inside the unit disk, and has a unique power series expansion $\Phi_S(q) = \sum_{n\geq 0} d_n q^n$ around the origin. In Theorem 1, $1/F_S(q)$ has the power series expansion $(1-q)\left(1 + \sum_{n\in S} q^n\right) = (1-q)f_S(q)$, analytic for $|q| < 1$. Then by uniqueness of the power series expansion of $1/F_S(q)$ on its domain of analyticity, $\Phi_S(q) = (1-q)f_S(q)$ on the unit disk. If L is infinite, then $\lim_{q\to 1} \Phi_S(q) := \lim_{q\to 1} 1/F_S(q) = 0$ which is equal to d_S by (6). If $L \geq 1$ is finite, then $1/L = \lim_{q\to 1} 1/F_S(q) = \lim_{q\to 1} \Phi_S(q) = d_S$, also by (6). If the case $L < 1$ were to occur under these hypotheses, it would mean $1/F_S(q) = (1-q)f_S(q) \to 1/L > 1$ as $q \to 0$. But with all coëfficients of $\sum_{n\geq 0} a_n q^n = 1 + \sum_{n\in S} q^n$ being 0 or 1, then by (6), $\lim_{q\to 1}(1-q)f_S(q) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} a_n \leq 1$, if the limit exists. Thus $L < 1$ cannot occur. □

3. Further Remarks

Let $c(n) = 2^{n-1}$ denote the number of compositions of size $n \geq 1$ [8, p. 151], with $c(0) := 1$, and let $c_S(n)$ denote the number of size-n compositions having all parts from $S \subseteq \mathbb{N}$. Considering the results in Section 1, one wonders about the "non-alternating" variant of (2):

$$C_S^+(n) := \sum_{\substack{\lambda\in\mathcal{P}_S \\ 0\leq|\lambda|\leq n}} \frac{\ell(\lambda)!}{m_1!m_2!m_3!\cdots m_n!} = \sum_{0\leq j\leq n} c_S(j), \tag{8}$$

the number of compositions $\gamma \in \mathcal{C}_S$ having sizes $0 \leq |\gamma| \leq n$. This is a fairly natural statistic, e.g., for $S = \mathbb{N}$ one has $C_{\mathbb{N}}^+(n) = \sum_{0\leq j\leq n} c(j) = 1 + (1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}) = 2^n$.

What are the analytic properties of the power series $F_S^+(q) := \sum_{n\geq 0} C_S^+(n)q^n$, where $|q| < 1$? Let $g_S(q) := f_S(q) - 1$ with $f_S(q)$ analytic on $|q| < 1$. Similar multinomial, geometric series, Cauchy product and analytic arguments as above, applied to $\sum_{r\geq 0} (g_S(q))^r$, $|g_S(q)| < 1$, prove the generating function formula $\sum_{n\geq 0} c_S(n)q^n = \sum_{\gamma\in\mathcal{C}_S} q^{|\gamma|} = (2 - f_S(q))^{-1} = (1 - g_S(q))^{-1}$ (see, e.g., [2, Thm. 1.1]), for $q \in \mathbb{C}$ such that $|f_S(q)| < 2$. Then by (5), together with the Cauchy product for power series as used in (7) above, we deduce the identity

$$F_S^+(q) \ = \ \sum_{n=0}^{\infty} C_S^+(n)q^n \ = \ \frac{1}{(1-q)\,(2 - f_S(q))}. \tag{9}$$

However, limiting formulas analogous to Theorem 1 do not result in this case, as $F_S^+(q)$ is not analytic on the unit disk. To see this, note that if S is a finite nonempty subset, then $|f_S(q)| \leq 1 + \#S$ when $|q| \leq 1$ since there are $\#S$ terms of the form q^n, with $f_S(1) = 1 + \#S$ exactly. If S is an infinite subset of $\mathbb{N}$, then $|f_S(q)| \to \infty$ as $q \to 1$. Thus $|f_S(q)| < 2$ for all $|q| < 1$ if and only if the subset S has one element. For a subset S with two or more elements, $F_S^+(q)$ converges on a disk strictly smaller than the unit disk, and $\lim_{q\to 1} F_S^+(q)$ does not exist.

Remark 1. It is possible to find composition-theoretic limiting formulas in a smaller disk, e.g.,

$$\lim_{q\to 1/2} \frac{1-2q}{1-q} \sum_{n\in S} c(n)q^n \;=\; \lim_{q\to 1/2} \frac{\sum_{n\in S} c(n)q^n}{\sum_{n\geq 0} c(n)q^n} \;=\; d_S. \tag{10}$$

This can be deduced from (6):

$$d_S = \lim_{q\to 1} (1-q)\sum_{n\in S} q^n \;=\; \lim_{q\to 1/2} (1-2q)\sum_{n\in S} (2q)^n \;=\; \lim_{q\to 1/2} \frac{1-2q}{1-q}\sum_{n\in S} 2^{n-1}q^n,$$

noting that $\sum_{n\geq 0} c(n)q^n = 1 + \sum_{n\geq 1} 2^{n-1}q^n = (1-q)/(1-2q)$.

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References

[1] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, no. 2, Addison–Wesley, Reading, Massachusetts, 1976. Reissued, Cambridge University Press, 1998.

[2] M. Bicknell, and V. E. Hoggatt, Palindromic compositions, *Fibonacci Quarterly* **13** (1975), 350–356.

[3] S. Bloch and A. Okounkov, *The character of the infinite wedge representation*, Adv. Math. **149** (2000), 1–60.

[4] N. J. Fine, *Basic Hypergeometric Series and Applications*, Mathematical Surveys and Monographs, no. 27, American Mathematical Society, Providence, Rhode Island,1988.

[5] G. Frobenius, Über die Leibnitzsche Reihe, *J. Reine Angew. Math.* **89** (1880), 262–264.

[6] G. H. Hardy and J. E. Littlewood, Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive, *Proc. London Math. Soc. Ser. 2* **13** (1914), 174–191.

[7] K. K. Kataria, A probabilistic proof of the multinomial theorem, *Amer. Math. Monthly* **123** (2016), 94–96.

[8] P. A. MacMahon, *Combinatory Analysis*, vol. I, Cambridge University Press, 1915; Reissued (with volumes I and II bound in one volume), AMS Chelsea, 2001.

[9] P. A. MacMahon, *Combinatory Analysis*, vol. II, Cambridge University Press, 1916; Reissued (with volumes I and II bound in one volume), AMS Chelsea, 2001.

[10] K. Ono, R. Schneider, and I. Wagner, Partition-theoretic formulas for arithmetic densities, *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proc. Math. Stat. **221** (2017), 611–624.

[11] K. Ono, R. Schneider, and I. Wagner, Partition-theoretic formulas for arithmetic densities, II, *Hardy–Ramanujan Journal* **43** (2020), 1–16.

[12] A. Salem, Reciprocal of infinite series and partition functions, *Integral Transforms and Special Functions* **22** (2011), 443–452.

[13] L. E. Snyder, Continuous Stolz extensions and boundary functions, *Trans. Amer. Math. Soc.* **119** (1965), 417–427.

[14] D. Zagier, Partitions, quasimodular forms, and the Bloch–Okounkov theorem, *Ramanujan J.* **4** (2016), 345–368.