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k-FIBONACCI NUMBERS AND MÖBIUS FUNCTIONS

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Abstract

The aim of this paper is to bring to the fore a different and effective way of approaching k-Fibonacci numbers. This approach uses the tools of multiplicative arithmetic functions. Some simplest examples were chosen for illustration. The Dirichlet convolution, the Bell series, the Busche-Ramanujan identities and the Möbius functions play an essential role in our development.

1. Introduction

Generalizations of Fibonacci numbers have been studied by many authors. These generalizations are based on classes of recurrence relations that often occur in applied mathematics. *Horadam's sequence* $\mathcal{H}_{a,b,p,q} = \{H_n\}_{n\geq 0}$ is defined by $H_n = pH_{n-1} - qH_{n-2}$ $(n > 1)$ with initial conditions $H_0 = a$ and $H_1 = b$ (see [8, 9]). The *k-Fibonacci sequence* $\mathcal{F}_k = \{F_{k,n}\}_{n\geq 0}$ (for k a positive integer in [4], for k a real number in [5], and for k a complex number in this paper) is a sequence, with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$, that satisfies the following second-order linear recurrence relation:

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2} \quad \text{if} \quad n > 1.$$

Obviously, $\mathcal{F}_1$ is the well-known Fibonacci sequence [A000045 in the OEIS] and $\mathcal{F}_2$ is the Pell sequence [A000129 in the OEIS]. In [4, 5], Falcon and Plaza give the basic properties of $\mathcal{F}_k$.

In [7] and [12], technical approaches from arithmetic functions were used in the study of generalized Fibonacci sequences concluding that certain properties of these sequences reflect properties of arithmetic functions. In this paper, we propose a similar approach, and concepts as arithmetic convolution, completely and specially multiplicative functions, Möbius inversion, etc. will be used to emphasize the conveniences offered by the theory of multiplicative arithmetic functions. For example, sums of k-Fibonacci numbers are obtained in Section 4 using an inversion theorem (Theorem 2) which point out the significance of Fibonacci-Möbius function of rank k. The theory of arithmetic functions, an active and classical part of mathematics, brings a helpful and straightforward approach to studying k-Fibonacci numbers. We will insert only the simplest examples (applications).

2. The Fibonacci Arithmetic Function and the Binet Formula

An *arithmetic function* is a complex-valued function defined on the set of positive integers. An arithmetic function f is said to be *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$, where (m, n) is the g.c.d. of m and n. If $f(mn) = f(m)f(n)$ holds for all pairs of positive integers then f is called *completely multiplicative*. The *Dirichlet convolution* $f * g$ of two arithmetic functions f and g is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}),$$

where the summation is over the positive divisors d of n. The set of all multiplicative functions with this operation is a commutative group, denoted by $(\mathfrak{M}, *)$. The *identity* $\mathbf{1}$ of this group is defined by $\mathbf{1}(n) = 1$ if $n = 1$, and $\mathbf{1}(n) = 0$ otherwise.

The *Bell series* (as a formal power series) of an arithmetic function f modulo a prime p is defined by

$$f(p, x) = \sum_{m=0}^{\infty} f(p^m)x^m.$$

Two multiplicative arithmetic functions are identical if all their Bell series are equal. Furthermore, for any two arithmetic functions f and g, we have:

$$(f * g)(p, x) = f(p, x) \cdot g(p, x),$$

for every prime p (see, for example, Theorems 2.24 and 2.25 of Apostol's book [2]).

It is straightforward to see that a multiplicative arithmetic function is completely determined by its values at the prime powers. We will take this into account in the following definition.

Definition 1. We will say that the multiplicative arithmetic function $\mathfrak{F}_k$ defined by

$$\mathfrak{F}_k(p^m) = F_{k,m+1},$$

for every prime p, is the *Fibonacci arithmetic function of rank k*.

Theorem 1. *If Φ_k and Ψ_k are the arithmetic functions defined by*

$$\Phi_k(n) = \varphi_k^{\Omega(n)} \qquad \textit{and} \qquad \Psi_k(n) = \psi_k^{\Omega(n)},$$

respectively, where

$$\varphi_k = \frac{k + \sqrt{k^2+4}}{2} \ ; \ \psi_k = \frac{k - \sqrt{k^2+4}}{2},$$

$\Omega(n)$ being the number of prime factors of n counted with multiplicity, then

$$\mathfrak{F}_{\mathrm{k}} = \Phi_k * \Psi_k.$$

Proof. The arithmetic functions Φ_k and Ψ_k are also multiplicative, and moreover, they are completely multiplicative.

For any prime p, the Bell series

$$\Phi_k(p,x) = 1 + \varphi_k x + \varphi_k^2 x^2 + \cdots + \varphi_k^m x^m + \cdots$$

and

$$\Psi_k(p,x) = 1 + \psi_k x + \psi_k^2 x^2 + \cdots + \psi_k^m x^m + \cdots$$

are geometric series and therefore,

$$\Phi_k(p,x) = \frac{1}{1-\varphi_k x} \qquad \text{and} \qquad \Psi_k(k,x) = \frac{1}{1-\psi_k x}.$$

The Bell series $\mathfrak{F}_k(p,x)$ of the arithmetic function $\mathfrak{F}_k$ for all primes p is given by

$$\mathfrak{F}_k(p,x) = \frac{1}{1-kx-x^2}.$$

That is because:

$$\mathfrak{F}_k(p,x) = \sum_{m=0}^{\infty} F_{k,m+1}x^m = F_{k,1} + F_{k,2}x + \sum_{m=2}^{\infty}(kF_{k,m} + F_{k,m-1})x^m$$

$$= 1 + kx + kx\sum_{m=2}^{\infty} F_{k,m}x^{m-1} + x^2\sum_{m=2}^{\infty} F_{k,m-1}x^{m-2} = 1 + kx\mathfrak{F}_k(p,x) + x^2\mathfrak{F}_k(p,x).$$

Now, since $(1-\varphi_k x)\cdot(1-\psi_k x) = 1 - kx - x^2$, it follows that

$$\Phi_k(p,x)\cdot\Psi_k(p,x) = \mathfrak{F}_k(p,x)$$

for all primes p and therefore $\mathfrak{F}_k = \Phi_k * \Psi_k$. □

Now, Binet's formula appears surprisingly obvious using the definition of the Dirichlet convolution.

Corollary 1 (Binet's formula [5, Proposition 2]). *The k-Fibonacci number $F_{k,m}$ is given by*

$$F_{k,m} = \frac{\varphi_k^m - \psi_k^m}{\varphi_k - \psi_k}.$$

Proof. Since $F_{k,m} = \mathfrak{F}(p^{m-1})$ for all $m > 0$ and p prime, Theorem 1 tells us that

$$F_{k,m} = (\Phi_k * \Psi_k)(p^{m-1}) = \sum_{i=0}^{m-1} \varphi_k^i \psi_k^{m-1-i} = \frac{\varphi_k^m - \psi_k^m}{\varphi_k - \psi_k}.$$

□

3. The Busche-Ramanujan Identities for k-Fibonacci Numbers

An arithmetic function f is called *specially multiplicative* if it is the Dirichlet convolution

$$f = g * h$$

of two completely multiplicative arithmetic functions g and h. If f is specially multiplicative then each of the equivalent identities

$$f(m)f(n) = \sum_{d|(m,n)} f(\frac{mn}{d^2})g(d)h(d) \qquad (1)$$

and

$$f(mn) = \sum_{d|(m,n)} f(\frac{m}{d})f(\frac{n}{d})\mu(d)g(d)h(d) \qquad (2)$$

is called a *Busche-Ramanujan identity* (see [11, Theorem 1.12 and p.25]), where μ is the classical Möbius function.

Since Φ_k and Ψ_k are completely multiplicative, the Fibonacci arithmetic function $\mathfrak{F}_k$ is specially multiplicative.

Proposition 1 (The Busche-Ramanujan identity for k-Fibonacci numbers)**.** *The product of two k-Fibonacci numbers is given by*

$$F_{k,m}F_{k,n} = \sum_{j=1}^{min\{m,n\}} (-1)^{j-1} F_{k,m+n+1-2j}.$$

Proof. The equality $F_{k,m}F_{k,n} = \mathfrak{F}_k(p^{m-1})\mathfrak{F}_k(p^{n-1})$ holds for an arbitrary prime p, and

$$\Phi_k(d)\Psi_k(d) = (\varphi_k\psi_k)^{\Omega(d)} = (-1)^{\Omega(d)}.$$

Now, using the Busche-Ramanujan identity (1) for the specially multiplicative arithmetic function $\mathfrak{F}_k$ we obtain

$$F_{k,m}F_{k,n} = \mathfrak{F}_k(p^{m-1})\mathfrak{F}_k(p^{n-1}) = \sum_{d|min\{p^{m-1},p^{n-1}\}} \mathfrak{F}_k(\frac{p^{m+n-2}}{d^2})(-1)^{\Omega(d)}$$

$$= \sum_{i=0}^{min\{m-1,n-1\}} (-1)^i F_{m+n-1-2i} = \sum_{j=1}^{min\{m,n\}} (-1)^{j-1} F_{m+n+1-2j}.$$

□

The above identity can be used successfully to prove well-known k-Fibonacci identities. We will exemplify this with d'Ocagne's identity.

Corollary 2 (d' Ocagne's identity [5, Proposition 4]). *For nonnegative integers $m < n$,*

$$F_{k,m+1}F_{k,n} - F_{k,m}F_{k,n+1} = (-1)^m F_{k,n-m}.$$

Proof. We have

$$\begin{aligned} F_{k,m+1}F_{k,n} - F_{k,m}F_{k,n+1} &= \sum_{j=1}^{m+1}(-1)^{j-1}F_{k,m+n+2-2j} - \sum_{j=1}^{m}(-1)^{j-1}F_{k,m+n+2-2j} \\ &= (-1)^m F_{k,m+n+2-2(m+1)} = (-1)^m F_{k,n-m}. \end{aligned}$$

□

Remark 1. From the Busche-Ramanujan identity (2) we also get the following identities:

$$\begin{aligned} F_{k,2m+1} &= F_{k,m+1}^2 + F_{k,m}^2 \qquad && (m > 0); \\ F_{k,2m} &= F_{k,m}(F_{k,m+1} + F_{k,m-1}) && (m > 1). \end{aligned}$$

4. The Fibonacci Möbius Function of Rank k

The *classical Möbius function* μ is an important multiplicative arithmetic function and it is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } \ n = 1 \\ 0 & \text{if } \ p^2|n \text{ for some prime } p \\ (-1)^{\omega(n)} & \text{otherwise,} \end{cases}$$

where $\omega(n)$ is the number of distinct prime factors of the positive integer n. Note that the Möbius function μ is the (Dirichlet) convolution inverse of the function ζ: $\zeta(n) = 1$ for all n.

Definition 2. We will say that the convolution inverse of the Fibonacci arithmetic function $\mathfrak{F}_k$, denoted μ_k, is the *Fibonacci Möbius function of rank k.*

For the Bell series $\mu_k(p,x)$ we have $\mu_k(p,x) = \frac{1}{\mathfrak{F}_k(p,x)}$ for every p prime. Hence

$$\mu_k(p,x) = 1 - kx - x^2,$$

and therefore

$$\mu_k(p^m) = \begin{cases} 1 & \text{if } m = 0 \\ -k & \text{if } m = 1 \\ -1 & \text{if } m = 2 \\ 0 & \text{if } m > 2. \end{cases}$$

What this means is the following.

Proposition 2. *The Fibonacci Möbius function μ_k is given by*

$$\mu_k(n) = \begin{cases} 1 & \textit{if } n = 1 \\ 0 & \textit{if } p^3 | n \textit{ for some prime } p \\ (-1)^{\omega(n)} k^{\overline{\omega}(n)} & \textit{otherwise,} \end{cases}$$

where $\omega(n)$ is the number of distinct prime factors of n, and $\overline{\omega}(n)$ is the number of square-free prime divisors of n.

If the k-Fibonacci sequence $(\mathcal{F}_k)$ is the classical one (that is, $k = 1$) then

$$\mu_1(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^3 | n \text{ for some prime } p \\ (-1)^{\omega(n)} & \text{otherwise.} \end{cases}$$

Since $(\mathfrak{M}, *)$ is a group and μ_k is the convolution inverse of $\mathfrak{F}_k$ it follows the inversion theorem below.

Theorem 2 (The Möbius inversion)**.** *Let f and g be two arithmetic functions. Then*

$$g = \mathfrak{F}_k * f \quad \textit{if and only if} \quad f = \mu_k * g.$$

An instant application of the Möbius inversion is available.

Corollary 3 ([4, Proposition 8], [10, Theorem 1]). *The sum of the first n k-Fibonacci numbers is given by*

$$\sum_{j=1}^{n} F_{k,j} = \frac{F_{k,n} + F_{k,n+1} - 1}{k} \qquad (k \neq 0).$$

Proof. Taking $f = \zeta$, we get $g(p^n) = \sum_{j=1}^{n+1} F_{k,j}$ for a prime p. Hence

$$\begin{aligned} 1 = \zeta(p^n) = (\mu_k * g)(p^n) &= \mu_k(1)g(p^n) + \mu_k(p)g(p^{n-1}) + \mu_k(p^2)g(p^{n-2}) \\ &= \sum_{j=1}^{n+1} F_{k,j} - k\sum_{j=1}^{n} F_{k,j} - \sum_{j=1}^{n-1} F_{k,j} = F_{k,n} + F_{k,n+1} - k\sum_{j=1}^{n} F_{k,j}. \end{aligned}$$

The proof is complete. □

If $k = 1$ we obtain the well-known formula involving the sum of the classical Fibonacci numbers:

$$\sum_{j=1}^{n} F_j = F_{n+2} - 1.$$

Our proof in Corollary 3 is constructive in the sense that it will lead us directly to the proposed formula for all possible k. We notice that other well known formulas involving sums of the classical Fibonacci numbers as the following sum of falling powers of 2:

$$\sum_{j=1}^{n} 2^{n-j} F_j = 2^{n+1} - F_{n+3}, \qquad \text{(etc.)},$$

can be easily generalized for k-Fibonacci numbers using the Möbius inversion theorem (Theorem 2). If, in Theorem 2, we choose $f = 2^{\Omega(n)}$, then $g(p^{n-1}) = (\mathfrak{F}_k * f)(p^{n-1}) = \sum_{j=1}^{n} 2^{n-j} F_{k,j}$ for all primes p. It follows

$$\begin{aligned} 2^n = f(p^n) = (\mu_k * g)(p^n) &= \sum_{j=1}^{n+1} 2^{n+1-j} F_{k,j} - k \sum_{j=1}^{n} 2^{n-j} F_{k,j} - \sum_{j=1}^{n-1} 2^{n-1-j} F_{k,j} \\ &= 2 \sum_{j=1}^{n} 2^{n-j} F_{k,j} + F_{k,n+1} - k \sum_{j=1}^{n} 2^{n-j} F_{k,j} - \frac{\sum_{j=1}^{n} 2^{n-j} F_{k,j} - F_{k,n}}{2}; \end{aligned}$$

that is,

$$\sum_{j=1}^{n} 2^{n-j} F_{k,j} = \frac{2^{n+1} - 2F_{k,n+1} - F_{k,n}}{3 - 2k}.$$

There is no doubt that the above steps can be repeated literally (obviously, using the Möbius inversion theorem) in a more general context.

Proposition 3. *Let f be a completely multiplicative arithmetic function. We have:*

$$\sum_{j=1}^{n} f(p)^{n-j} F_{k,j} = \frac{f(p)^{n+1} - f(p) F_{k,n+1} - F_{k,n}}{f(p)^2 - k f(p) - 1},$$

for every prime p.

Clearly, this formula gives the formula in Corollary 3 if $f = \zeta$. Proposition 3 leads us to a number of generalizations to sums with k-Fibonacci numbers. At the end of the section, we enunciate the simplest (with the falling powers of k).

Corollary 4. *We have*

$$\sum_{j=1}^{n} k^{n-j} F_{k,j} = F_{k,n+2} - k^{n+1}$$

and therefore $\sum_{j=1}^{n} 2^{n-j} P_j = P_{n+2} - 2^{n+1}$, where P_j are the Pell numbers and $\sum_{j=1}^{n} F_j = F_{n+2} - 1$, where F_j are the classical Fibonacci numbers.

Proof. Consider a completely multiplicative arithmetic function f with $f(p) = k$ for a given prime p. The rest is given by Proposition 3. □

5. Apostol's, Sastry's, and Cohen's Generalized Möbius Functions; The Binet Formulas

In the previous Section, starting with a k-Fibonacci sequence we defined the Fibonacci Möbius function μ_k as the convolution inverse of a specially multiplicative arithmetic function $\mathfrak{F}_k$. This action can be reversed by starting from a (Fibonacci) Möbius function to get to the corresponding sequence.

Several different number-theoretical generalizations of the Möbius function have been studied in the literature (for a comprehensive survey see [13]). They are arithmetic functions "of order a positive integer ℓ" which are all multiplicative. The first order ones coincide with the classical Möbius function. Such generalizations are *Apostol's* [1], *Sastry's* [14] and *Cohen's* [3] *Möbius functions of order* ℓ defined respectively by:

$$\mu_{A_\ell}(n) = \begin{cases} 1 & \text{if } \ n = 1 \\ 0 & \text{if } \ p^{\ell+1} | n \text{ for some prime } p \\ (-1)^r & \text{otherwise,} \end{cases}$$

$$\mu_{B_\ell}(n) = \begin{cases} 1 & \text{if } \ n = 1 \\ 0 & \text{if } \ p^{\ell+1} | n \text{ for some prime } p \\ (-1)^{\Omega(n)} & \text{otherwise,} \end{cases}$$

($\Omega(n)$ is the number of prime factors of n counted with multiplicity)

$$\mu_{C_\ell}(n) = \begin{cases} 1 & \text{if } \ n = 1 \\ 0 & \text{if } \ p^{\ell+1} | n \text{ for some prime } p \\ (-1)^{\omega(n)} & \text{otherwise,} \end{cases}$$

($\omega(n)$ is the number of distinct prime factors of n)

where r is the number of prime divisors p of n with ℓ the highest power of the prime p such that p^ℓ divides n. They are completely determined by their values at the prime powers. The case $\ell = 2$ is shown below (not including the index 2) completed with the Fibonacci Möbius function μ_k at the end:

$$\mu_A(p^m) = \begin{cases} 1 & \text{if } \ m = 0, 1 \\ -1 & \text{if } \ m = 2 \\ 0 & \text{if } \ m > 2, \end{cases} \qquad \mu_B(p^m) = \begin{cases} 1 & \text{if } \ m = 0, 2 \\ -1 & \text{if } \ m = 1 \\ 0 & \text{if } \ m > 2, \end{cases}$$

$$\mu_C(p^m) = \begin{cases} 1 & \text{if } \ m = 0 \\ -1 & \text{if } \ m = 1, 2 \\ 0 & \text{if } \ m > 2, \end{cases} \qquad \mu_k(p^m) = \begin{cases} 1 & \text{if } \ m = 0 \\ -k & \text{if } \ m = 1 \\ -1 & \text{if } \ m = 2 \\ 0 & \text{if } \ m > 2. \end{cases}$$

Just like the convolution inverse μ_k^{-1} give rise to the k-Fibonacci sequence $\{F_{k,n}\}_{n\geq 1}$ (since $\mu_k^{-1} = \mathfrak{F}_k$), the convolution inverses μ_A^{-1}, μ_B^{-1} and μ_C^{-1} lead us to three sequences $\{A_n\}_{n\geq 1}$, $\{B_n\}_{n\geq 1}$ and $\{C_n\}_{n\geq 1}$, respectively ($A_n = \mu_A^{-1}(p^{n-1})$, $B_n = \mu_B^{-1}(p^{n-1})$ and $C_n = \mu_C^{-1}(p^{n-1})$). Now, it is straightforward to see that both the *Apostol sequence* $(A_n = \mu_A^{-1}(p^{n-1}))_{n\geq 1}$ and the *Cohen sequence* $(C_n = \mu_C^{-1}(p^{n-1}))_{n\geq 1}$ are k-Fibonacci sequences (see $\mu_k(p^m)$ before Proposition 2) and $\mu_A = \mu_{-1}$, $\mu_C = \mu_1$. So, the Cohen sequence $\{C_n\}_{n\geq 1}$ is the classical Fibonacci sequence and Apostol's sequence $\{A_n\}_{n\geq 1}$ is the sequence of Fibonacci numbers with alternate negatives $\{(-1)^{n+1}F_n\}_{n\geq 1}$ (A039834 in the OEIS - called the *negaFibonacci sequence*). Apostol's (the negaFibonacci) sequence is given by

$$A_1 = 1,\ A_2 = -1, \text{ and } \ A_n = -A_{n-1} + A_{n-2} \ \text{ if } n > 2 \qquad (\mathcal{A}).$$

and, in the corresponding Binet's formula $A_n = \frac{\varphi_A^n - \psi_A^n}{\varphi_A - \psi_A}$ we have

$$\varphi_A = \varphi_{-1} = \frac{-1+\sqrt{5}}{2} \text{ and } \ \psi_A = \psi_{-1} = \frac{-1-\sqrt{5}}{2}.$$

Note that Sastry's Möbius function μ_B is not a Fibonacci Möbius function. So, *Sastry's sequence* $(B_n = \mu_B^{-1}(p^{n-1}))_{n\geq 1}$ is not a k-Fibonacci sequence.

Proposition 4. *For the nth term of Sastry's sequence* $\{B_n\}_{n\geq 1}$ *we have*

$$B_n = \frac{\varphi_B^n - \psi_B^n}{\varphi_B - \psi_B},$$

where

$$\varphi_B = \frac{1+i\sqrt{3}}{2} \quad \textit{and} \quad \psi_B = \frac{1-i\sqrt{3}}{2}.$$

Proof. The formal power series $\sum_{m=0}^{\infty} B_{m+1}x^m$ is the Bell series $\mu_B^{-1}(p,x)$ for every prime p. Since

$$\mu_B(p,x) = 1 - x + x^2$$

for every prime p, it follows that

$$\sum_{m=0}^{\infty} B_{m+1}x^m = \frac{1}{1-x+x^2}.$$

But,

$$\frac{1}{1-\varphi_B x} \cdot \frac{1}{1-\psi_B x} = \frac{1}{1-x+x^2},$$

and taking into account that $\frac{1}{1-\varphi_B x} = \varphi_B^{\Omega}(p,x)$ and $\frac{1}{1-\psi_B x} = \psi_B^{\Omega}(p,x)$ for every prime p, we get

$$\varphi_B^{\Omega} * \psi_B^{\Omega} = \mu_B^{-1}.$$

Thus,

$$B_n = \mu_B^{-1}(p^{n-1}) = (\varphi_B^{\Omega} * \psi_B^{\Omega})(p^{n-1}) = \sum_{j=0}^{n-1} \varphi_B^j \psi_B^{n-1-j} = \frac{\varphi_B^n - \psi_B^n}{\varphi_B - \psi_B},$$

which is what we started out to prove. □

Corollary 5. *The Sastry sequence is given by*

$$1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, \cdots \qquad (A010892 \text{ in the OEIS})$$

that is,

$$B_1 = 1, \; B_2 = 1, \text{ and } \; B_n = B_{n-1} - B_{n-2} \;\text{ if } n > 2.$$

Proof. Since $\varphi_B - \psi_B = i\sqrt{3}$ and $\varphi_B^n - \psi_B^n = 2isin\frac{n\pi}{3}$ we get

$$B_n = \frac{2sin\frac{n\pi}{3}}{\sqrt{3}} = \begin{cases} 0 & \text{if } \; 3 \mid n \\ (-1)^{[\frac{n}{3}]} & \text{otherwise,} \end{cases}$$

where $[\frac{n}{3}]$ denotes the greatest integer in $\frac{n}{3}$. Obviously, the corollary follows from the values of B_n above. □

Remark 2. In [6, Theorem 2] it is shown that a certain sequence $\mathcal{G} = \{G_n\}$ whose exponential generating function satisfies the functional differential equation $\frac{d}{dx}g(x) = e^{kx}g(-x)$ is a second-order linear recurence sequence, namely $G_{n+2} = kG_{n+1} - G_n$. The sequence associated with $k = 1$ (and $G_1 = G_2 = 1$) is A010892 and its basic properties are detailed in [6, Section 3]. We call this sequence Sastry's sequence because in our case it derives from Sastry's generalized Möbius function of order 2.

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