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ON THE 2-ADIC VALUATION OF GENERALIZED FIBONACCI SEQUENCES

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Abstract

Let $F_n^{(k)} = F_n$ denote the generalized Fibonacci number of order k defined by the recurrence $F_n = F_{n-1}+F_{n-2}+\cdots+F_{n-k}$, with initial conditions $F_1 = 1$ and $F_n = 0$ for $-(k-2) \le n \le 0$. In the light of recent study of such sequences, we characterize the 2-adic valuation of the sequences $(F_n^{(k)})$, and draw some conclusions concerning their zero-multiplicity. In addition to the theory of 2-adic analytic functions, our method also incorporates an adaptation of a classical formula of Ferguson.

1. Introduction

In recent years various Diophantine equations involving generalized Fibonacci sequences have been studied by several authors [6, 8, 7, 13, 14, 11, 4, 1, 2, 10]. For many of these questions, p-adic properties, especially for $p = 2$, have played an important role. For example, equations involving the generalized Fibonacci sequence $(T_n^{(k)}) = (T_n)$, defined by the k-th order recurrence $T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k}$ with initial conditions $(0, 1, \ldots 1)$ have been studied in [8, 7, 13, 11], leading to the characterization of its 2-adic valuation, modulo the determination of the nature of the zero of a 2-adic analytic function when the order k is odd. Specifically, the 2-adic valuation $\nu_2(T_n^{(k)})$ is given [6, 8, 13, 14] by integral polynomial formulas in congruence classes modulo $2k + 2$ (in accordance with Conjecture 8 of [8]), except that when $k \geqslant 5$ is odd and n is of the form $n = (2k + 2)m + k + 1$, we have

$$\nu_2(T_n) = 2 + \nu_2(m - z), \tag{1.1}$$

where z is some 2-adic integer which may be computed to high 2-adic precision. For even order k, the zero-multiplicity (that is, the number of integers $n \in \mathbb{Z}$ such that $T_n^{(k)} = 0$) of the sequence $(T_n^{(k)})$ is 1, whereas for odd $k \geqslant 5$ the sequence $(T_n^{(k)})$ has zero-multiplicity 1 or 2 according to whether the 2-adic integer z in (1.1) is a rational integer. In [11] it was shown that z is not an integer for $k = 5$, together with an upper bound on $|n|$ for any integer solutions to $T_{-n}^{(k)} = 0$ for odd $k > 5$. It is conjectured that no z value in (1.1) is a rational integer, which is equivalent to the sequence $(T_n^{(k)})$ having zero-multiplicity 1 for all $k \geqslant 4$. However, this conjecture seems to differ in spirit from the integrality hypothesis of [8, Conjecture 8].

There has likewise been much interest in the generalized Fibonacci sequence $(F_n^{(k)}) = (F_n)$, defined by the same recurrence $F_n = F_{n-1} + F_{n-2} + \cdots + F_{n-k}$, but with initial conditions $(0, \ldots, 0, 1)$ [7, 4, 1, 2, 10]. These initial conditions give the sequence $(F_n^{(k)})$ a larger zero-multiplicity than that of $(T_n^{(k)})$; in particular, in [4, Theorem 5.1] it is observed that the zero-multiplicity of $(F_n^{(k)})$ is at least $k(k-1)/2$ for all $k \geqslant 2$. In fact, for indices near 0, the sequence $(F_n^{(k)})$ has the structure

$$[\ldots, -48, 16, \underbrace{0, \ldots, 0}_{k-5}], [1, -7, 18, -20, 8, \underbrace{0, \ldots, 0}_{k-4}], [-1, 5, -8, 4, \underbrace{0, \ldots, 0}_{k-3}],$$

$$[1, -3, 2, \underbrace{0, \ldots, 0}_{k-2}], [-1, 1, \underbrace{0, \ldots, 0^*}_{k-1}], [1, \underbrace{1, 2, 4, 8, \ldots, 2^{k-1}}_{k}], \ldots$$

where each bracket [ ] consists of $k+1$ terms, and the starred term 0^* is $F_0^{(k)} = 0$. This stands in contrast to the corresponding structure

$$[13 - 4k, -3, \underbrace{1, \ldots, 1}_{k-3}, k-1, 6-2k], [-1, \underbrace{1, \ldots, 1}_{k-2}, 3-k, 0^*], [\underbrace{1, \ldots, 1}_{k-1}, k-1, 2k-2]$$

for $(T_n^{(k)})$ near $n = 0$, where the value $T_0^{(k)} = 0^*$ is the only apparent zero (except when $k = 3$, where $T_n^{(3)} = F_n^{(3)}$). In [4, Corollary 1.2] it is proved that the zero-multiplicity of $(F_n^{(k)})$ is exactly $k(k-1)/2$ for all $k \in \{4, \ldots, 500\}$, along with an upper bound on $|n|$ for any integer solutions to $F_{-n}^{(k)} = 0$ for all orders k. It is also conjectured that the zero-multiplicity of $(F_n^{(k)})$ is exactly $k(k-1)/2$ for all $k \geqslant 4$.

In the light of these developments, here we determine the 2-adic valuation of the sequence $(F_n^{(k)})$. On the congruence classes modulo $2k+2$ where the valuation is non-constant, we find that the valuation coincides with valuations of binomial coefficients, except that on certain congruence classes there is also an "extra" term $\nu_2(m - z_c)$, similar to that in (1.1), which depends a 2-adic integer z_c whose nature is unknown, but may be computed to high precision. The zeros of the binomial coefficients correspond precisely to the $k(k-1)/2$ known solutions [4, Theorem 5.1] to $F_{-n}^{(k)} = 0$, so that the conjectured zero-multiplicity $k(k-1)/2$ for $(F_n^{(k)})$ is equivalent to the condition that no z_c value is a rational integer. The result is:

Theorem 1. *Suppose $k \geqslant 4$. If $n = 2m(k+1)+b$ with $1 \leqslant b \leqslant k+1$, then*

$$\nu_2(F_n^{(k)}) = \begin{cases} 0, & \text{if } b = 1, 2, \\ 1, & \text{if } b = 3, \\ \nu_2(\binom{m+c-1}{c-1}) + 2c - 2, & \text{if } b = 2c, \\ \nu_2(\binom{m+2c-2}{2c-2}) + 4c - 3, & \text{if } b = 4c-1, \\ \nu_2(\binom{m+2c-1}{2c-1}) + \nu_2(m - z_c) + 4c - 1 - \nu_2(c), & \text{if } b = 4c+1, \end{cases}$$

where z_c is some 2-adic integer 2-adically close to $-c$. If $n = (2m+1)(k+1)+b$ with $1 \leqslant b \leqslant k+1$, then

$$\nu_2(F_n^{(k)}) = \begin{cases} 0, & \text{if } b = 1, 2, \\ \nu_2(\binom{m+c-1}{c-1}) + 2c - 2, & \text{if } b = 2c, \\ \nu_2(\binom{m+2c}{2c}) + 4c, & \text{if } b = 4c+1, \\ \nu_2(\binom{m+2c-1}{2c-1}) + \nu_2(m - z_c) + 4c - 1, & \text{if } b = 4c-1, \end{cases}$$

where z_c is some 2-adic integer 2-adically close to $-c$.

Although there are many similarities with the 2-adic valuation of $(T_n^{(k)})$, several notable differences may also be observed. As with $(T_n^{(k)})$, the valuation of $(F_n^{(k)})$ is given by polynomial formulas in congruence classes modulo $2k+2$. However, there are only 5 congruence classes on which the valuation is constant, and $\lfloor k/2 \rfloor$ classes on which the valuation depends on a (presumably) non-integral z value, and the degrees of the polynomials are as large as $\lfloor (k+1)/2 \rfloor$. The "extra" z-values occur only in specific congruence classes, regardless of the parity of k; namely, either $n = 2m(k+1)+b$ with $1 < b \leqslant k+1$ and $b \equiv 1 \pmod 4$, or $n = (2m+1)(k+1)+b$ with $1 < b \leqslant k+1$ and $b \equiv -1 \pmod 4$. In particular, for any odd k this theorem gives the 2-adic valuation of $F_n^{(k)}$ completely explicitly for all even n.

2. Demonstration of 2-Adic Valuation

We consider the sequence $F_n^{(k)} = F_n$ defined by the k-th order recurrence $F_n = F_{n-1} + F_{n-2} + \cdots + F_{n-k}$, with initial conditions $F_1 = 1$ and $F_n = 0$ for $-(k-2) \leqslant n \leqslant 0$. The 2-adic valuation $\nu_2(n)$ of an integer n denotes the highest power of 2 which divides n; this valuation extends uniquely to $\mathbb{C}_2$, the completion of an algebraic closure of the field $\mathbb{Q}_2$ of 2-adic numbers. A binomial coefficient $\binom{m}{n}$ is regarded as a polynomial $m(m-1)\cdots(m-n+1)/n!$ in m of degree n if $n \geqslant 0$, and vanishes if n is a negative integer or if m is an integer with $0 \leqslant m < n$.

As in previous work [14], our analysis relies crucially on the principle [14, Theorem 3] that the functions $m \mapsto F_{m(2k+2)+b}^{(k)}$ are 2-adically analytic on $\mathbb{Z}_2$; that is, given by rapidly converging 2-adic power series on large disks in $\mathbb{C}_2$ containing the ring $\mathbb{Z}_2$ of 2-adic integers. (The proof given in [14] for $(T_n^{(k)})$ depends only on the recurrence and not on the initial conditions). But rather than estimating the power series coefficients by the method described in [14, §4], we here employ an adaptation of the classical formula of Ferguson [3]:

Theorem 2 (Generalized Ferguson's Formula)**.** *Let $k \geqslant 2$ and $1 \leqslant b \leqslant k+1$. Then for any integer $a \in \mathbb{Z}$ we have*

$$F_{a(k+1)+b} = 2^{b-2} \sum_{i=0}^{\infty} (-1)^{a-i} 2^{i(k+1)} \left[\binom{ik+a+b-1}{ik+i+b-1} + \binom{ik+a+b-2}{ik+i+b-1} \right]$$

For integers $a \geqslant 0$ the sum terminates as soon as $i > a$; for integers $a < 0$ the sum terminates as soon as $ik > |a|$.

Proof. This theorem was derived for positive integers $n = a(k+1)+b$ by Ferguson [3, p. 272]. However, $a \mapsto F_{a(k+1)+b}$ is a continuous function on $\mathbb{Z}_2$ by [14, Theorem 3], and as a uniformly convergent series of polynomials, the right side is also continuous for $a \in \mathbb{Z}_2$. Since the positive integers, and the negative integers, are dense in $\mathbb{Z}_2$, the equality holds for negative integers a as well. □

Proof of Theorem 1. We first dispose of the five trivial congruence classes modulo $2k+2$ on which the 2-adic valuation is constant. For $b = 1, 2$, Theorem 2 implies

$$F_{a(k+1)+1} \equiv (-1)^a 2^{-1} \left[\binom{a}{0} + \binom{a-1}{0} \right] \equiv (-1)^a \pmod{2^k \mathbb{Z}_2} \tag{2.1}$$

and

$$F_{a(k+1)+2} \equiv (-1)^a \left[\binom{a+1}{1} + \binom{a}{1} \right] \equiv (-1)^a (2a+1) \pmod{2^{k+1} \mathbb{Z}_2}, \tag{2.2}$$

and therefore $\nu_2(F_{a(k+1)+b}) = 0$ for $b = 1, 2$. For $b = 3$ and a even,

$$\begin{aligned} F_{2m(k+1)+3} &\equiv (-1)^a 2 \left[\binom{2m+2}{2} + \binom{2m+1}{2} \right] \\ &= (-1)^a (2m+1)(4m+2) \pmod{2^{k+2} \mathbb{Z}_2}, \end{aligned} \tag{2.3}$$

so $\nu_2(F_{2m(k+1)+3}) = 1$. These are the only congruence classes of constant valuation.

For the remainder of the proof we assume $3 \leqslant b \leqslant k+1$ and rewrite the identity of Theorem 2 in the more convenient form

$$\begin{aligned} F_{a(k+1)+b} &= 2^{b-2} \sum_{i=0}^{\infty} (-1)^{a-i} 2^{i(k+1)} \left[\binom{ik+a+b-2}{ik+i+b-2} \times \frac{2a+ik-i+b-1}{ik+i+b-1} \right] \\ &= 2^{b-2} \sum_{i=0}^{\infty} A_i^{(b)}(a). \end{aligned} \tag{2.4}$$

In this expression, each term $A_i^{(b)}(a)$ is a polynomial in a with rational coefficients and zeros in $\frac{1}{2}\mathbb{Z}$; specifically we observe that each $A_i^{(b)}(a)$ vanishes for all $a \in \{-1,\ldots,2-b\}$, and is therefore divisible in $\mathbb{Q}[a]$ by the polynomial $\binom{a+b-2}{b-2}$. Dividing each term in (2.4) by this binomial coefficient, we consider the functions $\tilde{A}^{(b)}(a) = \sum_{i=0}^{\infty} \tilde{A}_i^{(b)}(a)$ defined by

$$\begin{aligned} F_{a(k+1)+b}\Big/\binom{a+b-2}{b-2} &= \left(2^{b-2}\sum_{i=0}^{\infty} A_i^{(b)}(a)\right)\Big/\binom{a+b-2}{b-2} \\ &= \frac{2^{b-2}}{b-1}\sum_{i=0}^{\infty}\tilde{A}_i^{(b)}(a) = \frac{2^{b-2}}{b-1}\tilde{A}^{(b)}(a), \end{aligned} \tag{2.5}$$

where

$$\tilde{A}_i^{(b)}(a) = (-1)^{a-i}2^{i(k+1)}\frac{(b-1)(a)_i}{(b+i-2)_i}\frac{2a+ik-i+b-1}{ik+i+b-1}\frac{(ik+a+b-2)_{ik}}{(ik+i+b-2)_{ik}}. \tag{2.6}$$

Here $(x)_m$ denotes the falling factorial $(x)_m = x(x-1)\cdots(x+1-m)$ for $m>0$, with $(x)_0 = 1$. Because of the analyticity of $F_{2m(k+1)+b}$ and $F_{(2m+1)(k+1)+b}$ [14, Theorem 3], we conclude that both $\tilde{A}^{(b)}(2m)$ and $\tilde{A}^{(b)}(2m+1)$ are 2-adic analytic functions of m on a large 2-adic disk in $\mathbb{C}_2$ which contains $\mathbb{Z}_2$; that is, they are given by rapidly convergent power series on $\mathbb{Z}_2$. We note that

$$\tilde{A}_0^{(b)}(a) = (-1)^a(2a+b-1), \tag{2.7}$$

so that both

$$\tilde{A}_0^{(b)}(2m) = 4m+b-1 \qquad \text{and} \qquad \tilde{A}_0^{(b)}(2m+1) = -(4m+b+1),$$

as polynomials in m, have linear coefficient of valuation 2. We will show that all coefficients of the polynomials $\tilde{A}_i^{(b)}(2m)$ and $\tilde{A}_i^{(b)}(2m+1)$, for $i>0$, have valuation larger than 2. This implies that $\tilde{A}^{(b)}(2m+j) \equiv \tilde{A}_0^{(b)}(2m+j) \pmod{2^3\mathbb{Z}_2[[m]]}$ for $j=0,1$, where $\mathbb{Z}_2[[m]]$ denotes the ring of formal power series over $\mathbb{Z}_2$; this will suffice to determine their valuations.

We first write $\tilde{A}_i^{(b)}(2m) = \sum_j a_{i,j}^{(b)} m^j$, and let $\pi_i^{(b)}$ denote the product of the zeros of $\tilde{A}_i^{(b)}(2m)$ which have finite nonnegative 2-adic valuation. For $i>0$, the constant term $a_{i,0}^{(b)}$ of the polynomial $\tilde{A}_i^{(b)}(2m)$ in (2.6) is zero, and the linear coefficient is

$$a_{i,1}^{(b)} = (-1)^i 2^{i(k+1)}\frac{2(b-1)(i-1)!(ik-i+b-1)}{(ik+i+b-1)_{i+1}}. \tag{2.8}$$

For $i>0$, the 2-adic valuation of this coefficient is

$$\begin{aligned} \nu_2(a_{i,1}^{(b)}) &= i(k+1)-1+\nu_2(b-1)+\nu_2(ik-i+b-1) \\ &\quad +S_2(ik+i+b-1)-S_2(i-1)-S_2(ik+b-2). \end{aligned} \tag{2.9}$$

Here we have used the well-known fact that $\nu_2(n!) = n - S_2(n)$, where $S_2(n)$ denotes the sum of the base 2 digits of n. Excluding zeros at $a = 0$ and at $-2a = ik - i + b - 1$, all the remaining zeros of the polynomial $\tilde{A}_i^{(b)}(a)$ in (2.6) are integers whose product, up to sign, is

$$\frac{(i-1)!(ik+b-2)!}{(b-2)!} \tag{2.10}$$

and the 2-adic valuation of this product is

$$ik + i - 1 + S_2(b-2) - S_2(i-1) - S_2(ik+b-2). \tag{2.11}$$

Every zero of $\tilde{A}_i^{(b)}(2m)$ of finite nonnegative valuation is one-half of an even integer zero of $\tilde{A}_i^{(b)}(a)$; since there are at least $\lfloor \frac{i-1}{2} \rfloor + \lfloor \frac{ik}{2} \rfloor$ such zeros, the product of the integral zeros of $\tilde{A}_i^{(b)}(2m)$ has valuation satisfying

$$\begin{aligned} \nu_2(\pi_i^{(b)}) \quad &\leqslant \lceil \frac{i-1}{2} \rceil + \lceil \frac{ik}{2} \rceil + S_2(b-2) - S_2(i-1) - S_2(ik+b-2) \\ &\quad + \max\{0, \nu_2((ik-i+b-1)/4)\}. \end{aligned} \tag{2.12}$$

For a polynomial or power series $f(x) = \sum_{i \geqslant 0} a_i x^i \in \mathbb{C}_2[[x]]$, the *Newton polygon* of f is the upper convex hull of the set of points $\{(i, \nu_2(a_i)) : i \geqslant 0\}$. A basic property ([5, Ch. IV.3, Lemma 4]; [9, Theorem 9.1]) is that the Newton polygon of f has a side of slope m and horizontal run l if and only if f has l zeros (counted with multiplicity) $\alpha_i \in \mathbb{C}_2$ with $\nu_2(\alpha_i) = -m$. Therefore the above quantity (2.12) is an upper bound for the maximum vertical drop in the Newton polygon of $\tilde{A}_i^{(b)}(2m)$ from its vertex at degree 1. Subtracting (2.12) from (2.9), the valuation of any coefficient therefore satisfies

$$\begin{aligned} \nu_2(a_{i,j}^{(b)}) \quad &\geqslant \nu_2(a_{i,1}^{(b)}) - \nu_2(\pi_i^{(b)}) \\ &\geqslant \lfloor \frac{i-1}{2} \rfloor + \lfloor \frac{ik}{2} \rfloor + 1 + \min\{2, \nu_2(ik-i+b-1)\} \\ &\quad + S_2(ik+i+b-1) - S_2(b-1) \\ &\geqslant \lfloor \frac{i-1}{2} \rfloor + \lfloor \frac{ik}{2} \rfloor + 3 - S_2(b-1) := r_{i,b} \end{aligned} \tag{2.13}$$

Since $1 \leqslant S_2(n) \leqslant \lfloor \lg(n+1) \rfloor$, we can estimate

$$r_{i,b} \geqslant r := \lfloor \frac{k}{2} \rfloor + 3 - \lfloor \lg(k+1) \rfloor \rfloor \tag{2.14}$$

for $i > 0$ and $3 \leqslant b \leqslant k+1$, which satisfies $r \geqslant 3$ for all $k \geqslant 4$. Therefore all coefficients $a_{i,j}^{(b)}$ for $i > 0$ have valuation $\nu_2(a_{i,j}^{(b)}) \geqslant 3$.

We have therefore shown that

$$F_{2m(k+1)+b} = \binom{2m+b-2}{b-2}\frac{2^{b-2}}{b-1}\tilde{A}^{(b)}(2m) \tag{2.15}$$

where $\tilde{A}^{(b)}(2m)$ is an analytic function (i.e., a convergent power series) for $m \in \mathbb{Z}_2$ which satisfies $\tilde{A}^{(b)}(2m) \equiv b-1+4m \pmod{2^r\mathbb{Z}_2[[m]]}$, with $r \geqslant 3$. From this it is clear that if b is even, then $\nu_2(\tilde{A}^{(b)}(2m)) = 0$ for all $m \in \mathbb{Z}_2$, and if $b \equiv -1 \pmod 4$, then $\nu_2(\tilde{A}^{(b)}(2m)) = 1$ for all $m \in \mathbb{Z}_2$. In either of these cases, from (2.15) we have

$$\nu_2(F_{2m(k+1)+b}) = \nu_2\left(\binom{2m+b-2}{b-2}\right) + b - 2. \tag{2.16}$$

If $b = 4c+1$, then $\tilde{A}^{(b)}(2m) \equiv 4(c+m) \pmod{2^r\mathbb{Z}_2[[m]]}$. In this case the Newton polygon of $\tilde{A}^{(b)}(2m)$ has vertices at $(0, \nu_2(4c))$ and $(1, 2)$ which determine its unique side of non-positive integer slope, corresponding to its unique root z_c of nonnegative valuation. By Hensel's Lemma [5, Ch. I.5, Theorem 3], this root z_c lies in $\mathbb{Z}_2$. By [5, Ch. IV.4, Theorem 14], the power series $\tilde{A}^{(b)}(2m)$ factors in $\mathbb{Z}_2[[m]]$ as the product of $m - z_c$ and a power series of constant valuation on $\mathbb{Z}_2$, so that $\nu_2(\tilde{A}^{(b)}(2m)) = 2 + \nu_2(m - z_c)$ for all $m \in \mathbb{Z}_2$. Therefore from (2.15) we have

$$\nu_2(F_{2m(k+1)+b}) = \nu_2\left(\binom{2m+b-2}{b-2}\right) + \nu_2(m - z_c) + b - 2 - \nu_2(c). \tag{2.17}$$

These valuations (2.16), (2.17) are equivalent to those given in the statement of the theorem, and complete the proof of the first statement, that is the case of $a = 2m$.

The analysis in the case of $a = 2m+1$ is quite similar. The only substantive difference from the analysis of (2.8) - (2.13) is that the constant coefficient of $\tilde{A}_1^{(b)}(2m+1)$ (as a polynomial in m) does not vanish, but equals 2^{k+1}. For $i > 1$ the constant coefficient of $\tilde{A}_i^{(b)}(2m+1)$ vanishes, and the analysis of (2.8) - (2.13) may be repeated almost identically; we omit the details. So in similar manner as for $a = 2m$, we deduce that

$$F_{(2m+1)(k+1)+b} = \binom{2m+b-1}{b-2}\frac{2^{b-2}}{b-1}\tilde{A}^{(b)}(2m+1) \tag{2.18}$$

with $\tilde{A}^{(b)}(2m+1) \equiv -(4m+b+1) \pmod{2^r\mathbb{Z}_2[[m]]}$ and $r \geqslant 3$. The valuations of the second statement of the theorem may then be determined as in (2.16), (2.17), completing the proof. $\square$

3. Zero-Multiplicity of $(F_n^{(k)})$

In [4, Theorem 5.1] it was observed that the zero-multiplicity of the sequence $(F_n^{(k)})$ is always at least $k(k-1)/2$; in particular, we have

$$F_{-n}^{(k)} = 0 \quad \text{for all integers} \quad n \in \bigcup_{0 \leqslant m \leqslant k-2} [m(k+1), (m+1)k-2]. \tag{3.1}$$

This may be seen from the generalized Ferguson's formula (Theorem 2). As a consequence of our analysis we can bound the size and number of exceptional zeros.

Theorem 3. *The 2-adic functions $m \mapsto F_{2m(k+1)+b}^{(k)}$ and $m \mapsto F_{(2m+1)(k+1)+b}^{(k)}$ have no multiple zeros. For $k \geqslant 4$, if n is an integer not included in (3.1) with $F_{-n}^{(k)} = 0$ then $|n| > 2^{\lfloor k/2 \rfloor + 2}$, and there are at most $\lfloor k/2 \rfloor$ such zeros.*

Proof. From Theorem 1 the first sentence is clear in all cases except when $a = 2m$ and $b = 4c+1$, or when $a = 2m+1$ and $b = 4c-1$. In the first case where $b = 4c+1$, the zeros of $m \mapsto F_{2m(k+1)+b}^{(k)}$ are $-m \in \{1, \dots, 2c-1\}$ and $m = z_c$. From (2.15) we have $\tilde{A}^{(b)}(2m) \equiv 4(c+m) \pmod{2^r \mathbb{Z}_2[[m]]}$, where r is as in (2.14). So if $\tilde{A}^{(b)}(2m) = 0$, then $m \equiv -c \pmod{2^{r-2}}$. We first show that $\tilde{A}^{(b)}(-2c) \neq 0$. From (2.6) we have $\tilde{A}_0^{(b)}(-2c) = 0$ and

$$\begin{aligned} \tilde{A}_1^{(b)}(-2c) &= 2^{(k+1)} 2c \frac{k-1}{k+b} \frac{(k+2c-1)_k}{(k+b-1)_k} \\ &= 2^{(k+1)} 2c \frac{(k-1)(4c)\cdots(2c)}{(k+4c+1)\cdots(k+2c)}, \end{aligned} \tag{3.2}$$

whose valuation is

$$\begin{aligned} \nu_2(\tilde{A}_1^{(b)}(-2c)) &= k+1+\nu_2(c(k-1)) - S_2(4c) + S_2(2c-1) \\ &\quad + S_2(k+4c+1) - S_2(k+2c-1). \end{aligned} \tag{3.3}$$

Similarly, from (2.6) we find

$$\begin{aligned} \nu_2(\tilde{A}_2^{(b)}(-2c)) &= 2k+3+\nu_2(c(k-1)) - S_2(4c+1) + S_2(2c-1) \\ &\quad + S_2(2k+4c+2) - S_2(2k+2c-1), \end{aligned} \tag{3.4}$$

so that the difference in valuations $\nu_2(\tilde{A}_2^{(b)}(-2c)) - \nu_2(\tilde{A}_1^{(b)}(-2c))$ is

$$\begin{aligned}&k+2+S_2(2k+4c+2)-S_2(2k+2c-1)-S_2(4c+1)+S_2(4c)\\&-S_2(k+4c+1)+S_2(k+2c-1)\\&\geqslant k+4-S_2(2k+2c-1)-S_2(k+4c+1)\\&\geqslant k+4-\lg(5k/2)-\lg(2k+2)>0\end{aligned}\tag{3.5}$$

for all $k \geqslant 4$. Similarly we find that $\nu_2(\tilde{A}_1^{(b)}(-2c)) < \nu_2(\tilde{A}_i^{(b)}(-2c))$ for all $i > 1$ when $k \geqslant 4$, which shows that $\tilde{A}^{(b)}(-2c) \neq 0$ when $b = 4c+1$. A similar analysis shows that $\tilde{A}^{(b)}(2m+1) \neq 0$ when $b = 4c-1$ and $m = -2c$.

From our analysis, when $a = 2m$ and $b = 4c+1$ we have $z_c \equiv -c \pmod{2^{r-2}\mathbb{Z}_2}$ and $z_c \neq -c$. Therefore if $n = 2m(k+1)+b$ is a negative integer with $\tilde{A}^{(b)}(2m) = 0$ then $|m| \geqslant c + 2^{r-2} \geqslant 1 + 2^{r-2} \geqslant \lfloor k/2 \rfloor$ for any $k \geqslant 4$; since no such n is included in (3.1), the analytic function $m \mapsto F_{2m(k+1)+b}$ has no multiple zeros. If $-n = 2m(k+1)+b$ is an integer with $F_{-n}^{(k)} = 0$, and is not included in (3.1), then $m \geqslant 2^{r-2}+1$ and thus

$$|n| \geqslant |(2m-1)(k+1)| > 2^{r-1}(k+1) \geqslant 2^{\lfloor k/2 \rfloor + 2}. \tag{3.6}$$

The case of $m \mapsto F_{(2m+1)(k+1)+b}$ follows from similar estimates. □

Corollary 1. *For $k \in \{4, \ldots, 500\}$ no value of z_c in Theorem 1 is a rational integer.*

Proof. If any $z_c \in \mathbb{Z}$, then $F_n^{(k)} = 0$ for the corresponding integer value $n = 2z_c(k+1)+b$ or $n = (2z_c+1)(k+1)+b$, which is not included in (3.1); hence the zero-multiplicity of $(F_n^{(k)})$ would exceed $k(k-1)/2$, contradicting [4, Corollary 1.2]. □

The value $F_{-17}^{(3)} = 0$ is the only known exceptional value not on the list (3.1) (cf. Theorem 5 below, [8]). Since there can be no other exceptions for $k \leqslant 500$, this implies via Theorem 3 that there are no other exceptions with $|n| < 2^{252}$. Thus any exception to (3.1) must be large, there cannot be very many such, and they can only occur in certain congruence classes.

Corollary 2. *For $k > 500$, the zero-multiplicity of $(F_n^{(k)})$ is at most $k(k-1)/2 + \lfloor k/2 \rfloor$; any exceptional value n not included in (3.1) with $F_n^{(k)} = 0$ must occur either in a congruence class $n = 2m(k+1)+b$ with $b = 4c+1$ or $n = (2m+1)(k+1)+b$ with $b = 4c-1$, and there can be at most one such value n in each such class.*

4. Small Values of k

We conclude with some illustrations of these theorems for specific small values of k, including some numerical computations of the z values. We begin by restating the previously known valuations for $k = 2, 3$ for purposes of comparison.

Theorem 4 (Lengyel [6], 1995)**.** *For order $k = 2$, the 2-adic valuation of the n-th Fibonacci number F_n is given by*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2, 4, 5 \pmod 6, \\ 1, & \text{if } n \equiv 3 \pmod 6, \\ 3 + \nu_2(m+1), & \text{if } n = 6m + 6. \end{cases}$$

Theorem 5 (Marques and Lengyel [8], 2014)**.** *For order $k = 3$, the 2-adic valuation of the n-th Tribonacci number $T_n = F_n^{(3)}$ is given by*

$$\nu_2(F_n^{(3)}) = \begin{cases} 0, & \text{if } n \equiv 1, 2, 5, 6 \pmod 8, \\ 1, & \text{if } n \equiv 3 \pmod 8, \\ 2 + \nu_2(m+1), & \text{if } n = 8m + 4 \text{ or } n = 8m + 8, \\ 3 + \nu_2((m+1)(m+3)), & \text{if } n = 8m + 7. \end{cases}$$

Remark. It will be observed that this formula fits the pattern in our Theorem 1 by taking the value $z_1 = -3$ for $n = 8m + 7$.

Corollary 3. *For $k = 4$, for all integers n we have*

$$\nu_2(F_n^{(4)}) = \begin{cases} 0, & \text{if } n \equiv 1, 2, 6, 7 \pmod{10}, \\ 1, & \text{if } n \equiv 3 \pmod{10}, \\ 2 + \nu_2(m+1), & \text{if } n = 10m + 4 \text{ or } n = 10m + 9, \\ 3 + \nu_2((m+1)(m+2)), & \text{if } n = 10m + 10, \\ 3 + \nu_2((m+1)(m-z)), & \text{if } n = 10m + 5 \text{ or } n = 10m + 8, \end{cases}$$

where the 2-adic integer $z \equiv 206263 \pmod{2^{25}\mathbb{Z}_2}$ for $n = 10m + 5$ and $z \equiv 706567 \pmod{2^{21}\mathbb{Z}_2}$ for $n = 10m + 8$.

Corollary 4. *For $k = 5$, for all integers n we have*

$$\nu_2(F_n^{(5)}) = \begin{cases} 0, & \text{if } n \equiv 1, 2, 7, 8 \pmod{12}, \\ 1, & \text{if } n \equiv 3 \pmod{12}, \\ 2 + \nu_2(m+1), & \text{if } n = 12m + 4 \text{ or } n = 12m + 10, \\ 3 + \nu_2((m+1)(m+2)), & \text{if } n = 12m + 6, 11, 12 \\ 3 + \nu_2((m+1)(m-z)), & \text{if } n = 12m + 5 \text{ or } n = 12m + 9, \end{cases}$$

where the 2-adic integer $z \equiv 439743 \pmod{2^{20}\mathbb{Z}_2}$ for $n = 12m + 5$ and $z \equiv 651063 \pmod{2^{22}\mathbb{Z}_2}$ for $n = 12m + 9$.

Corollary 5. *For $k = 6$, for all integers n we have*

$$\nu_2(F_n^{(6)}) = \begin{cases} 0, & \text{if } n \equiv 1,2,8,9 \pmod{14}, \\ 1, & \text{if } n \equiv 3 \pmod{14}, \\ 2+\nu_2(m+1), & \text{if } n = 14m+4 \text{ or } n = 14m+11, \\ 3+\nu_2((m+1)(m+2)), & \text{if } n = 14m+6,12,13, \\ 4+\nu_2((m+1)(m+2)), & \text{if } n = 14m+7, \\ 3+\nu_2((m+1)(m-z)), & \text{if } n = 14m+5 \text{ or } n = 14m+10, \\ 7+\nu_2\left(\binom{m+3}{3}(m-z)\right), & \text{if } n = 14m+14, \end{cases}$$

where the 2-adic integer $z \equiv 225759 \pmod{2^{20}\mathbb{Z}_2}$ for $n = 14m+5$, $z \equiv 842695 \pmod{2^{22}\mathbb{Z}_2}$ for $n = 14m+10$, and $z \equiv 205214 \pmod{2^{20}\mathbb{Z}_2}$ for $n = 14m+14$.

Corollary 6. *For $k = 7$, for all integers n we have*

$$\nu_2(F_n^{(7)}) = \begin{cases} 0, & \text{if } n \equiv 1,2,9,10 \pmod{16}, \\ 1, & \text{if } n \equiv 3 \pmod{16}, \\ 2+\nu_2(m+1), & \text{if } n = 16m+4 \text{ or } n = 16m+12, \\ 3+\nu_2((m+1)(m+2)), & \text{if } n = 16m+6,13,14 \\ 4+\nu_2((m+1)(m+2)), & \text{if } n = 16m+7, \\ 6+\nu_2\left(\binom{m+3}{3}\right), & \text{if } n = 16m+8,16, \\ 3+\nu_2((m+1)(m-z)), & \text{if } n = 16m+5 \text{ or } n = 16m+11, \\ 7+\nu_2\left(\binom{m+3}{3}(m-z)\right), & \text{if } n = 16m+15, \end{cases}$$

where the 2-adic integer $z \equiv 848639 \pmod{2^{25}\mathbb{Z}_2}$ for $n = 16m+5$, $z \equiv 603535 \pmod{2^{23}\mathbb{Z}_2}$ for $n = 16m+11$, and $z \equiv 131902 \pmod{2^{20}\mathbb{Z}_2}$ for $n = 16m+15$.

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