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IRREDUCIBILITY OF GENERALIZED FIBONACCI POLYNOMIALS

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Abstract

A second order polynomial sequence is of *Fibonacci-type* $\mathcal{F}_n$ (*Lucas-type* $\mathcal{L}_n$) if its Binet formula has a structure similar to that for Fibonacci (Lucas) numbers. Under certain conditions these polynomials are irreducible if and only if n is a prime number. For example, the Fibonacci polynomials, Pell polynomials, Fermat polynomials, Lucas polynomials, Pell-Lucas polynomials, Fermat-Lucas polynomials are irreducible when n is a prime number; and Chebyshev polynomials (second kind), Morgan-Voyce polynomials (Fibonacci type), and Vieta polynomials are reducible when n is a prime number. In this paper we give some theorems to determine whether the Fibonacci type polynomials and Lucas type polynomials are irreducible when n is prime.

1. Introduction

The Fibonacci polynomials F_n are defined as $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, where $F_0(x) = 0$ and $F_1(x) = 1$. Webb et al. [21] proved that F_p is irreducible if and only if p is a prime number. Hogatt et al. [11] defined a bivariate generalized Fibonacci polynomial $u_n(x, y)$ and proved that $u_p(x, y)$ is irreducible over $\mathbb{Q}$ if and only if p is a prime number.

The Lucas polynomials L_n are defined as $L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$, where $L_0(x) = 2$ and $L_1(x) = x$. Bergum and Hoggatt [3] proved that $L_p(x)/L_1(x)$ is irreducible if and only if $p > 2$ is a prime number. They also defined a bi-

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variate generalized Lucas polynomial $v_n(x,y)$ and proved that $v_p(x,y)/v_1(x,y)$ is irreducible over $\mathbb{Q}$ if and only if $p>2$ is a prime number.

A second order polynomial sequence is of *Fibonacci-type* (*Lucas-type*) if its Binet formula has a structure similar to that for Fibonacci (Lucas) numbers. Those are known as *generalized Fibonacci polynomials* GFP (see [5–9]). Some known examples are: Pell polynomials, Fermat polynomials, Chebyshev polynomials, Morgan-Voyce polynomials, Lucas polynomials, Pell-Lucas polynomials, Fermat-Lucas polynomials, Chebyshev polynomials, Vieta polynomials, and Vieta-Lucas polynomials. Other generalized Fibonacci polynomials are in [1, 3, 11].

From the discussion in the first two paragraphs above, we have two natural questions: is it true that $\mathcal{F}_p(x)$ is irreducible if and only if p is a prime number? And is it true that $\mathcal{L}_p(x)/\mathcal{L}_1(x)$ is irreducible if and only if $p>2$ is a prime number? In this paper we give precise conditions to determine whether some families of GFP are irreducible when p is a prime number and give precise conditions to determine whether some families of GFP are reducible when p is a prime number. As a corollary of the theorems proved here, we obtain that the Fibonacci polynomials, the Pell polynomials, and the Fermat polynomials are irreducible when $p>0$ is a prime number. A second corollary is that Chebyshev polynomials (second kind), Morgan-Voyce polynomials (Fibonacci type), and Vieta polynomials are reducible when p is a prime number. As a third corollary we have that $\mathcal{L}_p(x)/\mathcal{L}_1(x)$ is irreducible, where $p>2$ is a prime number and $\mathcal{L}_p(x)$ is one of these: Lucas polynomials, Pell-Lucas polynomials, or Fermat-Lucas polynomials.

2. Second Order Polynomial Sequences

In this section we reproduce the definitions by Flórez et al. [5–9] for generalized Fibonacci polynomials. The definitions here give rise to some known polynomial sequences (see for example, Table 1 or [5–9, 11, 13, 16, 17]). Throughout the paper we consider polynomials in $\mathbb{Q}[x]$ or in $\mathbb{Z}[x]$.

We now give the two second order polynomial recurrence relations in which we divide the generalized Fibonacci polynomials (GFP):

$$\mathcal{F}_0(x)=0,\ \mathcal{F}_1(x)=1,\ \text{and}\ \mathcal{F}_n(x)=d(x)\mathcal{F}_{n-1}(x)+g(x)\mathcal{F}_{n-2}(x), \tag{1}$$

for $n\geq 2$, where $d(x)$ and $g(x)$ are fixed non-zero polynomials in $\mathbb{Z}[x]$ satisfying $\gcd(d(x),g(x))=1$.

We say that a polynomial recurrence relation is of *Fibonacci-type* if it satisfies the relation given in (1), and of *Lucas-type* if:

$$\mathcal{L}_0(x)=p_0,\ \mathcal{L}_1(x)=p_1(x),\ \ \text{and}\ \ \mathcal{L}_n(x)=d(x)\mathcal{L}_{n-1}(x)+g(x)\mathcal{L}_{n-2}(x), \tag{2}$$

for $n\geq 2$, where $|p_0|=1$ or 2 and $p_1(x)$, $d(x)=\alpha p_1(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Z}[x]$ with α an integer of the form $2/p_0$. Some known examples of Fibonacci-type polynomials and Lucas-type polynomials are in Table 1 or in [5–9, 11, 13, 16, 17].

If G_n is either $\mathcal{F}_n$ or $\mathcal{L}_n$ for all $n \geq 0$ and $d^2(x) + 4g(x) > 0$, then the explicit formula for the recurrence relations in (1) and (2) is given by

$$G_n(x) = t_1 a^n(x) + t_2 b^n(x),$$

where $a(x)$ and $b(x)$ are the solutions of the quadratic characteristic equation associated with the second-order recurrence relation $G_n(x)$. That is, $a(x)$ and $b(x)$ are the solutions of $z^2 - d(x)z - g(x) = 0$. If $\alpha = 2/p_0$, then the Binet formula for Fibonacci-type polynomials is stated in (3) and the Binet formula for Lucas-type polynomials is stated in (4) (for details on the construction of the two Binet formulas see [7])

$$\mathcal{F}_n(x) = \frac{a^n(x) - b^n(x)}{a(x) - b(x)} \tag{3}$$

and

$$\mathcal{L}_n(x) = \frac{a^n(x) + b^n(x)}{\alpha}. \tag{4}$$

Since $a(x)$ and $b(x)$ are solutions of $z^2 - d(x)z - g(x) = 0$, we have

$$a(x) + b(x) = d(x), \quad a(x)b(x) = -g(x), \quad \text{and} \quad a(x) - b(x) = \sqrt{d^2(x) + 4g(x)},$$

where $d(x)$ and $g(x)$ are the polynomials defined in (1) and (2). These give that

$$a(x) = \frac{d(x) + \sqrt{d^2(x) + 4g(x)}}{2} \quad \text{and } b(x) = \frac{d(x) - \sqrt{d^2(x) + 4g(x)}}{2}. \tag{5}$$

A sequence of Lucas-type (Fibonacci-type) is *equivalent* or *conjugate* to a sequence of Fibonacci-type (Lucas-type), if their recursive sequences are determined by the same polynomials $d(x)$ and $g(x)$. Notice that two equivalent polynomials have the same $a(x)$ and $b(x)$ in their Binet representations. In [7, 8, 17, 18] there are examples of some known equivalent polynomials with their Binet formulas. The polynomials in Tables 1 and 2 are organized by pairs of equivalent polynomials. For instance, Fibonacci and Lucas, Pell and Pell-Lucas, and so on.

We use $\deg(P)$ and $\operatorname{lc}(P)$ to mean the degree and the leading coefficient of a polynomial P, respectively. Most of the following conditions were required in the papers that we are citing. Therefore, we require here that $\gcd(d(x), g(x)) = 1$ and $\deg(g(x)) < \deg(d(x))$ for both types of sequences and that the conditions in (6) also hold for Lucas type polynomials;

$$\gcd(p_0, p_1(x)) = \gcd(p_0, d(x)) = \gcd(p_0, g(x)) = 1, \text{ and that } \deg(\mathcal{L}_1) \geq 1. \tag{6}$$

Notice that in the definition of Pell-Lucas we have $Q_0(x) = 2$ and $Q_1(x) = 2x$. Thus, the $\gcd(2, 2x) = 2 \neq 1$. Therefore, Pell-Lucas does not satisfy the extra

Polynomial	Initial value $G_0(x) = p_0(x)$	Initial value $G_1(x) = p_1(x)$	Recursive Formula $G_n(x) = d(x)G_{n-1}(x) + g(x)G_{n-2}(x)$
Fibonacci	0	1	$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$
Lucas	2	x	$D_n(x) = xD_{n-1}(x) + D_{n-2}(x)$
Pell	0	1	$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$
Pell-Lucas	2	$2x$	$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x)$
Pell-Lucas-prime	1	x	$Q'_n(x) = 2xQ'_{n-1}(x) + Q'_{n-2}(x)$
Fermat	0	1	$\Phi_n(x) = 3x\Phi_{n-1}(x) - 2\Phi_{n-2}(x)$
Fermat-Lucas	2	$3x$	$\vartheta_n(x) = 3x\vartheta_{n-1}(x) - 2\vartheta_{n-2}(x)$
Chebyshev second kind	0	1	$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$
Chebyshev first kind	1	x	$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$
Morgan-Voyce	0	1	$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x)$
Morgan-Voyce	2	$x+2$	$C_n(x) = (x+2)C_{n-1}(x) - C_{n-2}(x)$
Vieta	0	1	$V_n(x) = xV_{n-1}(x) - V_{n-2}(x)$
Vieta-Lucas	2	x	$v_n(x) = xv_{n-1}(x) - v_{n-2}(x)$

Table 1: Recurrence relation of some GFP.

Polynomial Lucas type	Polynomial of Fibonacci type	$a(x)$	$b(x)$
Lucas	Fibonacci	$(x+\sqrt{x^2+4})/2$	$(x-\sqrt{x^2+4})/2$
Pell-Lucas-prime	Pell	$x+\sqrt{x^2+1}$	$x-\sqrt{x^2+1}$
Fermat-Lucas	Fermat	$(3x+\sqrt{9x^2-8})/2$	$(3x-\sqrt{9x^2-8})/2$
Chebyshev 1st kind	Chebyshev 2nd kind	$x+\sqrt{x^2-1}$	$x-\sqrt{x^2-1}$
Morgan-Voyce	Morgan-Voyce	$(x+2+\sqrt{x^2+4x})/2$	$(x+2-\sqrt{x^2+4x})/2$
Vieta-Lucas	Vieta	$(x+\sqrt{x^2-4})/2$	$(x-\sqrt{x^2-4})/2$

Table 2: $\mathcal{L}_n(x)$ and its conjugate $\mathcal{F}_n(x)$.

conditions that we imposed in (6). So, to resolve this inconsistency we use $Q'_n(x) = Q_n(x)/2$ instead of $Q_n(x)$.

For the rest of this paper we assume $\deg(d) > \deg(g)$. For instance, the familiar examples in Tables 1 and 2 satisfy this condition. Notice that Jacobsthal and Jacobsthal-Lucas polynomials defined as $j_n(x) = j_{n-1}(x) + 2xj_{n-1}(x)$ are GFP but they do not satisfy the mentioned condition. So, we do not study those polynomials here in this paper.

3. Fibonacci Type Polynomials' Irreducibility

In this section we discuss the irreducibility and reducibility of GFP of Fibonacci type. In particular, we give a complete classification (reducible and irreducible) for the familiar polynomials of Fibonacci type given in Table 1. In the end of the section we give a more general theorem to determine whether a GFP of Fibonacci type is irreducible.

The following lemma generalizes [11, Lemma 5]. The proof can be done by induction, so we omit it.

Lemma 1. *If $\mathcal{F}_n(x)$ is a GFP of Fibonacci type, with $n > 0$, then*

$$\mathcal{F}_n(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} d(x)^{n-2i-1} g(x)^i.$$

Hogatt et al. [11] defined the bivariate generalized Fibonacci polynomial

$$u_n(x,y) = xu_{n-1}(x,y) + yu_n(x,y), \quad \text{with } u_0(x,y) = 0 \quad \text{and} \quad u_n(x,y) = 1.$$

In their version of Lemma 1 for $u_n(x,y)$ it holds that $u_n(x,y^2)$ is a homogeneous polynomial. Webb et al. [21] proved that $u_p(x,1)$ is irreducible over $\mathbb{Q}$ if and only if p is a prime number. These two results were used in [11] to prove that $u_p(x,y)$ is irreducible over $\mathbb{Q}$ if and only if p is a prime number. However, we need some caution on the interpretation of these results. For example, in the result proved by Webb we cannot substitute x by any polynomial. Thus, if instead of x we take x^3 we obtain that $u_3(x^3,1) = (x+1)(x^2-x+1)$, so this new polynomial is reducible. Similarly, we can construct examples to show that $u_p(x,y)$ is not always irreducible for every prime and for every choice of y. For instance, if instead of y we take $-y^{2k}$, it holds that $u_p(x,-y^{2k})$ is not always irreducible when p is a prime number, with $k \geq 0$. For example, $u_5(x,-y^2) = (-x^2-xy+y^2)(-x^2+xy+y^2)$. In general, this gives a factoring for Chebyshev polynomials of second kind $U_p(x)$. Thus, if $p = 2k+1$, then $U_p(x) = (U_{k+1}(x) - y^k U_k(x))(U_{k+1}(x) + y^k U_k(x))$ (see Proposition 3). Some other examples, in which $u_p(x,y)$ is reducible, occur when taking $y = -1, -4, -5, -9, -20$. In particular, $u_5(x,-5) = (x^2-5x+5)(x^2+5x+5)$ and $u_5(x+2,-1) = (x^2+3x+1)(x^2+5x+5)$.

We now recall the first of our main questions in this paper. Is it true that $\mathcal{F}_p(x)$ is irreducible if and only if p is prime? From the above discussion and Proposition 3, we can see some counterexamples to determine that the question is not true in general. Since there are some families of the generalized Fibonacci polynomial that are irreducible if and only if p is a prime number, the question is still valid. In this section we explore the question for families of GFP of the Fibonacci type. (From the definition (1), we know that families of GFP of Fibonacci type depend on their initial conditions.) Thus, we reformulate the question as: under what conditions on $d(x)$ and $g(x)$ are the families of GFP of the Fibonacci type irreducible when p is a prime number.

Note that from [7, Proposition 6] we know that $\mathcal{F}_n(x)$ is reducible if n is a composite number. For the remaining part of the paper we use $F_n(x)$ to denote the classic Fibonacci polynomial as defined in the introduction.

Lemma 2 ([21])**.** *The Fibonacci polynomial $F_p(x)$ is irreducible over $\mathbb{Q}$ if and only if p is a prime number.*

Proposition 3. *Let $m(x)$ be a polynomial in $\mathbb{Z}[x]$ and let p be an odd number. If $g(x) = -m(x)^2$, then $\mathcal{F}_p(x)$ is reducible.*

Proof. If $p = 2k+1$, from [7] or [9, Proposition 1] we have that $\mathcal{F}_p(x) = \mathcal{F}_{k+1}^2(x) + g(x)\mathcal{F}_k^2(x)$. Since $g(x) = -m(x)^2$ the conclusion follows. □

The previous proposition shows that Chebyshev polynomials of the second kind, Morgan-Voyce polynomials and Vieta polynomials are reducible over $\mathbb{Q}$ when p is an odd prime number.

Proposition 4. *If $g(x) = 1$ and $d(x) = ax+b$ with $a \neq 0$, then $\mathcal{F}_p(x)$ is irreducible over $\mathbb{Q}$.*

Proof. First of all, we observe that if $g(x) = 1$ and $d(x) = ax + b$, then $\mathcal{F}_p(x) = (F_p \circ d)(x)$. Since both $F_p(x)$ and $ax + b$ are irreducible, we have that $\mathcal{F}_p(x)$ is irreducible. □

Lemma 5. *If $g(x) = k \in \mathbb{Z}_{>0}$ and $d(x) = ax$ with $a \neq 0$ and $k \in \mathbb{Z}_{>0}$, then $\mathcal{F}_p(x)$ is irreducible in $\mathbb{Q}[x]$.*

Proof. Since both $F_p(x)$ and ax are irreducible over $\mathbb{Q}$, we have that $\mathcal{F}_p^*(x) := (F_p \circ d)(x)$ is irreducible.

To complete this proof we need the following lemma. This lemma is an adaptation, to what we need here, of a result that is well-known in the literature (see for example [2, 4]).

Lemma 6 ([2, 4])**.** *Let $f(x)$ be a polynomial of degree n with $f(0) \neq 0$. Then $f(x)$ is irreducible if and only if $t^n f(1/t)$ is irreducible.*

This lemma implies that

$$\mathcal{F}_p^*(x) \text{ is irreducible } \iff s(t) := (k^{1/2}t)^{p-1}\mathcal{F}_p^*\left(\frac{1}{k^{1/2}t}\right) \text{ is irreducible.}$$

Therefore,

$$s(t) \text{ is irreducible } \iff h(r) := (r)^{p-1}s\left(\frac{1}{r}\right) \text{ is irreducible.}$$

Taking $g(x) = k$ and $d(x) = ax$ with $a \neq 0$ and $k \in \mathbb{Z}_{>0}$, we obtain $\mathcal{F}_p(x)$. This and Lemma 1 imply that $h(x) = \mathcal{F}_p(x)$. □

Propositions 3 and 4 show whether or not the familiar polynomials of Fibonacci type are irreducible (see Table 1) when p is prime.

Lemma 7 ([12, 21])**.** *Let $i := \sqrt{-1}$ and let $\gamma_j = 2i\cos\frac{j\pi}{p}$ for $j = 1, 2, \ldots, p-1$, where p is a prime number. Then $\Gamma = \{\gamma_1, \ldots, \gamma_{p-1}\}$ are the roots of the Fibonacci polynomial $F_p(x)$.*

The following lemma is a generalization of Capelli's (see Lemma 10) to what we need here in this paper.

Lemma 8. *Let $f(x)$, $g(x)$, $h(x)$ in $K[x]$, with K a field,* $\deg(h(x)) > \deg(g(x))$*, and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, where $a_0 \neq 0$. Let γ be any root of $f(x)$ in an algebraic closure of K. The polynomial $p(x) = a_n h(x)^n + a_{n-1}g(x)h(x)^{n-1} + a_{n-2}g(x)^2h(x)^{n-2} + \ldots + a_0 g(x)^n$ is irreducible over K if and only if $f(x)$ is irreducible over K and $h(x) - g(x)\gamma$ is irreducible over $K(\gamma)$.*

Proof. Let θ be a root of $h(x)-g(x)\gamma$ in the algebraic closure of K. So, $h(\theta) = g(\theta)\gamma$. If $h(\theta) = 0$, then $g(\theta) = 0$, since $\gamma \neq 0$. Therefore, $p(\theta) = 0$. If $h(\theta) \neq 0$, then we have

$$p(\theta) = g(\theta)^n f\left(\frac{h(\theta)}{g(\theta)}\right) = g(\theta)^n f(\gamma) = 0.$$

Notice that $\deg(p(x)) = n\deg(h(x))$. Also, we have

$$[K(\theta) : K] = [K(\theta) : K(\gamma)][K(\gamma) : K].$$

Since $\deg(h(x)) > \deg(g(x))$, we have $\deg(h(x)-g(x)\gamma) = \deg(h(x))$. Therefore, this gives that $[K(\theta) : K(\gamma)] \leq \deg(h(x))$ and $[K(\gamma) : K] \leq n$. Thus, $p(x)$ is irreducible over K if and only if $[K(\theta) : K] = n\deg(h(x))$, which is the case if and only if $[K(\theta) : K(\gamma)] = \deg(h(x))$ and $[K(\gamma) : K] = n$. This holds if and only if $f(x)$ is irreducible over K and $h(x) - g(x)\gamma$ is irreducible over $K(\gamma)$. $\square$

Theorem 9. *Let $p > 2$ be a prime number and let $\Gamma = \{\gamma_1, \ldots, \gamma_{p-1}\}$ be the set of roots of $F_p(x)$. A GFP $\mathcal{F}_p(x)$ is irreducible over $\mathbb{Q}$ if and only if $d(x)^2 - g(x)\gamma^2$ is irreducible over $\mathbb{Q}\left(\gamma^2\right)$ for some $\gamma \in \Gamma$.*

Proof. For all $z \in \mathbb{C}$ such that $g(z) \neq 0$, we can deduce that

$$\mathcal{F}_p(z) = g(z)^{\frac{p-1}{2}} F_p\left(\frac{d(z)}{g(z)^{1/2}}\right).$$

From Lemma 1 we know that $F_n(x) \in \mathbb{Z}[x^2]$. So, we let $S_p(x)$ be a polynomial in $\mathbb{Z}[x]$ such that $S_p(x^2) = F_p(x)$. Since $F_p(x)$ is irreducible over $\mathbb{Q}$, it follows that $S_p(x)$ is irreducible over $\mathbb{Q}$. For all $z \in \mathbb{C}$ such that $g(z) \neq 0$, we deduce that

$$\mathcal{F}_p(z) = g(z)^{\frac{p-1}{2}} S_p\left(\frac{d(z)^2}{g(z)}\right).$$

The conclusion follows from Lemma 8. $\square$

Lemma 10 ([20])**.** *Let $f(x)$, $r(x)$ in $K[x]$, where K is a field. Let γ be any root of $f(x)$ in an algebraic closure of K. The composition $f(r(x))$ is irreducible over K if and only if $f(x)$ is irreducible over K and $r(x) - \gamma$ is irreducible over $K(\gamma)$.*

Corollary 11. *Let $\Gamma = \{\gamma_1, \dots, \gamma_{p-1}\}$ be the set of roots of the Fibonacci polynomial $F_p(x)$, where p is a prime number. Let $g(x) \in \mathbb{Z}_{>0}$. The polynomial $\mathcal{F}_p(x)$ is irreducible in $\mathbb{Q}$ if and only if $d(x) - \gamma$ is irreducible over $\mathbb{Q}(\gamma)$ for some $\gamma \in \Gamma$.*

Proof. We consider the generalized Fibonacci polynomial $\mathcal{F}_p^*(x)$ defined when $g^*(x)$ is a constant and $d^*(x) = x$ (we use * to avoid any ambiguity with the upcoming analysis, using similar notation). From Lemma 5 we have that $\mathcal{F}_p^*(x)$ is irreducible.

Now consider the generalized Fibonacci polynomial $\mathcal{F}_p(x)$, where $g(x)$ is a positive constant (integer) and $d(x)$ is a polynomial that satisfies that $d(x) - \gamma$ is irreducible over $\mathbb{Q}(\gamma)$ for some $\gamma \in \Gamma$. Note that $\mathcal{F}_p(x)$ is the composition of $\mathcal{F}_p^*(x)$ with $d(x)$, i.e., $\mathcal{F}_p(x) = (\mathcal{F}_p^* \circ d)(x)$. This and Lemma 10 imply that $\mathcal{F}_p(x)$ is irreducible if and only if $\mathcal{F}_p^*(x)$ is irreducible. □

As a corollary of the previous results we have that if $\mathcal{F}_n$ satisfies any of the conditions given in Propositions 3, 4, Theorem 9, and Corollary 11, we have this. Suppose that the prime-power factorization of n is given by $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_s}$, where $p_1, p_2, \cdots, p_s$ are distinct odd primes. Then $\mathcal{F}_{p_i}$ is an irreducible factor of $\mathcal{F}_n$ for $i = 1, 2, \dots, s$. The proof of this fact follows straightforwardly using Propositions 3, 4, Theorem 9, Corollary 11, and [7, Proposition 6].

4. Lucas Type Polynomials' Irreducibility

In this section we discuss the irreducibility of GFP of Lucas type. In particular, we show that the familiar polynomials of Lucas type given in Table 1 are irreducible. In the end of the section we give a more general theorem to determine whether a GFP of Lucas type is irreducible.

Lemma 12 and Proposition 15 are generalizations of Bergum and Hoggatt's results in [3]. The proof of both cases follows by a natural adaptation of their proof to the GFP of Lucas type given in this paper.

Lemma 12. *If $\mathcal{L}_n(x)$ is the conjugate of $\mathcal{F}_n(x)$, then*

$$\mathcal{L}_n(x) = \frac{1}{\alpha} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} d(x)^{n-2i} g(x)^i.$$

Proof. From [9, Proposition 3, Part 2] we obtain that $\alpha \mathcal{L}_n(x) = g(x)\mathcal{F}_{n-1}(x) + \mathcal{F}_{n+1}(x)$. This and Lemma 1 give that

$$\begin{aligned} \alpha \mathcal{L}_n(x) &= g(x)\mathcal{F}_{n-1}(x) + \mathcal{F}_{n+1}(x) \\ &= \sum_{i=1}^{\lfloor n/2 \rfloor} \left(\binom{n-i-1}{i-1} + \binom{n-i}{i} \right) d(x)^{n-2i} g(x)^i + d(x)^n. \end{aligned}$$

This completes the proof. □

Bergum and Hoggatt [3] defined a bivariate generalized Lucas polynomial by $v_n(x,y) = xv_{n-1}(x,y) + yv_{n-2}(x,y)$ with $v_0(x,y) = 2$ and $v_1(x,y) = x$. They proved that the Lucas polynomials $v_n(x,1)$ are irreducible over $\mathbb{Q}$ if and only if n is a power of 2, and proved that the polynomials $v_p(x,1)/x$ are irreducible over $\mathbb{Q}$ if and only if $p > 2$ is a prime number. In their version of Lemma 12 for $v_n(x,y)$ it holds that $v_n(x,y^2)$ are homogeneous polynomials. This implies that $v_n(x,y)$ is irreducible over $\mathbb{Q}$ if and only if n is a power of 2; and that $v_p(x,y)/x$ is irreducible over $\mathbb{Q}$ if and only if $p > 2$ is a prime number. Again, we need some caution on the interpretation of these results. Thus, if instead of x we take $x^2 + x$, we obtain that $v_3(x^2 + x, 1)/(x^2 + x) = (x^2 - x + 1)(x^2 + 3x + 3)$. Similarly, we can construct examples to show that $v_n(x,y)$ is not always irreducible for n a power of 2 and for every choice of y. For instance, if instead of y we take $-y^{2k}$, with k even, it holds that $v_2(x, -2y^{2k}) = (x - 2y^k)(x + 2y^k)$. Similar results hold when y is replaced by $-(t/2)y^{2k}$, where t is an even perfect square. Another example is $v_4(x, x-2) = (x^2 - 2)(x^2 + 4x - 4)$. Examples, to show that $v_p(x,y)/x$ is not always irreducible for every choice of y and for every prime, can be constructed by replacing y by $-py^2$ in $v_p(x,y)$ with $p = 3 \bmod 4$. For instance, $v_3/x = (x - 3y)(x+3y)$, $v_7/x = (x^3+7x^2y-49y^3)(x^3-7x^2y+49y^3)$, and $v_{11}/x = (x^5+11x^4y-363x^2y^3 - 1331xy^4 - 1331y^5)(x^5 - 11x^4y + 363x^2y^3 - 1331xy^4 + 1331y^5)$. Taking these examples, from the point of view of GFP of Lucas type, states as, for a fixed prime $q \equiv 3 \pmod 4$, and picking $g(x) = -q$, then $\mathcal{L}_q(x)/p_1(x)$ is reducible over $\mathbb{Q}$ (see Proposition 23).

All the above examples (in Section 3 and in Section 4) show that there is no clarity on both the quantifiers and the initial conditions on the results in [3, 11, 21]. So, one of the motivations for this paper is to revisit some of the main results given by Hoggatt, Bergum, Long, Parberry, and Webb in the mentioned papers, and then use those results to give more general theorems.

We recall again our second main question in this paper. Is it true that $\mathcal{L}_p(x)/p_1(x)$ is irreducible if and only if $p > 2$ is a prime number? From the above discussion, we can see some counterexamples to determine that the question is not true in general. Again, since there are some families of the generalized Fibonacci polynomial that are irreducible if and only if p is prime, the question is still valid.

In this section we explore the question for families of GFP of Lucas type $\mathcal{L}_p(x)$. Thus, in this section we explore the conditions that we have to impose on $p_1(x)$, $d(x)$ and $g(x)$ to obtain $\mathcal{L}_p(x)/p_1(x)$ is irreducible when p is a prime number. Note that from [7, Proposition 7] we know that $\mathcal{L}_n(x)/p_1(x)$ is reducible if n is a composite number with an odd divisor.

The Proposition 15 gives enough conditions to prove whether $\mathcal{L}_q(x)/p_1(x)$, polynomials of Lucas type of the form as shown in Table 1, are irreducible when q is prime. The proof uses the Eisenstein criterion [14].

The following two propositions are known as the Schönemann and the Eisenstein criteria, respectively.

Proposition 13 ([2, 19])**.** *Let q be a prime number. If $f(x) \in \mathbb{Z}[x]$ has the form $f(x) = g(x)^n + qm(x)$ with $g(x)$ an irreducible polynomial in $\mathbb{F}_q[x]$ and does not divide $m(x) \bmod q$, then $f(x)$ is irreducible.*

Proposition 14 ([2,14])**.** *Let q be a prime number and let $f(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial in $\mathbb{Z}[x]$. If $a_n \bmod q \neq 0$, $a_0 \bmod q^2 \neq 0$, and $a_i \bmod q = 0$ for $i = 0, 1, \ldots, n-1$, then $f(x)$ is irreducible over $\mathbb{Q}$.*

Proposition 15. *Let $q > 2$ be a prime number, with $\gcd(q, g(x)) = 1$.*

1. *If $d(x) = ax^t$, with $\gcd(q, a) = 1$, then $\mathcal{L}_q(x)/p_1(x)$ is irreducible over $\mathbb{Q}$.*
2. *If $d(x) = cx + b$, then $\mathcal{L}_q(x)/p_1(x)$ is irreducible over $\mathbb{Q}$.*

Proof. Proof of Part 1. From Lemma 12 and the fact that $d(x) = \alpha p_1(x)$ we have that

$$\mathcal{L}_q(x)/p_1(x) = \sum_{i=0}^{\frac{q-1}{2}} \frac{q}{q-i}\binom{q-i}{i} d(x)^{q-2i-1} g(x)^i.$$

It is well known that $\sum_{k=i}^{\frac{q-1}{2}} \binom{q}{2k}\binom{k}{i} 2^{-q+2i+1} = \frac{q}{q-i}\binom{q-i}{i}$ (see for example, [3, 10, 15]). Since $q \big| \binom{q}{2i}$ for $i = 1, 2, \ldots, (q-1)/2$, we have that q divides $\frac{q}{q-i}\binom{q-i}{i}$. This implies that q divides $\frac{q}{q-i}\binom{q-i}{i} a^{q-2i-1} g(x)^i$ for $1 \leq i \leq (q-1)/2$. Since $\frac{q}{q-i}\binom{q-i}{i} g(x)^{q-2i-1} = qg(x)^{q-2i-1}$ when $i = (q-1)/2$ and $\gcd(q, g(x)) = 1$, we have that q^2 does not divide the independent term of $\mathcal{L}_q(x)/p_1(x)$. The fact that $\gcd(q, a) = 1$ gives that q does not divide a^{q-1}, the leading coefficient of $\mathcal{L}_q(x)/p_1(x)$. These and the Eisenstein criterion (Proposition 14) complete the proof of Part 1.

Proof of Part 2. Let $G_q(x)$ be equal to $\mathcal{L}_q(x)/p_1(x)$ as defined in Part 1 with $d(x) = x$ and let $H_q(x)$ be equal to $\mathcal{L}_q(x)/p_1(x)$ as defined in Part 2. Note that the composition of $G_q(x)$ with $cx + b$, gives $H_q(x)$. Therefore, $H_q(x) = G_q(ax + b)$ is irreducible if and only if $G_q(x)$ is irreducible. Since $G_q(x)$ is irreducible for $q > 2$, this completes the proof. $\square$

As a corollary of the previous proposition we obtain that the following polynomials (from Table 1) are irreducible when p is an odd prime number: Lucas $D_p(x)$; Pell-Lucas $Q_p(x)$; Fermat-Lucas $\vartheta_p(x)$; Chebyshev first kind $T_p(x)$; Morgan-Voyce $C_p(x)$; Vieta-Lucas $v_p(x)$.

Proposition 16. *Let $q > 2$ be a prime number. If $d(x) \in \mathbb{Z}[x]$ is irreducible $\bmod q$, then $\mathcal{L}_q(x)/p_1(x)$ is irreducible over $\mathbb{Q}$.*

Proof. Since q is a factor of $\frac{q}{q-i}\binom{q-i}{i}$ for $i = 1, \ldots, (q-1)/2$, from Lemma 12 we have that there is a polynomial $h(x) \in \mathbb{Z}[x]$ such that

$$qh(x) = \sum_{i=1}^{\frac{q-1}{2}} \frac{q}{q-i}\binom{q-i}{i} d(x)^{q-2i-1} g(x)^i.$$

This and Lemma 12 imply that

$$\mathcal{L}_q(x)/p_1(x) = d(x)^{q-1} + qh(x). \tag{7}$$

This decomposition and the fact that $\deg(g(x)) < \deg(d(x))$, imply that

$$\deg(h(x)) < \deg(d(x)^{q-1}) = \deg(\mathcal{L}_q(x)/p_1(x)).$$

If we let

$$qt(x) := \sum_{i=1}^{\frac{q-1}{2}-1} \frac{q}{q-i}\binom{q-i}{i} d(x)^{q-2i-2} g(x)^i,$$

then $h(x)$ can be written in the form $h(x) = d(x)t(x) + g(x)^{\frac{q-1}{2}}$. This, the irreducibility of $d(x) \bmod q$, the fact that $\gcd(d(x), g(x)) = 1$, and that $\deg g(x) < \deg d(x)$, imply that $\gcd(d(x), h(x)) = \gcd(d(x), g(x)) = 1 \pmod q$. The desired conclusion follows by Proposition 13. ☐

Proposition 17. *Let $q > 2$ be a prime number and let $d(x)$ be $a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$, where $a_k \bmod q \neq 0$, and $a_i \bmod q = 0$ for $i = 0, 1, \ldots, k-1$. If $g(0) \bmod q \neq 0$, then $\mathcal{L}_q(x)/p_1(x)$ is irreducible over $\mathbb{Q}$.*

Proof. Since $a_i \bmod q = 0$ for $i = 0, 1, \ldots, k-1$, we have that there is a polynomial $p(x) \in \mathbb{Z}[x]$ such that $d(x)^{q-1} = (a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0)^{q-1} = a_k^{q-1}x^{k(q-1)} + qp(x)$. (Note that $p(x)$ can be zero.) This and Lemma 12 imply that

$$\mathcal{L}_q(x)/p_1(x) = \sum_{i=1}^{\frac{q-1}{2}-1} \frac{q}{n-i}\binom{q-i}{i} d(x)^{q-2i-1} g(x)^i + (a_k^{q-1}x^{k(q-1)} + qp(x)) + qg(x)^{\frac{q-1}{2}}$$

Since q is a factor of $\frac{q}{q-i}\binom{q-i}{i} d(x)^{q-2i-1} g(x)^i$ for every for $i = 1, \ldots, (q-1)/2 - 1$, we have that there is a polynomial $h(x) \in \mathbb{Z}[x]$ such that

$$qh(x) = \sum_{i=1}^{\frac{q-1}{2}-1} \frac{q}{q-i}\binom{q-i}{i} d(x)^{q-2i-1} g(x)^i.$$

(Note that $q \mid h(0)$ and $q \mid p(0)$ because $q \mid a_0$.) Therefore,

$$\mathcal{L}_q(x)/p_1(x) = qh(x) + a_k^{q-1}x^{k(q-1)} + qp(x) + qg(x)^{\frac{q-1}{2}}. \tag{8}$$

This decomposition of $\mathcal{L}_q(x)/p_1(x)$ and the fact that $\deg(g(x)) < \deg(d(x))$ imply that $\mathrm{lc}(\mathcal{L}_q(x)/p_1(x)) = a_k^{q-1}$. This and (8) prove that q divides all coefficients of $\mathcal{L}_q(x)/p_1(x)$ except its leading coefficient.

To complete the proof using the Eisenstein criterion (Proposition 14), we prove that q^2 does not divide the independent term of $\mathcal{L}_q(x)/p_1(x)$. From (8) we can see that the independent coefficient of $\mathcal{L}_q(x)/p_1(x)$ has the form $q\big(h(0) + p(0) + g(0)^{\frac{q-1}{2}}\big)$. This and the fact that q does not divide $g(0)$ imply that q^2 does not divide $q\big(h(0) + p(0) + g(0)^{\frac{q-1}{2}}\big)$. This completes the proof. $\square$

The statement of the previous proposition can be generalized to this with the same proof. Let $q > 2$ be a prime number and let $b_k x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$ be the coefficients of $d(x)^{q-1} + qg(x)^{\frac{q-1}{2}}$, where $b_k \bmod q \neq 0$, $b_0 \bmod q^2 \neq 0$, and $b_i \bmod q = 0$ for $i = 0, 1, \ldots, k-1$. Then $\mathcal{L}_q(x)/p_1(x)$ is irreducible over $\mathbb{Q}$.

In this corollary we give a partial irreducible decomposition of $\mathcal{L}_n$ when n a composite number.

Corollary 18. *Let $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ be the prime-power factorization of n, where p_1, p_2,$\cdots$, p_k are distinct odd prime numbers.*

1. *If $d(x) = ax^t$, with $\gcd(p_i, a) = \gcd(p_i, g(x)) = 1$, then $\mathcal{L}_{p_i}/d(x)$ is an irreducible factor of $\mathcal{L}_n$.*
2. *If $d(x) = cx + b$, with $\gcd(p_i, g(x)) = 1$, then $\mathcal{L}_{p_i}/d(x)$ is an irreducible factor of $\mathcal{L}_n$.*
3. *If $d(x) \in \mathbb{Z}[x]$ is irreducible $\bmod p_i$, then $\mathcal{L}_{p_i}/d(x)$ is an irreducible factor of $\mathcal{L}_n$.*
4. *If $g(0) \bmod p_i \neq 0$ and $d(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$, where $a_k \bmod p_i \neq 0$, and $a_i \bmod p_i = 0$, for $i = 0, 1, \ldots, k-1$, then $\mathcal{L}_{p_i}/d(x)$ is an irreducible factor of $\mathcal{L}_n$.*

Proof. It follows straightforwardly using [7, Corollary 2] and Propositions 15, 16, and 17. $\square$

The proof of Parts 1 and 2 of the following proposition follows, again, by the Eisenstein criterion and Lemma 12, where $p = 2$. The proof of Part 3 is similar to the proof of Proposition 16. So, we omit details.

Proposition 19. *Let $n = 2^k$ for $k \geq 1$.*

1. *If $d(x) = ax^t$, with a and $g(x) \not\equiv 0 \bmod 2$ odd integers, then $\mathcal{L}_{2^t}(x)$ is irreducible over $\mathbb{Q}$, for $t \geq 1$.*
2. *If $d(x) = cx + b$, with c and $g(x)$ odd integers, then $\mathcal{L}_{2^t}(x)$ is irreducible over $\mathbb{Q}$, for $t \geq 1$.*

3. *If* $d(x) \in \mathbb{Z}[x]$ *is irreducible* mod 2*, then* $\mathcal{L}_{2^t}(x)$ *is irreducible over* $\mathbb{Q}$*, for* $t \geq 1$.

4. *If* $g(0)$ *is an odd integer and* $d(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$*, where* a_k *is an odd integer and* a_i *is a even integer, for* $i = 0, 1, \ldots, k-1$*, then* $\mathcal{L}_{2^t}(x)$ *is irreducible over* $\mathbb{Q}$ *for* $t \geq 1$.

Lemma 20 ([12])**.** *Let* $p = 2k+1$ *be a prime number, let* $i := \sqrt{-1}$*, and let* $\tau_j = \pm 2i \sin \frac{j\pi}{p}$ *for* $j = 1, 2, \ldots, k$*. Then* $T = \{\tau_1, \ldots, \tau_{p-1}\}$ *are the roots of* $L_p(x)/x$.

Theorem 21. *Let* $p > 2$ *be a prime number and let* $T = \{\tau_1, \ldots, \tau_{p-1}\}$ *be the set of roots of* $L_p(x)/x$*, where* $L_p(x)$ *is the Lucas polynomial. The polynomial* $\alpha\mathcal{L}_p(x)/d(x) = \mathcal{L}_p(x)/p_1(x)$ *is irreducible over* $\mathbb{Q}$ *if and only if* $d(x)^2 - g(x)\tau^2$ *is irreducible over* $\mathbb{Q}\left(\tau^2\right)$ *for some* $\tau \in T$.

Proof. This proof is similar to the proof of Theorem 9 (see also the proof of Theorem 24), using Lemma 12 instead of Lemma 1. Then setting $Jp(x) := L_p(x)/x$ and then replacing $F_p(x)$ in the proof of Theorem 9 by $Jp(x)$, we obtain the desired result. $\square$

Corollary 22. *Let* $g(x) = 1$ *and let* $T = \{\tau_1, \ldots, \tau_{p-1}\}$ *be the set of roots of* $L_p(x)/x$*, where* $L_p(x)$ *is the Lucas polynomial. The polynomial* $\alpha\mathcal{L}_p(x)/d(x) = \mathcal{L}_p(x)/p_1(x)$ *is irreducible over* $\mathbb{Q}$ *if and only if* $d(x) - \tau$ *is irreducible over* $\mathbb{Q}(\tau)$ *for some* $\tau \in T$.

Proof. Consider $\mathcal{L}_p(x)/d(x)$ with $g(x) = 1$ and $d(x)$ a polynomial such that $d(x) - \tau$ is irreducible over $\mathbb{Q}(\tau)$ for some $\tau \in T$. Note that $\mathcal{L}_p(x)/d(x)$ is the composition of $L_p(x)/x$ with $d(x)$, i.e., $\mathcal{L}_p(x) = (L_p \circ d)(x)$. This and Lemma 10 imply that $\mathcal{L}_p(x)/d(x)$ is irreducible if and only if $L_p(x)/x$ is irreducible. $\square$

Proposition 23. *If* $g(x) = -qh(x)^2$*, where* $h(x) \in \mathbb{Z}[x]$ *and* $q \equiv 3 \pmod 4$*, then* $\mathcal{L}_q(x)/p_1(x)$ *is reducible over* $\mathbb{Q}$.

Proof. By Theorem 21, we only need to show that $d(x)^2 + qh(x)^2\tau_j^2$ is reducible over $\mathbb{Q}\left(\tau_j^2\right)$ for some $\tau_j \in T$ (see Lemma 20). Since

$$d(x)^2 + q\tau_j^2 h(x)^2 = \big(d(x) - \sqrt{q}i\tau_j h(x)\big)\big(d(x) + \sqrt{q}i\tau_j h(x)\big),$$

it suffices to show that $\sqrt{q}i\tau_j \in \mathbb{Q}\left(\tau_j^2\right)$. From Lemma 20, we conclude that $\sqrt{q}i\tau_j = \pm 2\sqrt{q}\sin\frac{j\pi}{q}$. So, $\tau_j^2 = -4\sin^2\frac{j\pi}{q}$. The fact that $\cos\frac{2j\pi}{q} = 1 - 2\sin^2\frac{j\pi}{q}$ implies that $\mathbb{Q}\big(\tau_j^2\big) = \mathbb{Q}\big(\cos\frac{2j\pi}{q}\big)$.

Since $\gcd(2j, q) = 1$, we have $\big[\mathbb{Q}\big(\cos\frac{2j\pi}{q}\big) : \mathbb{Q}\big] = \frac{q-1}{2}$. So, we only need to show that $\sqrt{q}\sin\frac{j\pi}{q} \in \mathbb{Q}\big(\cos\frac{2j\pi}{q}\big)$. Thus, we know that $\cos\frac{2j\pi}{q} \in \mathbb{Q}\big(\sqrt{q}\sin\frac{j\pi}{q}\big)$, and therefore, we just need to show that $\big[\mathbb{Q}\big(\sqrt{q}\sin\frac{j\pi}{q}\big) : \mathbb{Q}\big] \leq \frac{q-1}{2}$.

Since $q \equiv 3 \bmod 4$, we have the quadratic Gauss sum

$$\sum_{n=0}^{q-1} e^{\frac{2\pi i n^2}{q}} = i\sqrt{q}.$$

So,

$$\sqrt{q}\sin\frac{j\pi}{q} = \frac{\left(e^{\frac{ij\pi}{q}} - e^{-\frac{ij\pi}{q}}\right)}{2}\sum_{n=0}^{q-1} e^{\frac{2\pi i n^2}{q}} = \frac{1}{2}\sum_{n=0}^{q-1}\left(\zeta^{n^2+j/2} - \zeta^{n^2-j/2}\right),$$

where $\zeta = e^{\frac{2i\pi}{q}}$. Since $\sin\frac{j\pi}{q} = \sin\frac{(q-j)\pi}{q}$, we may assume that j is even. Then, since the conjugates of ζ over $\mathbb{Q}$ are $\zeta^2, \zeta^3, \ldots, \zeta^{q-1}$, it therefore follows that the conjugates of $\sqrt{q}\sin\frac{j\pi}{q}$ over $\mathbb{Q}$ are

$$\frac{1}{2}\sum_{n=0}^{q-1}\left(\zeta^{m\left(n^2+j/2\right)} - \zeta^{m\left(n^2-j/2\right)}\right), \quad m = 1, 2, \ldots, q-1.$$

Also notice, however, that

$$\frac{1}{2}\sum_{n=0}^{q-1}\left(\zeta^{m\left(n^2+j/2\right)} - \zeta^{m\left(n^2-j/2\right)}\right) = \frac{1}{2}\sum_{n=0}^{q-1}\left(\zeta^{m(q-1)\left(n^2+j/2\right)} - \zeta^{m(q-1)\left(n^2-j/2\right)}\right)$$

for all $m \in \mathbb{Z}$, since ζ and ζ^{q-1} are complex conjugates. Hence, the number of conjugates of $\sqrt{q}\sin\frac{2j\pi}{q}$ over $\mathbb{Q}$ is at most $\frac{q-1}{2}$, and so $\left[\mathbb{Q}\left(\sqrt{q}\sin\frac{2j\pi}{q}\right) : \mathbb{Q}\right] \leq \frac{q-1}{2}$. This completes the proof. □

This previous proposition in combination with Proposition 17 give rise to infinite families of GFP of Lucas type that have special behavior. For example, the Lucas polynomials, with $\alpha = 1$, $d(x) = x$ and $g(x) = -3$, give that $\mathcal{L}_3(x)/d(x) = x^2 - 9 = (x-3)(x+3)$ and that $\mathcal{L}_p(x)/d(x)$ is irreducible for every prime $p \neq 3$; the Lucas polynomials, with $d(x) = x$ and $g(x) = -7$, give that $\mathcal{L}_7(x)/d(x) = x^6 - 49x^4 + 686x^2 - 2401 = (x^3 - 7x^2 + 49)(x^3 + 7x^2 - 49)$ and that $\mathcal{L}_p(x)/d(x)$ is irreducible for every prime $p \neq 7$; the Lucas polynomials, with $d(x) = x$ and $g(x) = -11$, give that $\mathcal{L}_{11}(x)/d(x) = x^{10} - 121x^8 + 5324x^6 - 102487x^4 + 805255x^2 - 1771561 = (x^5 - 11x^4 + 363x^2 - 1331x + 1331)(x^5 + 11x^4 - 363x^2 - 1331x - 1331)$ and that $\mathcal{L}_p(x)/d(x)$ is irreducible for every prime $p \neq 11$.

Since $d(x) = \alpha p_1(x)$, we have $\mathcal{L}_p(x)/d(x) = \mathcal{L}_p(x)/p_1(x)$ in Theorem 21 and Corollary 22 when $\alpha = 1$.

Theorem 24. *Let $m \in \mathbb{Z}_{>0}$ and $R = \{\rho_1, \ldots, \rho_{2^m}\}$ be the set of roots of $L_{2^m}(x)$. The polynomial $\mathcal{L}_{2^m}(x)$ is irreducible over $\mathbb{Q}$ if and only if $d(x)^2 - g(x)\rho^2$ is irreducible over $\mathbb{Q}\left(\rho^2\right)$ for some $\rho \in R$.*

Proof. For all $z \in \mathbb{C}$ such that $g(z) \neq 0$, we can deduce that

$$\alpha \mathcal{L}_{2^m}(z) = g(z)^{2^{m-1}} L_{2^m}\left(\frac{d(z)}{g(z)^{1/2}}\right).$$

From Lemma 12 we know that $L_{2^m}(x) \in \mathbb{Z}[x^2]$. Let $S_{2^m}(x) \in \mathbb{Z}[x]$ such that $S_{2^m}(x^2) = L_{2^m}(x)$. Since $L_{2^m}(x)$ is irreducible over $\mathbb{Q}$, it follows that $S_{2^m}(x)$ is irreducible over $\mathbb{Q}$. For all $z \in \mathbb{C}$ such that $g(z) \neq 0$, we deduce

$$\alpha \mathcal{L}_{2^m}(z) = g(z)^{2^{m-1}} S_{2^m}\left(\frac{d(z)^2}{g(z)}\right).$$

The conclusion follows from Lemma 8. □

Computer experimentation shows that there are many other polynomials $d(x)$ and $g(x)$ such that $\mathcal{L}_q(x)/p_1(x)$ and $\mathcal{F}_q(x)$ are irreducible for primes greater than 2 and $\mathcal{L}_{2^k}(x)$ is irreducible over $\mathbb{Q}$.

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