# INVERSE PROBLEMS RELATED TO SOME WEIGHTED ZERO-SUM CONSTANTS FOR CYCLIC GROUPS

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**Abstract**

We here take up some inverse problems related to a weighted generalization of the Davenport constant for some particular weights for a finite cyclic group.

## 1. Introduction

Let $G$ be a finite abelian group (written additively) of exponent $m$ and $A \subset [1, m-1]$. By a sequence over $G$, we mean a finite sequence of terms from $G$ which is unordered

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and in which repetition of terms is allowed. We view sequences over $G$ as elements of the free abelian monoid $\mathcal{F}(G)$ and use multiplicative notation (so our notation is consistent with [10], [11] and [12]). The *Davenport constant of $G$ with weight $A$*, denoted by $D_A(G)$, is defined to be the least positive integer $k$ such that any sequence $x_1 \cdot \ldots \cdot x_k \in \mathcal{F}(G)$ has a non-empty *$A$-weighted zero-sum subsequence*; that is, there exist a non-empty subsequence $x_{j_1} \cdot \ldots \cdot x_{j_l}$ and $a_1, \ldots, a_l \in A$ such that $\sum_{i=1}^{l} a_i x_{j_i} = 0$.

The case $A = \{1\}$ corresponds to the classical *Davenport constant* $D(G)$, which was initiated by Rogers [21] and Davenport (see [20], for instance) in connection with factorization in an algebraic number field $K$. For an expository account of $D(G)$, one may look into [10] and [11], for instance.

The weighted version $D_A(G)$ of the Davenport constant (and similar generalizations of some other zero-sum constants) were initiated in some papers of Adhikari, Chen, Friedlander, Konyagin and Pappalardi [3], Adhikari and Rath [5], Thangadurai [22], Adhikari and Chen [2] and Adhikari, Balasubramanian, Pappalardi and Rath [1].

Many people worked (see Yuan and Zeng [24], Grynkiewicz, Marchan and Ordaz [14], Halter-Koch [15], Marchan, Ordaz and Schmid [18], Marchan, Ordaz, Santos and Schmid [17], etc.) on questions, conjectures and applications related to these generalizations.

We write $\mathbb{Z}_n$ to denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and shall use the notation $\mathbb{Z}_n^*$ to denote the group of units of $\mathbb{Z}_n$.

Here we take up the inverse problems related to $D_A(\mathbb{Z}_n)$ for a couple of weight sets $A$. For a real number $x$, we shall use the notation $\lceil x \rceil$ to denote the least integer $\geq x$. For positive integers $m$, $n$ and $r$, the notation $m^r \mid\mid n$ will be used to indicate that $m^r \mid n$ and $m^{r+1} \nmid n$.

In one of the early papers [5], it was proved that for a prime $p$ and $A = \{1, 2, \ldots, r\}$, where $r$ is an integer satisfying $1 < r < p$, one has

$$D_A(\mathbb{Z}_p) = \left\lceil \frac{p}{r} \right\rceil. \tag{1}$$

For general $n$, with $A = \{1, 2, \ldots, r\}$, where $r$ is an integer such that $1 < r < n$, Xia and Li [23] and Adhikari, David and Urroz [4] independently showed that by the argument in [5], one can establish the following:

$$D_A(\mathbb{Z}_n) = \left\lceil \frac{n}{r} \right\rceil. \tag{2}$$

Addressing the inverse problem corresponding to (2), we prove the following.

**Theorem 1.** *Let $n$ and $r$ be positive integers such that $1 < r < n$ and $A = \{1, 2, \ldots, r\}$. Then a sequence $S : x_1 \cdot x_2 \cdot \ldots \cdot x_{\lceil \frac{n}{r} \rceil - 1}$ over $\mathbb{Z}_n$ has no non-empty*

*A-weighted zero-sum subsequence, if and only if it is of the form*

$$u^{(\lceil \frac{n}{r} \rceil - 1)}, \tag{3}$$

*where $u$ is a unit of $\mathbb{Z}_n$.*

For a prime $p$, if $A$ is the set of all the squares in $\mathbb{Z}_p^*$, then it was proved by Adhikari and Rath [5] that

$$D_A(\mathbb{Z}_p) = 3. \tag{4}$$

In what follows, we shall use the following notation

$$R_n = \{a^2 : a \in \mathbb{Z}_n^*\}.$$

That is, $R_n$ is the set of all squares in $\mathbb{Z}_n^*$.

Adhikari, David and Urroz [4] showed that if $n = p^r$ for a prime $p > 3$ or $n$ is square-free and coprime to 6, then

$$D_{R_n}(\mathbb{Z}_n) = 2\Omega(n) + 1, \tag{5}$$

where for any positive integer $n$, $\Omega(n)$ denotes the number of prime factors of $n$, multiplicity included.

Later, Chintamani and Moriya [7] proved that the equality (5) holds when $n$ is any integer coprime to $30 = 2 \times 3 \times 5$. They also proved that (5) holds when $n$ is a power of 3.

Regarding further results in this direction, Grynkiewicz and Hennecart [13] proved that if $A$ is the set of squares in the group of units in the cyclic group $\mathbb{Z}_n$, then for a prime $p$, writing $v_p(n)$ to denote the $p$-valuation of $n$, one has

$$D_A(\mathbb{Z}_n) = 2\Omega(n) + \min\{v_5(n), v_3(n)\} + 1, \tag{6}$$

for odd $n$ with either $v_3(n) = 0$ or $v_3(n) \geq v_5(n)$.

To state our next results, we shall need the following definition.

**Definition 1.** If the weight-set $A$ is a subgroup of $\mathbb{Z}_n^*$, two sequences $x_1 \cdot \ldots \cdot x_k$ and $y_1 \cdot \ldots \cdot y_k$ over $\mathbb{Z}_n$ are said to be *equivalent with respect to* $A$ if there are $a_1, \ldots, a_k \in A$, $c \in \mathbb{Z}_n^*$ and a permutation $\sigma \in S_k$ such that $y_i = ca_i x_{\sigma(i)}$, for $i = 1, 2, \ldots, k$.

It is clear from Definition 1 that if $x_1 \cdot \ldots \cdot x_k$ and $y_1 \cdot \ldots \cdot y_k$ are equivalent with respect to $A$, then $x_1 \cdot \ldots \cdot x_k$ does not have any non-empty $A$-weighted zero-sum subsequence if and only if $y_1 \cdot \ldots \cdot y_k$ does not have any non-empty $A$-weighted zero-sum subsequence. Thus, when the weight-set $A \subset \mathbb{Z}_n^*$ is a group under multiplication, we want to characterize the extremal sequences up to this equivalence.

In the following theorems, we characterize the extremal sequences for the result (5) of Adhikari, David and Urroz [4] and a generalization due to Chintamani and Moriya [7] mentioned in the introduction.

**Theorem 2.** *Let $p \geq 3$ be a prime and $n = p^s$. A sequence $S : x_1 \cdot \ldots \cdot x_{2s}$ of length $2\Omega(n) = 2s$ over $\mathbb{Z}_n$ does not have any non-empty $R_n$-weighted zero-sum subsequence if and only if $S$ is equivalent with respect to $R_n$ to the following sequence*

$$\prod_{i=0}^{s-1} \theta_i \cdot \prod_{i=0}^{s-1} \gamma_i,$$

*where $\theta_i = p^i$, $\gamma_i = (-v_i)p^i$ with $v_i \in \mathbb{Z}_n^* \setminus R_n$.*

**Theorem 3.** *Let $n = p_1 \cdots p_k$ be a product of $k$ distinct primes, with $\gcd(n, 30) = 1$. A sequence $S : x_1 \cdot \ldots \cdot x_{2k}$ over $\mathbb{Z}_n$ of length $2\Omega(n) = 2k$ does not have any non-empty $R_n$-weighted zero-sum subsequence if and only if it is equivalent with respect to $R_n$ to a sequence of the following form:*

$$\prod_{i=1}^{k} \alpha_i \cdot \prod_{i=1}^{k} \beta_i,$$

*with $\alpha_1 = u_1, \alpha_2 = q_1 u_2, \ldots, \alpha_k = q_1 q_2 \cdots q_{k-1} u_k$, $\beta_1 = -v_1, \beta_2 = -q_1 v_2, \ldots, \beta_k = -q_1 q_2 \cdots q_{k-1} v_k$ where $q_1, \ldots, q_k$ is a permutation of the primes $p_1, \ldots, p_k$, $u_i \in R_{q_i}$, and $v_i \in \mathbb{Z}_n^* \setminus R_{q_i}$ for each $i = 1, 2, \ldots, k$.*

## 2. Preliminaries

We shall need the following result ([6], [9], one may also see [19], for instance).

**Lemma 1.** (Cauchy–Davenport Inequality). *For a prime $p$, let $A$ and $B$ be two non-empty subsets of $\mathbb{Z}_p$. Then*

$$|A + B| \geq \min\{p, |A| + |B| - 1\},$$

*where*

$$A + B = \{x \in \mathbb{Z}_p \mid x = a + b,\ a \in A, b \in B\}$$

*and for a subset $K$ of $\mathbb{Z}_p$, $|K|$ denotes the cardinality of $K$.*

By iterating the above, for non-empty subsets $A_1, A_2, \ldots, A_h$ of $\mathbb{Z}_p$, one has

$$|A_1 + A_2 + \cdots + A_h| \geq \min\{p, \sum_{i=1}^{h} |A_i| - h + 1\}. \tag{7}$$

If $n = p \geq 7$, a prime, given $y_1, y_2, y_3 \in \mathbb{Z}_p^*$, and writing $A_i = y_i R_n$ for $i = 1, 2, 3$, by (7),

$$\begin{aligned} |A_1 + A_2 + A_3| &\geq \min\{p, \sum_{i=1}^{3} |A_i| - 2\} \\ &= \min\{p, 3\frac{p-1}{2} - 2\} \\ &= p, \text{ since } p \geq 7. \end{aligned}$$

Thus we have the following.

**Lemma 2.** *Let $p \geq 7$ be any prime. Then given any $y_1, y_2, y_3 \in \mathbb{Z}_p^*$, we have*

$$R_p y_1 + R_p y_2 + R_p y_3 = \mathbb{Z}_p.$$

We observe that from the above lemma, one can easily deduce the following.

**Lemma 3.** *Let $p \geq 7$ be any prime. If a sequence $y_1 \cdot y_2 \cdot \ldots \cdot y_l$ over $\mathbb{Z}_p$ has at least three non-zero elements, then there exist $\alpha_1, \alpha_2, \ldots, \alpha_l \in R_p$ such that*

$$\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_l y_l = 0.$$

*Proof.* Without loss of generality, let $y_1, y_2$ and $y_3$ be three terms which are not congruent to 0 modulo $p$. Then by Lemma 2, we have

$$R_p y_1 + R_p y_2 + R_p y_3 = \mathbb{Z}_p.$$

Therefore, there exist $\alpha_1, \alpha_2, \alpha_3 \in R_p$ such that

$$\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = -y_4 - y_5 - \cdots - y_l.$$

Now choosing $\alpha_i = 1$ for $i = 4, 5, \ldots, l$, we obtain our desired result. □

We shall also need the following result of Chowla [8] (one may also see [16], [19]) which provides a generalization of the Cauchy-Davenport inequality to composite numbers.

**Lemma 4.** (Chowla). *Let $n$ be a natural number and $A$ and $B$ be nonempty subsets of $\mathbb{Z}_n$. If $0 \in B$ and $\gcd(b, n) = 1$ for all $b \in B \setminus \{0\}$, then*

$$|A + B| \geq \min\{n, |A| + |B| - 1\}.$$

## 3. Proofs of Our Theorems

*Proof of Theorem 1.* Clearly, a sequence of the form (3) cannot have a non-empty $A$-weighted zero-sum subsequence.

Let us now assume that $S : x_1 \cdot x_2 \cdot \ldots \cdot x_{\lceil \frac{n}{r} \rceil - 1}$ has no non-empty $A$-weighted zero-sum subsequence.

First, we show that the sequence $S$ cannot have two distinct elements in it.

Suppose that $S$ has two distinct elements and without loss of generality, let $x_1 \neq x_2$. Write $k = \lceil \frac{n}{r} \rceil - 1$ and $D = \sum_{i=1}^{k} x_i$.

Now, consider the following elements:

$$\begin{aligned}
&x_1, \quad x_2, \quad x_1 + x_2, \quad x_1 + x_2 + x_3, \ldots, x_1 + x_2 + \cdots + x_k = D,\\
&D + x_1, \quad D + x_2, \quad D + x_1 + x_2, \quad D + x_1 + x_2 + x_3, \ldots, 2D,\\
&\vdots\\
&(r-1)D + x_1, \quad (r-1)D + x_2, \quad (r-1)D + x_1 + x_2, \ldots, rD.
\end{aligned}$$

We observe that among the $r(k+1) = r\lceil \frac{n}{r} \rceil \geq n$ elements listed above, $tD + x_1 \neq tD + x_2$ for $t = 0, 1, 2, \ldots, r-1$.

If none of the listed elements is zero in $\mathbb{Z}_n$, by the pigeonhole principle, two of them must be equal and by the above observation, any possible equality will give us a non-empty $A$-weighted zero-sum subsequence.

Therefore, all the elements of $S$ must be equal. That is,

$$x_1 = x_2 = \cdots = x_k = u.$$

If $u$ is a non-unit, then $\gcd(u, n) = d > 1$.

Write $\frac{n}{d} = m < n$ and $x_i = dy_i$ for $i = 1, 2, \ldots, k$.

If $m \leq r$, then $um$ being a multiple of $dm = n$, we are through.

In general, $k = \lceil \frac{n}{r} \rceil - 1 \geq \lceil \frac{m}{r} \rceil$, and therefore by (2), $S' : y_1 \cdot y_2, \cdot \ldots \cdot y_{\lceil \frac{n}{r} \rceil - 1}$ has a non-empty subsequence $y_{j_1} \cdot \ldots \cdot y_{j_l}$ such that $\sum_{i=1}^{l} a_i y_{j_i} \equiv 0 \pmod{m}$, with $a_1, \ldots, a_l \in A$, so that $\sum_{i=1}^{l} a_i x_{j_i} \equiv 0 \pmod{n}$ - contradiction to our assumption. Hence, $u$ cannot be a non-unit. Therefore, $S$ is of the form (3). □

*Proof of Theorem 2.* Let $S : x_1 \cdot \ldots \cdot x_k$ be a sequence of length $k = 2s$ over $\mathbb{Z}_n$ such that $S$ does not have any non-empty $R_n$-weighted zero-sum subsequence. Clearly, $x_i \neq 0$ for all $i$, and we can write

$$x_i = p^{j_i} \alpha_i, \tag{8}$$

where $0 \leq j_i < s$ and $\alpha_i \in \mathbb{Z}_n^*$.

If for more than two $i$'s, the corresponding $j_i$'s are equal, then without loss of generality, we may assume that for $i = 1, 2, 3$, we have $x_i = p^j \alpha_i$, where $\alpha_i \in \mathbb{Z}_n^*$. Since by Lemma 4 we have

$$|R_n\alpha_1 + (R_n \cup \{0\})\alpha_2 + (R_n \cup \{0\})\alpha_3| \geq \min\{n, 3|R_n|\} = n, \text{ as } p \geq 3,$$

we get a non-empty $R_n$-weighted zero-sum subsequence of the given sequence $S$.

Therefore, for not more than two $i$'s with $1 \leq i \leq k$, the corresponding $j_i$'s can be equal. But since $|S| = 2s$, for each $j$, $0 \leq j < s$, there are exactly two $x_i$'s such that $p^j \mid\mid x_i$.

Corresponding to a fixed $t$, $0 \leq t < s$, let $x_u$ and $x_v$ be the elements of $S$ such that with the notation as in (8), $j_u = j_v = t$ and $x_u = p^t\alpha_u$, $x_v = p^t\alpha_v$.

If $\alpha_u, -\alpha_v \in R_n$, then $R_n\alpha_u = R_n(-\alpha_v) = R_n$, so that an $R_n$-linear combination of $\alpha_u$ and $\alpha_v$ is zero in $\mathbb{Z}_n$, which will further imply that an $R_n$-linear combination of $x_u$ and $x_v$ is zero, a contradiction.

Similarly, it cannot happen that $\alpha_u \notin R_n$ and $-\alpha_v \notin R_n$.

Hence $S$ is equivalent with respect to $R_n$ to the sequence

$$\prod_{i=0}^{s-1} \theta_i \cdot \prod_{i=0}^{s-1} \gamma_i,$$

where $\theta_i = p^i$, $\gamma_i = (-v_i)p^i$ with $v_i \in \mathbb{Z}_n^* \setminus R_n$.

Now, suppose that we are given a sequence of the form

$$\prod_{i=0}^{s-1} \theta_i \cdot \prod_{i=0}^{s-1} \gamma_i,$$

where $\theta_i = p^i$, $\gamma_i = (-v_i)p^i$ with $v_i \in \mathbb{Z}_n^* \setminus R_n$. An $R_n$-weighted subsequence sum will be of the form

$$c_1p^{i_1} + c_2p^{i_2} + \cdots + c_lp^{i_l},$$

with $i_1 < i_2 < \cdots < i_l$, where for any particular $i$, $c_i = u_i$, $c_i = -v'_i$ or $c_i = u_i - v'_i$, where $u_i \in R_n$ and $v'_i \in \mathbb{Z}_n^* \setminus R_n$. If this subsequence sum is $0 \in \mathbb{Z}_n$, then we shall have $c_1 \equiv 0 \pmod{p}$. Since $u_{i_1} \in R_n$ and $v'_{i_1} \in \mathbb{Z}_n^* \setminus R_n$, this is not possible.

Thus, the sequence $S$ in the theorem does not have any $R_n$-weighted zero-sum subsequence. □

*Proof of Theorem 3.* Arguing as in Theorem 2, here we observe that a sequence of the form as given in the statement of the theorem cannot have any non-empty $R_n$-weighted zero-sum subsequence.

We proceed to prove the converse by induction on $k$. If $k = 1$ so that $n = p_1$, a prime, then a sequence of length 2 with no $R_n$-weighted zero-sum subsequence is clearly of the form $\alpha\beta$ where $\alpha = v_1$ is a quadratic residue modulo $p_1$ and $\beta = -v_2$ where $v_2$ is a quadratic non-residue modulo $p_1$.

Now, let $k \geq 2$ and let $S : x_1 \cdot \ldots \cdot x_{2k}$ be a sequence over $\mathbb{Z}_n$ such that it does not have any non-empty $R_n$-weighted zero-sum subsequence.

If for each prime $q \in \{p_1, p_2, \ldots, p_k\}$, there are three $x_i$'s, co-prime to $q$, then by Lemma 3 and the Chinese Remainder Theorem, $S$ is an $R_n$-weighted zero-sum sequence.

Therefore, there exists at least one prime $q_1 \in \{p_1, p_2, \ldots, p_k\}$, such that at most two $x_i$'s are co-prime to $q_1$.

First, we assume that the number of $x_i$'s co-prime to $q_1$ is not more than one. Without loss of generality, assume that $q_1$ divides $x_i$ for $i = 2, \ldots, 2k$. Let $x_i = q_1 y_i$ for $i = 2, \ldots, 2k$.

Writing $n_1 = \frac{n}{q_1}$, since the length of the sequence $y_2 \cdot \ldots \cdot y_{2k}$ is $2k - 1 = 2\Omega(n_1) + 1 = D_{R_{n_1}}(\mathbb{Z}_{n_1})$, there is a non-empty $R_{n_1}$-weighted zero-sum subsequence modulo $n_1$ of $y_2 \cdot \ldots \cdot y_{2k}$. Therefore, by the Chinese Remainder Theorem, $y_2 \cdot \ldots \cdot y_{2k}$ has a non-empty $R_n$-weighted zero-sum subsequence modulo $n_1$ and hence $x_2 \cdot \ldots \cdot x_{2k}$ has a non-empty $R_n$-weighted zero-sum subsequence modulo $n$.

Thus there are exactly two $x_i$'s which are co-prime to $q_1$. Without loss of generality, we assume that $x_1$ and $x_2$ are co-prime to $q_1$. So, $q_1$ divides $x_i$ for $i = 3, \ldots, 2k$. Let $x_i = q_1 y_i$ for $i = 3, \ldots, 2k$.

Then it cannot happen that both $x_1$ and $-x_2$ are squares or both of them are non-squares in $\mathbb{Z}_{q_1}^*$. Otherwise, there exist $\alpha_1, \alpha_2 \in R_{q_1}$ such that $\alpha_1 x_1 + \alpha_2 x_2 \equiv 0 \pmod{q_1}$. Therefore, by the Chinese Remainder Theorem, there exist $\beta_1, \beta_2 \in R_n$ such that $\beta_1 x_1 + \beta_2 x_2 \equiv 0 \pmod{q_1}$; let $\beta_1 x_1 + \beta_2 x_2 = q_1 z$. Now $z \cdot y_3 \cdot \ldots \cdot y_{2k}$ is of length $2k - 1 = 2\Omega(n_1) + 1$ and will have a non-empty $R_{n_1}$-weighted zero-sum subsequence modulo $n_1$ and by the previous argument $(\beta_1 x_1 + \beta_2 x_2) \cdot x_3 \cdot \ldots \cdot x_{2k}$ and hence $S$ will have a non-empty $R_n$-weighted zero-sum subsequence modulo $n$.

Now, using the induction hypothesis on $y_3 \cdot \ldots \cdot y_{2k}$ so that it does not have a non-empty $R_{n_1}$-weighted zero-sum subsequence modulo $n_1$, we are through. $\square$

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