A Q-ANALOGUE OF GRANVILLE'S CONGRUENCE

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Abstract

In this paper, we establish a q-analogue of Granville's congruence as follows:

$$\sum_{k=1}^{p-1} \frac{x^k}{[k]_q} \equiv \frac{1 - x^p - (x;q)_p}{[p]_q} + x^p \frac{(p-1)(1-q)}{2} \pmod{[p]_q},$$

for any real number x and odd prime p. Here $[n]_q = 1 + q + q^2 + ... + q^{n-1}$ and $(x;q)_n = (1-x)(1-xq)...\left(1-xq^{n-1}\right).$

1. Introduction

The harmonic numbers are given by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k} \text{ for } n \in \mathbb{Z}^+.$$

In [12], Wolstenholme discovered that for any prime number $p \geq 5$,

$$H_{p-1} \equiv 0 \pmod{p^2}.$$

In [6], Lehmer showed that for any prime number $p \geq 3$,

$$H_{(p-1)/2} \equiv -2q_p(2) + pq_p^2(2) \pmod{p^2},$$

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where Fermat's quotient with base 2 is $q_p(2) = (2^{p-1} - 1)/p$.

The q-analogue of harmonic numbers H_n is given by

$$H_n(q) = \sum_{k=1}^{n} \frac{q^k}{[k]_q},$$

where $[0]_q = 1$ and $[k]_q = (1 - q^k)/(1 - q) = 1 + q + q^2 + ... + q^{k-1}$. It is easy to see that $\lim_{q \to 1} [k]_q = k$.

The q-Pochhammer symbol is given by

$$(x; q)_0 = 1, \quad (x; q)_n = \prod_{k=0}^{n-1} \left(1 - xq^k\right), \quad \text{for } n \geq 1.$$

For any m, $n \in \mathbb{N}$, define the q-binomial coefficients by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}},$$

if $n \geq m$, and if $n < m$, then $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_q = 0$. It is clearly seen that

$$\lim_{q \to 1} \begin{bmatrix} n \\ m \end{bmatrix}_q = \binom{n}{m},$$

where $\binom{n}{m}$ is the usual binomial coefficient. The q-binomial coefficients satisfy the recurrence relation

$$\begin{bmatrix} n+1 \\ m \end{bmatrix}_q = q^m \begin{bmatrix} n \\ m \end{bmatrix}_q + \begin{bmatrix} n \\ m-1 \end{bmatrix}_q.$$

It is known that

$$\sum_{k=m}^{n} q^{k-m} \begin{bmatrix} k \\ m \end{bmatrix}_q = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q \tag{1}$$

and Rothe's formula (see [2]) is given by

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x; q)_n. \tag{2}$$

In [1, 9], the authors showed that for any prime number $p \geq 5$

$$H_{p-1}(q) \equiv \frac{p-1}{2}(q-1) + \frac{p^2-1}{24}(1-q)^2 \; [p]_q \pmod{[p]_q^2}. \tag{3}$$

In [3], Glaisher proved that for any prime number $p \geq 3$,

$$q_p(2) \equiv -\frac{1}{2} \sum_{k=1}^{p-1} \frac{2^k}{k} \pmod{p},$$

and Skula conjectured that for any prime number $p \geq 5$,

$$q_p^2(2) \equiv -\sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p}.$$

In [4], Granville showed for that any prime number $p \geq 5$,

$$\sum_{k=1}^{p-1} \frac{x^k}{k} \equiv \frac{1 - x^p + (x-1)^p}{p} \pmod{p}. \tag{4}$$

In [10], Sun showed that for any prime number $p \geq 3$,

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{x^k}{k^2} \quad \equiv \quad & \frac{1}{p}\left(\frac{1+(x-1)^p - x^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k - 1}{k}\right) \\ & + p\sum_{k=2}^{p-1} \frac{x^k}{k^2} H_{k-1} \pmod{p^2}.\end{aligned}$$

In [11], Tauraso obtained that for a positive integer α and $k = 1, 2, ..., p-1$,

$$\begin{bmatrix} \alpha p - 1 \\ k \end{bmatrix}_q \equiv (-1)^k \, q^{\alpha p k - \binom{k+1}{2}} \left(1 - [\alpha p]_q \, H_k(q)\right) \pmod{[p]_q^2}. \tag{5}$$

In [7], Pan and Cao defined the q-Fermat quotient by

$$Q_p\left(m, q\right) = \frac{\left(q^m; q^m\right)_{p-1} / \left(q; q\right)_{p-1} - 1}{[p]_q},$$

where m is any nonnegative integer such that $p \nmid m$.

In [8], Pan established that for any prime number $p \geq 5$,

$$\begin{aligned}& \sum_{j=1}^{p-1} \frac{q^j\left(-q; q\right)_j}{[j]_q^2} + Q_p^2(2, q) \\ \equiv \quad & -(p-1)Q_p(2, q)(1-q) - \frac{(7p-5)(p-1)}{24}\left(1-q\right)^2 \pmod{[p]_q},\end{aligned}$$

and for any prime number p,

$$q^{kp} \equiv 1 - k\left(1-q\right)[p]_q + \binom{k}{2}\left(1-q\right)^2[p]_q^2 \pmod{[p]_q^3}. \tag{6}$$

In [11], Tauraso showed that for any prime number $p \geq 3$, if a, b and d are integers such that a, $d > b$, $b \geq 0$ and $(a, p) = 1$, then

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q^d} \quad \equiv \quad & \frac{(1-q)^d}{p^d}\left((-1)^d p \sum_{s=0}^{d-1} c_s \binom{r_0 + sp}{d}\right. \\ & \left. - \sum_{s=0}^{d} (-1)^s \binom{d}{s}\binom{sp}{2d}\right) \pmod{[p]_q},\end{aligned}$$

where $r_0 = -b/a \pmod{p}$ such that $r_0 \in \{0,1,2,...,p-1\}$ and

$$c_s = \sum_{k=0}^{s} (-1)^{s-k} \binom{r_0 + kp + d - 1}{d-1} \binom{d}{s-k}.$$

In [5], He obtained that for any prime number $p \geq 5$,

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} \equiv\ & -2Q_p(2,q) - \frac{(p-1)(1-q)}{2} + Q_p^2(2,q)\,[p]_q \\ & + \left(Q_p(2,q)(1-q) + \frac{(p^2-1)(1-q)^2}{12} \right) [p]_q \pmod{[p]_q^2}. \end{aligned}$$

2. Some Congruences

In this section, firstly, the following lemma will be given for the proof of Theorem 1 .

Lemma 1. *Let $p \geq 3$ be a prime number. For all $x \in \mathbb{R}$,*

$$\sum_{k=1}^{p-1} \left(xq^{p-1}\right)^k H_k(q) \equiv \frac{1}{[p]_q} \left([p]_{xq^{p-1}} - (x;q)_{p-1} \right) \pmod{[p]_q}. \tag{7}$$

Proof. By (2) and (5), we write

$$\begin{aligned} (x;q)_{p-1} &= \sum_{k=0}^{p-1} q^{\binom{k}{2}} \begin{bmatrix} p-1 \\ k \end{bmatrix}_q (-x)^k \\ &\equiv \sum_{k=0}^{p-1} q^{pk-\binom{k+1}{2}+\binom{k}{2}} \left(1 - [p]_q H_k(q)\right) x^k \\ &= \sum_{k=0}^{p-1} \left(xq^{p-1}\right)^k - [p]_q \sum_{k=1}^{p-1} \left(xq^{p-1}\right)^k H_k(q) \\ &= \frac{1-\left(xq^{p-1}\right)^p}{1-xq^{p-1}} - [p]_q \sum_{k=1}^{p-1} \left(xq^{p-1}\right)^k H_k(q) \pmod{[p]_q^2}. \end{aligned}$$

So, the proof is obtained. □

Lemma 2. *For $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n-1} q^k \frac{(x;q)_k - 1}{[k]_q} = \sum_{k=1}^{n-1} q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \frac{(-x)^k}{[k]_q}. \tag{8}$$

Proof. From (2), we have

$$\begin{aligned}
\sum_{k=1}^{n-1} q^k \frac{(x;q)_k - 1}{[k]_q} &= \sum_{k=1}^{n-1} \frac{q^k}{[k]_q} \sum_{j=1}^{k} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q x^j \\
&= \sum_{j=1}^{n-1} q^{\binom{j}{2}+j} (-x)^j \sum_{k=j}^{n-1} \frac{q^{k-j}}{[k]_q} \begin{bmatrix} k \\ j \end{bmatrix}_q \\
&= \sum_{j=1}^{n-1} q^{\binom{j+1}{2}} \frac{(-x)^j}{[j]_q} \sum_{k=j}^{n-1} q^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q \\
&= \sum_{j=1}^{n-1} q^{\binom{j+1}{2}} \frac{(-x)^j}{[j]_q} \sum_{k=j-1}^{n-2} q^{k-(j-1)} \begin{bmatrix} k \\ j-1 \end{bmatrix}_q .
\end{aligned}$$

By (1), we have

$$\sum_{k=1}^{n-1} q^k \frac{(x;q)_k - 1}{[k]_q} = \sum_{j=1}^{n-1} q^{\binom{j+1}{2}} \frac{(-x)^j}{[j]_q} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q ,$$

so, we have the proof. □

Now, we will give the q-analogue of Granville's congruence in (4).

Theorem 1. *For all $x \in \mathbb{R}$ and a prime number $p \geq 3$, we have*

$$\sum_{k=1}^{p-1} \frac{x^k}{[k]_q} \equiv \frac{1 - x^p - (x;q)_p}{[p]_q} + x^p \frac{(p-1)(1-q)}{2} \pmod{[p]_q}. \tag{9}$$

Proof. Consider that

$$\begin{aligned}
&x^{p-1} q^{(p-1)^2} H_{p-1}(q) \\
&= \sum_{k=1}^{p-1} \left(xq^{p-1}\right)^k H_k(q) - \sum_{k=1}^{p-2} \left(xq^{p-1}\right)^k H_k(q) \\
&= \sum_{k=0}^{p-2} \left(xq^{p-1}\right)^{k+1} H_{k+1}(q) - \sum_{k=0}^{p-2} \left(xq^{p-1}\right)^k H_k(q) \\
&= \sum_{k=0}^{p-2} \left(xq^{p-1}\right)^{k+1} \left(H_k(q) + \frac{q^{k+1}}{[k+1]_q} \right) - \sum_{k=0}^{p-2} \left(xq^{p-1}\right)^k H_k(q) \\
&= \left(xq^{p-1}\right) \sum_{k=0}^{p-2} \left(xq^{p-1}\right)^k H_k(q) + \sum_{k=0}^{p-2} \frac{\left(xq^{p-1}\right)^{k+1} q^{k+1}}{[k+1]_q} - \sum_{k=0}^{p-2} \left(xq^{p-1}\right)^k H_k(q)
\end{aligned}$$

$$= \left(xq^{p-1}-1\right)\sum_{k=0}^{p-2}\left(xq^{p-1}\right)^{k}H_k(q)+\sum_{k=1}^{p-1}\frac{\left(xq^{p-1}\right)^{k}q^{k}}{[k]_q}$$
$$= \left(xq^{p-1}-1\right)\left(\sum_{k=0}^{p-1}\left(xq^{p-1}\right)^{k}H_k(q)-\left(xq^{p-1}\right)^{p-1}H_{p-1}(q)\right)+\sum_{k=1}^{p-1}\frac{x^kq^{pk}}{[k]_q}.$$

From necessary rearrangements, we write

$$\sum_{k=1}^{p-1}\frac{x^kq^{pk}}{[k]_q}=\left(1-xq^{p-1}\right)\sum_{k=0}^{p-1}\left(xq^{p-1}\right)^{k}H_k(q)+xq^{p-1}\left(xq^{p-1}\right)^{p-1}H_{p-1}(q).$$

Using (3) and (6), we get

$$\sum_{k=1}^{p-1}\frac{x^k}{[k]_q}\equiv\left(1-xq^{p-1}\right)\sum_{k=0}^{p-1}\left(xq^{p-1}\right)^{k}H_k(q)+x^p\frac{p-1}{2}(q-1)\pmod{[p]_q}.$$

With the help of (6) and (7), we have

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{x^k}{[k]_q} &\equiv \left(1-xq^{p-1}\right)\frac{1}{[p]_q}\left(\frac{1-\left(xq^{p-1}\right)^{p}}{1-xq^{p-1}}-(x;q)_{p-1}\right)+x^p\frac{p-1}{2}(q-1)\\ &= \frac{1-\left(xq^{p-1}\right)^{p}-\left(1-xq^{p-1}\right)(x;q)_{p-1}}{[p]_q}+x^p\frac{p-1}{2}(q-1)\\ &\equiv \frac{1-x^p-(x;q)_p}{[p]_q}+x^p\frac{(p-1)(1-q)}{2}\pmod{[p]_q}.\end{aligned}$$

Thus, the proof is complete. □

For example, when $x=-1$ and q^n in Theorem 1, respectively, we have that for any prime number $p\geq 3$,

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{[k]_q}\equiv -2Q_p(2,q)+\frac{(p-1)(q-1)}{2}\pmod{[p]_q}[5],$$

and for all $n\in\mathbb{Z}$,

$$\sum_{k=1}^{p-1}\frac{q^{nk}}{[k]_q}\equiv-\frac{(q^n;q)_p}{[p]_q}+(1-q)\left(n+\frac{p-1}{2}\right)\pmod{[p]_q}.$$

Corollary 1. *Let $p\geq 3$ be a prime number. We have*

$$\sum_{k=1}^{p-1}q^k\frac{(x;q)_k}{[k]_q}\equiv\frac{1-x^p-(x;q)_p}{[p]_q}+(x^p-1)\frac{(p-1)(1-q)}{2}\pmod{[p]_q}.$$

Proof. Taking p in place of n in Lemma 2 and by (5), we get

$$\begin{aligned}\sum_{k=1}^{p-1} q^k \frac{(x;q)_k}{[k]_q} &= \sum_{k=1}^{p-1} q^{\binom{k+1}{2}} \begin{bmatrix} p-1 \\ k \end{bmatrix}_q \frac{(-x)^k}{[k]_q} + \sum_{k=1}^{p-1} \frac{q^k}{[k]_q} \\ &\equiv \sum_{k=1}^{p-1} \frac{(xq^p)^k}{[k]_q} + H_{p-1}(q) \pmod{[p]_q}.\end{aligned}$$

By (6) and Theorem 1, we have

$$\sum_{k=1}^{p-1} q^k \frac{(x;q)_k}{[k]_q} \equiv \frac{1-x^p-(x;q)_p}{[p]_q} + x^p \frac{(p-1)(1-q)}{2} + H_{p-1}(q) \pmod{[p]_q}.$$

From (3), the desired result is given. □

For example, for any prime number $p \geq 3$, Corollary 1 is reduced as follows (see also [8] and [11]):

$$\sum_{k=1}^{p-1} q^k \frac{(-q;q)_k}{[k]_q} \equiv \frac{2-(-q;q)_p}{[p]_q} + p(q-1) \pmod{[p]_q}.$$

Corollary 2. *For all $x \in \mathbb{R}$ and a prime number $p \geq 3$, we have*

$$\sum_{1 \leq i \leq k \leq p-1} q^k \frac{x^i}{[i]_q} \equiv 1 - x[p]_x - (x;q)_p + x^p \frac{(p-1)(1-q)}{2}[p]_q \pmod{[p]_q^2}.$$

Proof. Observe that

$$\begin{aligned}\sum_{1 \leq i \leq k \leq p-1} q^k \frac{x^i}{[i]_q} &= \sum_{j=1}^{p-1} \frac{x^j}{[j]_q} \sum_{k=j}^{p-1} q^k \\ &= \sum_{j=1}^{p-1} \frac{x^j}{[j]_q} \left(\frac{1-q^p}{1-q} - \frac{1-q^j}{1-q} \right) \\ &= \sum_{k=1}^{p-1} \frac{x^k}{[k]_q}[p]_q - \sum_{j=0}^{p-1} x^j + 1.\end{aligned}$$

From Theorem 1, we have

$$\sum_{1 \leq i \leq k \leq p-1} q^k \frac{x^i}{[i]_q} \equiv 1 - x[p]_x - (x;q)_p + x^p \frac{(p-1)(1-q)}{2}[p]_q \pmod{[p]_q^2},$$

as claimed. □

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