# FULL $k$-COMPLETE PARTITIONS

**Augustine O. Munagi**
*School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa*
augustine.munagi@wits.ac.za

**Ntendeni A. Takalani**
*Department of Mathematics and Applied Mathematics, University of Limpopo, Sovenga, South Africa*
annah.takalani@ul.ac.za

*Received: 12/3/21, Accepted: 7/13/22, Published: 8/3/22*

**Abstract**

A partition of a positive integer is full $k$-complete if it contains $p(t)$ representations of every positive integer $t$, $1 \leq t \leq k$, where $p(n)$ is the number of all partitions of $n$. We study the enumeration properties of such partitions in this paper. We also discuss a further type of completeness defined by containment of perfect partitions.

## 1. Introduction

A *partition* of an integer $n > 0$ is a representation of $n$ as a weakly increasing sum of positive integers. The summands are called *parts*, and $n$ is the *weight*, of the partition.

We will denote partitions as vectors with positive-integer entries. Thus a partition of $n$ into $k$ parts will be generally expressed as

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k),\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k, \tag{1}$$

or

$$\lambda = (\lambda_1^{v_1}, \lambda_2^{v_2}, \ldots, \lambda_t^{v_t}),\ 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_t,\ t \leq k, \tag{2}$$

where $v_i > 0$ denotes the multiplicity of $\lambda_i$ for each $i$, and $v_1 + v_2 + \cdots + v_t = k$. The empty partition of 0 is denoted by (0) or $\emptyset$.

Hoggatt and King [4] defined an arbitrary sequence $\{f_i\}_{i=1}^{\infty}$ of positive integers to be complete, if and only if every positive integer $n$ can be represented in the form $n = \sum_{i=1}^{\infty} \alpha_i f_i$, where $\alpha_i$ is either 0 or 1. When translated to partitions, Park [9] defined a complete partition of an integer as follows.

**Definition 1** ([9])**.** A complete partition of $n$ is a weakly increasing partition $\lambda$ with $\lambda_1 = 1$, such that each integer $t$, $1 \le t \le n$, can be expressed as a sum of elements of $\lambda$, that is, each $t$ can be expressed as $\sum_{j=1}^{k} \alpha_j \lambda_j$, where $\alpha_j$ is either 0 or 1.

For example, of the 11 partitions of $n = 6$, five are complete partitions, namely, $(1^6), (1^3 3), (1^4 2), (123), (1^2 2^2)$. Subsequently Park [10] studied $r$-complete partitions whereby $\lambda$ is said to be $r$-complete if each $i$ with $1 \le i \le rn$ can be expressed as $\sum_{j=1}^{k} \alpha_j \lambda_j$, where $\alpha_j \in \{0, 1, 2, \ldots, r\}$. It follows that the complete partitions correspond to 1-complete partitions under the latter definition. For example, $(1^4)$ and $(1^2 2)$ are 1-complete partitions of $n = 4$, and $(1^4), (1^2 2)$ and $(1, 3)$ are 2-complete partitions.

In this paper we consider a more inclusive type of complete partition. The "full $k$-complete" partitions discussed here are complete by virtue of the fact that they contain $p(t)$ representations of every positive integer $t < k$, where $p(N)$ is the number of all partitions of $N$. In addition, we give brief discussions of two types of completeness according to the containment of perfect partitions called 'perfect-partition-complete' and 'full-perfect-partition-complete' partitions.

In Section 2 we explain two important notions for subsequent work. Then in Section 3 we define full $k$-complete partitions and explore their properties. Section 4 is devoted to a brief discussion of perfect-partition-complete partitions, followed by full-perfect-partition-complete partitions in Section 5.

## 2. Preliminaries

In this section we explain the essential concepts of derived partition and perfect partition. In the sequel we will also use the notation $\lambda \vdash n$ to mean that $\lambda$ is a partition of $n$.

**Definition 2.** Let $\lambda$ be a partition of $n$. A *derived partition* (or subpartition) of $\lambda$ is any partition $\pi$ of a nonnegative integer $m \le n$ whose parts form a sub-multiset of the multiset of the parts of $\lambda$.

The set of derived partitions of $\lambda$ will be denoted by $D(\lambda)$. It is clear that $(0) \in D(\lambda)$. So $D(\lambda) \ne \emptyset$ for all $\lambda$.

**Example 1.** Consider the partition $\lambda = (1, 2^2, 3) \vdash 8$. Then

$$D(\lambda) = \{(0), (1), (2), (2^2), (3), (1, 2), (1, 2^2), (1, 3), (2, 3), (2^2, 3), (1, 2, 3), \lambda\}.$$

Since $\lambda$ is a partition of a finite integer $n$ it has a finite number of derived partitions. So $D(\lambda)$ is a finite set. Moreover, in order to form a derived partition of $\lambda = (\lambda_1^{v_1}, \ldots, \lambda_k^{v_k})$, one selects $t$ copies of $\lambda_i$, where $0 \le t \le v_i$ for each $i$. Thus we have proved the following assertion.

**Proposition 1.** *We have* $|D((\lambda_1^{v_1}, \lambda_2^{v_2}, \ldots, \lambda_k^{v_k}))| = (v_1 + 1)(v_2 + 1) \cdots (v_k + 1)$.

Thus from Example 1 we have $|D((1, 2^2, 3))| = 2 \cdot 3 \cdot 2 = 12$, as expected.

### 2.1. Perfect Partitions

The definition of a perfect partition first appeared in the works of P. A. MacMahon [6, 7]. Subsequently other mathematicians studied and found several properties and generalizations of perfect partitions (see for example, [1, 5, 8]).

**Definition 3.** A *perfect partition* is a partition in which the parts contain exactly one partition of every positive integer smaller than or equal to $n$.

That is, if $\lambda \vdash n$ is a perfect partition, then $D(\lambda)$ contains exactly one partition of each number from 0 to $n$. This implies that $|D(\lambda)| = n + 1$.

For example, $(1^3, 4) \vdash 7$ is a perfect partition since

$$D((1^3, 4)) = \{(0), (1), (1^2), (1^3), (4), (1, 4), (1^2, 4), (1^3, 4)\},$$

which contains $|D((1^3, 4))| = 8$ partitions consisting of one partition of $0, 1, \ldots, 7$, respectively.

The enumeration of perfect partitions is facilitated by the enumeration of ordered factorizations of integers into products of factors greater than 1.

**Theorem 1** (MacMahon)**.** *The number of perfect partitions of* $n$ *is the same as the number of ordered factorizations of* $n + 1$ *without 1's.*

*Proof.* Let $n + 1 = g_1 g_2 \cdots g_r$, $g_i > 1$, be an ordered factorization of $n + 1$. Then the corresponding perfect partition of $n$ is given by

$$\lambda = \left(1^{g_1 - 1}, g_1^{g_2 - 1}, (g_1 g_2)^{g_3 - 1}, \ldots, (g_1 g_2 \cdots g_{r-1})^{g_r - 1}\right). \qquad (3)$$

By summing the parts of $\lambda$ in (3), one may verify that the weight of $\lambda$ is $n$. On the other hand, we may use Proposition 1 to verify that $|D(\lambda)| = n + 1$. □

**Example 2.** We give the ordered factorizations of 8 which correspond to the perfect partitions of 7 in Table 1.

| Perfect Partition of 7 | $(1^7)$ | $(1, 2^3)$ | $(1^3, 4)$ | $(1, 2, 4)$ |
|---|---|---|---|---|
| Ord. Factorization of 8 | 8 | $2 \cdot 4$ | $4 \cdot 2$ | $2 \cdot 2 \cdot 2$ |

Table 1: Perfect Partitions of 7 and factorizations of 8

## 3. Full $k$-Complete Partitions

**Definition 4.** A partition $\pi$ of a positive integer is called *full $k$-complete* if the parts of $\pi$ contain the parts of all partitions of every integer from 1 to $k$. Alternatively, $\pi$ is full $k$-complete if the set $D(\pi)$ contains all partitions of every positive integer from 1 to $k$.

For instance, (1) is full 1-complete, (1,1,1,2) and (1,1,2,3) are both full 2-complete while (1,1,1,2,3) is full 3-complete (and therefore full 1- and full 2-complete). A perfect partition is at most full 1-complete.

For example,

$$D((1,1,1,2)) = \{(1),(1,1),(2),(1,1,1),(1,2),(1,1,2),(1,1,1,2)\}$$

which is seen to contain every partition $\lambda$ of $m$ for $m = 1,2$. However, $(1,1,1,2)$ is not full 3-complete because $(3) \notin D((1,1,1,2))$.

Let $C_k(n)$ denote the set of full $k$-complete partitions of $n$ with $c_k(n) = |C_k(n)|$. Any partition $\pi \in C_k(n)$ is full $j$-complete, where $j = 1,2,\ldots,k$. Note that $C_k(0) = \{(0)\}$ by definition.

The set $C_k(n)$ is largest when $k = 1$ since $C_1(n)$ is the set of all partitions of $n$ that contain 1 as a part. The union of all full $k$-complete partitions of $n$ is $C_1(n)$. There is a strictly decreasing chain of sets of full $k$-complete partitions of $n$, namely

$$C_1(n) \supset C_2(n) \supset C_3(n) \supset \cdots.$$

An integer $n$ has a finite number of full $k$-complete partitions, but there is an infinite number of full $k$-complete partitions for any $k > 0$. For instance, a partition of the form $(1^a, 2^b)$, $a > 1, b > 0$, is full 2-complete. Table 2 shows all members of the sets $C_i(1), C_i(2), \ldots, C_i(10)$ for $i = 2,3$.

**Lemma 1.** *Let $\pi$ be a full $k$-complete partition, and let $m$ be an integer with $1 \le m \le k$. Then $\pi$ contains all positive integer divisors of $m$.*

*Proof.* If $d \mid m$ and $d \notin \pi$, then the partition $(d^{m/d}) \notin D(\pi)$. But this contradicts the fact that $\pi$ is full $k$-complete. □

### 3.1. Initial Full $k$-Complete Partitions

We isolate certain $k$-complete partitions which occur uniquely for certain minimal weights $n$.

**Definition 5.** The full $k$-complete partition with least weight is called the *initial full $k$-complete partition.*

| $n$ | $k$ | $C_k(n)$ |
|---|---|---|
| 1 | 2 | $\emptyset$ |
| 2 | 2 | $\emptyset$ |
| 3 | 2 | $\emptyset$ |
| 4 | 2 | $(1^2, 2)$ |
| 5 | 2 | $(1^3, 2)$ |
| 6 | 2 | $(1^4, 2), (1^2, 2^2)$ |
| 7 | 2 | $(1^5, 2), (1^3, 2^2), (1^2, 2, 3)$ |
| 7 | 3 | $\emptyset$ |
| 8 | 2 | $(1^6, 2), (1^4, 2^2), (1^3, 2, 3), (1^2, 2^3), (1^2, 2, 4)$ |
| 8 | 3 | $(1^3, 2, 3)$ |
| 9 | 2 | $(1^7, 2), (1^5, 2^2), (1^4, 2, 3), (1^3, 2^3), (1^3, 2, 4), (1^2, 2^2, 3), (1^2, 2, 5)$ |
| 9 | 3 | $(1^4, 2, 3)$ |
| 10 | 2 | $(1^8, 2), (1^6, 2^2), (1^5, 2, 3), (1^4, 2^3), (1^4, 2, 4), (1^3, 2^2, 3), (1^3, 2, 5),$ $(1^2, 2^4), (1^2, 2, 3^2), (1^2, 2^2, 4), (1^2, 2, 6)$ |
| 10 | 3 | $(1^5, 2, 3), (1^3, 2^2, 3)$ |
| 10 | 4 | $\emptyset$ |

Table 2: Sets of full $k-$complete partitions, $C_k(n)$ for $1 \leq n \leq 10$ and $k = 2, 3$

Let the initial full $k$-complete partition be denoted by $\pi_k$, and let the initial full $k$-complete partition of $n$ be $\pi_k(n)$.

For example, when $k = 1, 2, 3$, we have $\pi_1 = (1)$, $\pi_2 = (1^2, 2)$, $\pi_3 = (1^3, 2, 3)$.

If $\lambda$ is any full $k$-complete partition, then by definition $\pi_k \in D(\lambda)$. So the intersection of all sets of derived partitions of full $k$-complete partitions is $\pi_k$.

From Lemma 1 we see that the initial full $k$-complete partition $\pi_k$ depends on the divisors of all $m \in \{1, \ldots, k\}$. By definition $\pi_k$ does not contain all the divisors of $k+1$; in particular it does not contain $k+1$. The following lemma gives further characterizations of $\pi_k$, from which it may be computed. The proof may be deduced from Lemma 1.

**Lemma 2.** *The initial full $k$-complete partition $\pi_k$ satisfies the following properties:*

*(i) The partition $\pi_k$ is equivalent to the multiset union of the sets of divisors of the integers $1, 2, \ldots, k$. Thus if $V(n)$ denotes the set of divisors of $n$, then*

$$\pi_k \equiv \bigcup_{m=1}^{k} V(m). \tag{4}$$

*(ii) Let $pV(m)$ denote the set of partitions of $m$ into single divisors of $m$. Then $\pi_k$ is obtained by selecting the parts with maximal multiplicity from the set $pV(1) \cup pV(2) \cup \cdots \cup pV(k)$.*

Note that any $\lambda \in pV(m)$ has the form $\lambda = (d^{m/d})$. So we can write

$pV(m) = \{(1^{e_m}), (2^{e_m/2}), \ldots, (m^{e_m/m})\}$, where $e_m \equiv m$, $e_m/d = 0$ if $d \nmid m$. Then

$$\pi_k \equiv \left( t^{\max(e_t/t,\, e_{t+1}/t,\, \ldots,\, e_k/t)} \mid t = 1, 2, \ldots k \right). \tag{5}$$

**Example 3.** Since $V(m)$, $m = 1, 2, 3, 4$, are given by $\{1\}, \{1, 2\}, \{1, 3\}$, $\{1, 2, 4\}$ respectively, part (i) of Lemma 2 gives

$$\pi_4 \equiv (1) \cup (1, 2) \cup (1, 3) \cup (1, 2, 4) = (1, 1, 1, 1, 2, 2, 3, 4).$$

On the other hand, since $pV(m)$, $m = 1, 2, 3, 4$, are given by $\{(1)\}, \{(1^2), (2)\}$, $\{(1^3), (3)\}$, $\{(1^4), (2^2), (4)\}$ respectively, part (ii) of Lemma 2 gives

$$\pi_4 \equiv (1^{\max(1,2,3,4)}, 2^{\max(1,1,2)}, 3^{\max(1,0)}, 4^{\max(1)}) = (1^4, 2^2, 3, 4).$$

**Proposition 2.** *The weight $w_k$ of $\pi_k$ is given by*

$$w_k = \sum_{j=1}^{k} \sigma(j) = \sum_{r=1}^{k} r \left\lfloor \frac{k}{r} \right\rfloor, \tag{6}$$

*where $\sigma(N)$ is the sum of divisors of $N$.*

*Proof.* It follows from Equation (4), by taking the cardinalities of the sets, that $w_k = \sigma(1) + \sigma(2) + \cdots + \sigma(k)$.

The second equality is a known identity (see for example [3]). However, note that for each integer $m$, the quantity $\lfloor \frac{k}{m} \rfloor$ counts the multiples of $m$ not exceeding $k$, or equivalently, copies of $m$ in the multiset of divisors of $1, 2, \ldots, k$. So the full contribution of $m$ to $w_k$ is $m\lfloor \frac{k}{m} \rfloor$, and to $\pi_k$ is $\lfloor \frac{k}{m} \rfloor$ copies of $m$ or $m^{\lfloor \frac{k}{m} \rfloor}$. Hence $\pi_k$ has the following explicit form, which is compatible with (5).

$$\pi_k = (1^k, 2^{\lfloor k/2 \rfloor}, 3^{\lfloor k/3 \rfloor}, \ldots, \lfloor k/2 \rfloor^2, \lfloor k/2 \rfloor + 1, \lfloor k/2 \rfloor + 2, \ldots, k). \tag{7}$$

By summing the parts of $\pi_k$ we obtain the second equality in (6). □

From Proposition 2 we have $w_k = \sum_{j=1}^{k-1} \sigma(j) + \sigma(k)$ which implies the recurrence

$$w_k = w_{k-1} + \sigma(k),\ k > 0,\ \text{with } w_0 = 0. \tag{8}$$

The sequence $\{w_k\}_{k \geq 1}$ begins as follows:

$$1, 4, 8, 15, 21, 33, 41, 56, 69, 87, 99, 127, 141, 165, 189, 220, 238, 277, \ldots$$

This agrees with sequence $A024916$ in Sloane [11], wherein a comment describes the $k$-th term as "the total number of parts in all partitions of the positive integers less than or equal to $k$ into equal parts".

The following result may be deduced from the proof of Proposition 2, by summing up the multiplicities of $\pi_k$ in (7).

**Corollary 1.** *The length $\ell_k$ of the initial full $k$-complete partition $\pi_k$ is given by*

$$\ell_k = \sum_{j=1}^{k} \tau(j) = \sum_{r=1}^{k} \left\lfloor \frac{k}{r} \right\rfloor, \tag{9}$$

*where $\tau(N)$ is the number of divisors of $N$.*

Apart from (4), (5) and (7) $\pi_{k+1}$ may be obtained from $\pi_k$ using the following algorithm:

$$\text{"Insert the divisors of } k+1 \text{ into } \pi_k\text{"} . \tag{10}$$

For example, since the divisors of 4 are $1, 2, 4$, we have

$$\pi_4 = \pi_3 \cup (1, 2, 4) = (1^3, 2, 3) \cup (1, 2, 4) = (1^4, 2^2, 3, 4).$$

Since $C_1(n)$ is the set of partitions of $n$ containing 1, we have $c_1(n) = p(n-1)$. This fact is extended in the following statement.

**Theorem 2.** *The number of full $k$-complete partitions of $n$ is given by $p(n - w_k)$:*

$$c_k(n) = p(n - w_k). \tag{11}$$

*Proof.* The number of partitions of $n$ containing a positive integer $r$ is given by $p(n - r)$ since we can remove one copy of $r$ from a partition of $n$ containing $r$ to obtain an arbitrary partition of $n - r$.

By definition the multiset of parts of a full $k$-complete partition $\lambda \vdash n$ contains the multiset of the parts of $\pi_k$, that is, contains a partition of $w_k$. So we may replace the parts of $\pi_k$ in $\lambda$ with $w_k$ to obtain a partition of $n$ containing $w_k$. Hence, the result follows. □

Theorem 2 implies the generating function (see for example, Andrews [2])

$$\sum_{n=1}^{\infty} c_k(n) q^n = q^{w_k} \prod_{n=1}^{\infty} \frac{1}{1 - q^n}. \tag{12}$$

**Remark 1.** Denote the set of partitions of $N$ by $P(N)$, i.e., $p(N) = |P(N)|$. The proof of Theorem 2 guarantees that $C_k(n)$ may be obtained by inserting the parts of $\pi_k$ into partitions $\pi \in P(n - w_k)$ where $\pi$ may be empty. For example, $C_2(8) = (2, 1^2) \cup P(4)$ gives:

$$\begin{aligned} C_2(8) &= (2, 1^2) \cup \{(1^4), (1^2, 2), (1, 3), (2^2), (4)\} \\ &= \{(1^6, 2), (1^4, 2^2), (1^3, 2, 3), (1^2, 2^3), (1^2, 2, 4)\} \quad \text{(cf. Table 2).} \end{aligned}$$

Equation (11) implies

$$c_k(n) - c_k(n-1) = p(n-w_k) - p(n-1-w_k) = p(n-w_k)_{\geq 2},$$

where $p(N)_{\geq 2}$ is the number of partitions of $N$ without 1's. This gives the following recurrence on which the construction of the rows of Table 2 is based:

$$c_k(n) = c_k(n-1) + p(n-w_k)_{\geq 2}. \tag{13}$$

Thus to obtain $C_k(n)$ we either insert 1 into each $\pi \in C_k(n-1)$ or merge the parts of $\pi_k$ with the parts of any partition of $n - w_k$ that does not contain 1.

We can recover the generating function (12) from (13) as follows.

$$\begin{aligned}\sum_{n=1}^{\infty} (c_k(n-1) + p(n-w_k)_{\geq 2})\, q^n &= q^{w_k+1} \prod_{n=1}^{\infty} \frac{1}{1-q^n} + q^{w_k} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \\ &= q \cdot q^{w_k} \prod_{n=1}^{\infty} \frac{1}{1-q^n} + q^{w_k} \prod_{n=1}^{\infty} \frac{1}{1-q^n}(1-q) \\ &= (q + (1-q))\, q^{w_k} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \\ &= q^{w_k} \prod_{n=1}^{\infty} \frac{1}{1-q^n}.\end{aligned}$$

Lastly, we compare $p(0) + p(1) + \cdots + p(k)$ with the cardinality of $D(\pi_k)$. Using (7) and Proposition 1 we get

$$|D(\pi_k)| = \prod_{j=1}^{k} \left\lfloor \frac{k+j}{j} \right\rfloor, \ k > 1.$$

Note that $p(0) = 1$ counts the empty partition $\emptyset$ of 0, and $p(0) + p(1) = 1 + 1 = 2$, including the partition (1). On the other hand, $D(\emptyset) = \{\emptyset\}$ and $D((1)) = \{\emptyset, (1)\}$. Hence $p(0) + p(1) + \cdots + p(k) = |D(\pi_k)|$ when $k = 0, 1$. But when $k \geq 2$, we obtain the strong inequality

$$p(0) + p(1) + \cdots + p(k) < |D(\pi_k)|, \ k > 1. \tag{14}$$

Indeed the left side of (14) counts only partitions of $0, 1, \ldots, k$. On the other hand, $|D(\pi_k)|$ counts all partitions of $0, 1, \ldots, k$, and some partitions of $m, k+1 \leq m \leq w_k$:

$$D(\pi_k) = \bigcup_{j=1}^{k} P(j) \bigcup_{m=k+1}^{w_k} Q(m), \quad Q(m) \subset P(m).$$

In particular $\pi_k \in \bigcup_{m=k+1}^{w_k} Q(m) \neq \emptyset$.

The difference between the two quantities increases so rapidly that the following limit converges very fast to 0 (see Table 3).

$$\lim_{k\to\infty} \frac{p(0)+p(1)+\cdots+p(k)}{\prod_{j=1}^{k}\lfloor\frac{k+j}{j}\rfloor} = 0.$$

| $k$ | $p(0)+\cdots+p(k)$ | $\lvert D(\pi_k)\rvert$ | $(p(0)+\cdots+p(k))/\lvert D(\pi_k)\rvert$ |
|---|---|---|---|
| 3 | 7 | 16 | 0.3750000000 |
| 4 | 12 | 60 | 0.1833333333 |
| 5 | 19 | 144 | 0.1250000000 |
| 6 | 30 | 672 | 0.04315476190 |
| 7 | 45 | 1536 | 0.02864583333 |
| 8 | 67 | 6480 | 0.01018518519 |
| 9 | 97 | 19200 | 0.005000000000 |
| 10 | 139 | 76032 | 0.001815025253 |
| 11 | 195 | 165888 | 0.001169463735 |
| 12 | 272 | 1048320 | 0.0002585088523 |

Table 3: $p(0)+\cdots+p(k)$ versus $|D(\pi_k)|$ for some values of $k$

### 3.2. Strictly Full $k$-Complete Partitions

A partition $\pi$ is said to be *strictly full $k$-complete* if $\pi$ is full $k$-complete but not full $(k+1)$-complete. Thus $(1^2,2,3)$ is strictly full 2-complete and $(1^3,2,3,4)$ is strictly full 3-complete. The partition $(1,2^2,3,4,5^2,6,7)$ is strictly full 1-complete. Every perfect partition is strictly full 1-complete.

By definition, every initial full $k$-complete partition is strictly full $k$-complete. The number $s_k(n)$ of strictly full $k$-complete partitions of $n$ is given by

$$s_k(n) = |C_k(n)\setminus C_{k+1}(n)| = |C_k(n)| - |C_{k+1}(n)|.$$

Thus Equation (11) implies

$$s_k(n) = p(n-w_k) - p(n-w_{k+1}).$$

Hence using Equations (12) and (8) we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} s_k(n)q^n &= (q^{w_k} - q^{w_{k+1}})\prod_{n=1}^{\infty}\frac{1}{1-q^n}\\ &= q^{w_k}(1-q^{\sigma(k+1)})\prod_{n=1}^{\infty}\frac{1}{1-q^n}. \qquad (15)\end{aligned}$$

Thus if $f_t(n)$ denotes the number of partitions of $n$ that do not contain $t$ as a part, then

$$\sum_{n=0}^{\infty} s_k(n)q^n = q^{w_k} \sum_{n=0}^{\infty} f_{\sigma(k+1)}(n)q^n. \tag{16}$$

Hence we have proved the following partition theorem. But we also give a combinatorial proof below.

**Theorem 3.** *The number of strictly full $k$-complete partitions of $n$ is equal to the number of partitions of $n - w_k$ which do not contain $\sigma(k+1)$ as a part.*

*Proof.* Let $S_k(n)$ denote the set of strictly full $k$-complete partitions of $n$. In view of Equation (11), the set of full $k$-complete partitions of $n$ is equinumerous with the set $P(n - w_k)$ of partitions of $n - w_k$, with a generally disjoint union:

$$C_k(n) = P(n - w_k) = S_k(n) \cup C_{k+1}(n). \tag{17}$$

The parts of each $\pi \in C_{k+1}(n)$ consist of all the divisors of $1, 2, \ldots, k, k+1$. Thus if $n < w_{k+1}$, then $C_{k+1}(n) = \emptyset$ and we simply have $S_k(n) = P(n - w_k)$, and clearly no contained partition in the right-hand-side of (17) can contain $\sigma(k+1)$.

However, if $n \geq w_{k+1}$, we may replace the members of the set of divisors of $k+1$ in $\pi \in C_{k+1}(n)$ with their sum $\sigma(k+1)$. This transformation induces a bijection between $C_{k+1}(n)$ and the set of partitions of $n - w_k$ containing $\sigma(k+1)$. This then produces $P(n - w_k) = S_k(n)$ provided we exclude partitions of $n - w_k$ containing the set of divisors of $k+1$, which is in turn identified with partitions of $n - w_k$ containing $\sigma(k+1)$. The result follows. □

For example, from Table 2 we see that $C_2(7) = S_2(7) \cup C_3(7) = S_2(7)$ since $C_3(7) = \emptyset$; but

$$C_2(8) = S_2(8) \cup C_3(8) \equiv \{(1^6, 2), (1^4, 2^2), (1^2, 2^3), (1^2, 2, 4)\} \cup \{(1^3, 2, 3)\}.$$

**Remark 2.** Referring to Theorem 3, note that a partition $\pi \in S_k(n)$ may contain $\sigma(k+1)$ as a part. For example, when $n = 10, k = 2$ and $\sigma(3) = 1 + 3 = 4$, we find in Table 2 that $(1^4, 2, 4), (1^2, 2^2, 4) \in S_2(10) = C_2(10) \setminus C_3(10)$, where $C_3(10) = \{(1^5, 2, 3), (1^3, 2^2, 3)\}$. However, by replacing 1 and 3 with 4 we obtain $C_3(10) = \{(1^5, 2, 3), (1^3, 2^2, 3)\} \longmapsto \{(1^4, 2, 4), (1^2, 2^2, 4)\} \in S_2(10)$.

## 4. Perfect-Partition-Complete Partitions

A partition $\lambda$ of a positive integer is called *perfect-partition-complete for $n$* (*pp-complete* for short) if the parts of $\lambda$ contain all perfect partitions of $n$.

In other words, $\lambda$ is pp-complete for $n$ if $\pi \in D(\lambda)$ for every perfect partition $\pi \vdash n$.

As with $k$-complete partitions, there is an infinite number of pp-complete partitions for any positive integer $n$. For example, since the perfect partitions of 3 are $(1^3)$ and $(1, 2)$, we see that any partition of the form

$$(1^a, 2^b, 3^{c_1}, 4^{c_2}, \ldots),\ a \geq 3, b \geq 1, c_1 \geq 0, c_2 \geq 0, \ldots,$$

is a pp-complete partition for 3.

The pp-complete partition for $k$ with least weight will be called the $k$th *initial pp-complete* partition.

The $k$th initial pp-complete partition will be denoted by $\Lambda_k$, $k = 1, 2, 3, \ldots$ (see the third column of Table 4). The contained perfect partitions are also given in the table (see the second column). For example, the following inclusion holds for $k = 7$:

$$\{(1^7), (1, 2^3), (1^3, 4), (1, 2, 4)\} \subset D((1^7, 2^3, 4)).$$

Many perfect partitions have the form $(1^{x-1}, x^y)$ for a fixed $x > 0$. Consequently $x \in \Lambda_{xy+x-1}$. For example, if $x = 3$, then $3 \in \Lambda_{3y+2}$, $y \geq 1$. This motivates the following statement.

**Proposition 3.** *Let $k$ and $t$ be positive integers with $k > t$. Then $\Lambda_k$ contains $t$ as a part if and only if $k \equiv -1$ (mod $t$).*

*Proof.* We have

$$\begin{aligned} t \in \Lambda_k &\iff \text{there exists a perfect partition } \pi \vdash k \text{ with } t \in \pi \\ &\iff \text{there exists a factorization of the form } k+1 = t \cdot v,\ v > 0 \\ &\iff \text{there exists a perfect partition of the form } \pi = (1^{t-1}, t^{v-1}) \vdash k. \end{aligned}$$

The weight of $\pi$ is $t - 1 + t(v - 1) = tv - 1 = k$. Hence $k \equiv -1 \pmod{t}$. □

We consider few theorems concerning the initial pp-complete partitions in analogy to the theorems in Section 3.

**Theorem 4.** *The $k$th initial pp-complete partition has the explicit form:*

$$\Lambda_k = \left(d_1^{d_\tau - 1}, d_2^{d_{\tau-1}-1}, \ldots, d_{\tau-1}^{d_2-1}, d_\tau^{d_1-1}\right),\ \tau = \tau(k+1). \tag{18}$$

*Proof.* We give a constructive proof based on the bijection asserted by Theorem 1.

Let $\Lambda_k = (\lambda_1^{e_1}, \lambda_2^{e_2}, \ldots, \lambda_r^{e_r})$, $\lambda_1 < \cdots < \lambda_k$, $e_i > 0\ \forall\ i$. Then we observe that, in general, the $e_i$ are maximal among the multiplicities of distinct parts of perfect partitions of $k$. Such maximal multiplicities arise from perfect partitions of $k$ whose corresponding ordered factorizations have at most two factors.

| $k$ | Perfect Partitions of $k$ | $\Lambda_k$ | $w(\Lambda_k)$ |
|---|---|---|---|
| 1 | $(1)$ | $(1)$ | 1 |
| 2 | $(1^2)$ | $(1^2)$ | 2 |
| 3 | $(1^3),(1,2)$ | $(1^3,2)$ | 5 |
| 4 | $(1^4)$ | $(1^4)$ | 4 |
| 5 | $(1^5),(1,2^2),(1^2,3)$ | $(1^5,2^2,3)$ | 12 |
| 6 | $(1^6)$ | $(1^6)$ | 6 |
| 7 | $(1^7),(1,2^3),(1^3,4),(1,2,4)$ | $(1^7,2^3,4)$ | 17 |
| 8 | $(1^8),(1^2,3^2)$ | $(1^8,3^2)$ | 14 |
| 9 | $(1^9),(1,2^4),(1^4,5)$ | $(1^9,2^4,5)$ | 22 |
| 10 | $(1^{10})$ | $(1^{10})$ | 10 |
| 11 | $(1^{11}),(1,2^5),(1,2^2,6),(1,2,4^2),$ $(1^3,4^2),(1^5,6),(1^2,3^3),(1^2,3,6)$ | $(1^{11},2^5,3^3,4^2,6)$ | 44 |
| 12 | $(1^{12})$ | $(1^{12})$ | 12 |

Table 4: The $k$th initial pp-complete partitions for $1 \leq k \leq 12$

To see this, note that the trivial factorization $k+1 = 1\cdot(k+1)$ gives the maximal multiplicity in $(1^k) \in \Lambda_k$. Thus $e_1 = k$.

The 2-factorization $k+1 = a_1 \cdot a_2$, $a_i > 1$ corresponds to $(1^{a_1-1}, a_1^{a_2-1}) \vdash k$ which gives the maximal string $a_1^{a_2-1}$ for all first factors $a_1$. Correspondingly, the $a_2 - 1$ give the multiplicities in $\Lambda_k$ from left to right. We find that each string $\lambda_i^{e_i}$ originates from a factorization of the form $\lambda_i \cdot (e_i+1) = k+1$.

This may be clarified with an example. Let $k = 7$. Then $\{a_1a_2 \mid a_1a_2 = 8\} = \{2\cdot 4,\ 4\cdot 2\}$, and the partitions corresponding to these two factorizations, together with $8 = 1\cdot 8$, suffice to determine $\Lambda_7$:

$$\begin{aligned}\{1\cdot 8\}\cup\{2\cdot 4,\ 4\cdot 2\} &\mapsto \{(1^{8-1}),(1,2^{4-1}),(1^3,4^{2-1})\}\\ &= \{(1^7),(1,2^3),(1^3,4)\}\\ &\subset D((1^7,2^3,4)),\ \text{with } \Lambda_7 = (1^7,2^3,4).\end{aligned}$$

More directly we can write:

$$\begin{aligned}(1\cdot 8, 2\cdot 4,\ 4\cdot 2,\ 8\cdot 1) &\mapsto (1^{8-1},2^{4-1},4^{2-1},8^{1-1})\\ &= (1^7,2^3,4,\emptyset)\\ &= (1^7,2^3,4).\end{aligned}$$

To summarize the construction in general, let the increasing sequence of divisors of $k+1$ be

$$d_1, d_2, d_3, \ldots, d_{\tau(k+1)-1}, d_{\tau(k+1)}, \tag{19}$$

where $d_1 = 1$, $d_{\tau(k+1)} = k+1$ with $d_i \cdot d_{\tau(k+1)-i+1} = k+1$ for all $i$.

Then $\Lambda_k$ has the stated general form. □

**Corollary 2.** *Let $\ell(\Lambda_k)$ denote the length, and let $w(\Lambda_k)$ denote the weight, of $\Lambda_k$. Then*

$$\ell(\Lambda_k) = \sigma(k+1) - \tau(k+1). \tag{20}$$

$$w(\Lambda_k) = (k+1)\tau(k+1) - \sigma(k+1). \tag{21}$$

*Proof.* From Theorem 4 the number of parts is given by

$$\begin{aligned}\ell(\Lambda_k) &= (d_\tau - 1) + (d_{\tau-1} - 1) + (d_{\tau-2} - 1) + \cdots + (d_2 - 1) + (d_1 - 1)\\ &= (d_\tau + d_{\tau-1} + d_{\tau-2} + \cdots + d_2 + d_1) + (-1)\tau\\ &= \sigma(k+1) - \tau(k+1).\end{aligned}$$

Similarly,

$$\begin{aligned}w(\Lambda_k) &= (d_\tau - 1)1 + (d_{\tau-1} - 1)d_2 + \cdots + (d_2 - 1)d_{\tau-1} + (d_1 - 1)d_\tau\\ &= d_\tau 1 + d_{\tau-1}d_2 + \cdots + d_2 d_{\tau-1} + d_1 d_\tau - (1 + d_2 + \cdots + d_{\tau-1} + d_\tau)\\ &= \sum_{i=1}^{\tau(k+1)} (k+1) - \sigma(k+1) \qquad \because\ d_i d_{\tau+1-i} = k+1\\ &= (k+1)\tau(k+1) - \sigma(k+1).\end{aligned}$$

□

For example, let $k = 7$. Then

$$\sigma(8) - \tau(8) = (1 + 2 + 4 + 8) - |\{1, 2, 4, 8\}| = 15 - 4 = 11,$$

and from Table 4 we can verify that $\Lambda_7$ has 11 parts.

The sequence $\{w(\Lambda_k)\}_{k\geq 1}$ begins as follows:

$$1, 2, 5, 4, 12, 6, 17, 14, 22, 10, 44, 12, 32, 36, 49, \ldots$$

This agrees with sequence $A094471$ in [11]. A comment also describes the $k$th term as "The sum of all parts minus the total number of parts of all partitions of $k+1$ into equal parts".

**Corollary 3.** *The cardinality of the set of derived partitions of $\Lambda_k$ is given by*

$$|D(\Lambda_k)| = \prod_{d|(k+1)} d = (k+1)^{\tau(k+1)/2}. \tag{22}$$

*Proof.* The first equality follows immediately on applying Proposition (1) to the explicit form (18). The second equality is a known identity (see, for example, Apostol [3]). □

## 5. Full-Perfect-Partition-Complete Partitions

A partition $\lambda$ of a positive integer is said to be *full-perfect-partition-complete* for $n$ (*fpp-complete* for short) if the parts of $\lambda$ contain all perfect partitions of every integer from 1 to $n$.

In other words, $\lambda$ is fpp-complete for $n$ if $\pi \in D(\lambda)$ for every perfect partition $\pi$ of $m \in \{1, 2, \ldots, n\}$.

Again we concentrate on the $k$th *initial fpp-complete* partition, to be denoted by $\Pi_k$ for $k = 1, 2, 3, \ldots$.

Since each $\Lambda_k$ contains all perfect partitions of $k$, and $\Pi_k$ contains all perfect partitions of all integers from 1 to k, it follows that $\Pi_k$ contains all the partitions $\Lambda_k$, $k = 1, \ldots, k$. So we can write:

$$\Pi_k = \Lambda_1 \hat{U} \Lambda_2 \hat{U} \cdots \hat{U} \Lambda_k, \tag{23}$$

where $A\hat{U}B$ stands for taking every distinct part with maximum multiplicity between $A$ and $B$. Note that the binary operation $\hat{U}$ is associative.

The partitions $\Pi_1, \ldots, \Pi_{12}$ are given in Table 5 (see the second column). The corresponding lengths $\ell(\Pi_k)$ and weights $w(\Pi_k)$ appear in the third and fourth columns respectively.

| $k$ | $\Pi_k$ | $\ell(\Pi_k)$ | $w(\Pi_k)$ |
|---|---|---|---|
| 1 | $(1)$ | 1 | 1 |
| 2 | $(1^2)$ | 2 | 2 |
| 3 | $(1^3, 2)$ | 4 | 5 |
| 4 | $(1^4, 2)$ | 5 | 6 |
| 5 | $(1^5, 2^2, 3)$ | 8 | 12 |
| 6 | $(1^6, 2^2, 3)$ | 9 | 13 |
| 7 | $(1^7, 2^3, 3, 4)$ | 12 | 20 |
| 8 | $(1^8, 2^3, 3^2, 4)$ | 14 | 24 |
| 9 | $(1^9, 2^4, 3^2, 4, 5)$ | 17 | 32 |
| 10 | $(1^{10}, 2^4, 3^2, 4, 5)$ | 18 | 33 |
| 11 | $(1^{11}, 2^5, 3^3, 4^2, 5, 6)$ | 23 | 49 |
| 12 | $(1^{12}, 2^5, 3^3, 4^2, 5, 6)$ | 24 | 50 |

Table 5: The $k$th initial fpp-complete partitions for $1 \leq k \leq 12$

By comparing Tables 5 and 4 when $k = 7$ we find, for instance, that Equation (23) gives

$$\begin{aligned}\Pi_7 &= (1)\hat{U}(1^2)\hat{U}(1^3,2)\hat{U}(1^4)\hat{U}(1^5,2^2,3)\hat{U}(1^3)\hat{U}(1^7,2^3,4)\\ &= (1^7,2^3,3,4).\end{aligned}$$

We observe the following immediate properties of $\Pi_k$, from Table 5:

(1) $\Pi_k$ contains all of the integers $1, 2, \ldots, \lfloor \frac{k+1}{2} \rfloor$, for each $k \geq 5$.

(2) The sequences $\{\ell(\Pi_k)\}_{k\geq 1}$ and $\{w(\Pi_k)\}_{k\geq 1}$ are monotone increasing.

(3) The integer $k+1$ is prime if and only if $\ell(\Pi_k) = \ell(\Pi_{k-1}) + 1$ and $w(\Pi_k) = w(\Pi_{k-1}) + 1$.

As in previous agendas we compute the formulas for $\ell(\Pi_k)$ and $w(\Pi_k)$.

**Theorem 5.** *The kth initial fpp-complete partition has the explicit form:*

$$\Pi_k = (1^k, 2^{\lfloor (k-1)/2 \rfloor}, 3^{\lfloor (k-2)/3 \rfloor}, \ldots, \lfloor (k+1)/3 \rfloor^2, \lfloor (k+4)/3 \rfloor, \ldots, \lfloor (k+1)/2 \rfloor). \quad (24)$$

*Proof.* Note that $\Pi_k$ has the form

$$\Pi_k = (1^{e_1}, 2^{e_2}, 3^{e_3}, \ldots, h^{e_h}, \ldots, \lfloor (k+1)/2 \rfloor^1), \quad (25)$$

where $1 \leq h \leq \lfloor (k+1)/2 \rfloor$ and $e_h > 0$ for all $h$.

In order to determine the $e_i$ we consider the system of sets of proper divisors of 2, 3, ..., k+1, and let $dV(r)$ denote the set of divisors of $r$. Let

$$DV(k) = \{dV(r) \mid 2 \leq r \leq \lfloor (k+1)/2 \rfloor\}.$$

Clearly $1 \in dV(r)$ for every $r$, and $\lfloor (k+1)/2 \rfloor \in dV(r)$ for some $r$. The parts of $\Pi_k$ are obtained with the following rule:

*For each $h$ select all occurrences of $h$ as a proper divisor in $DV(k)$;*
*the number of occurrences is the multiplicity $e_h$.*

Since $1 \in dV(r)$ as a proper divisor, for all $r$, the multiplicity of 1 in $\Pi_k$ is $k$.

But $2 \leq h \in dV(r)$ as a proper divisor if and only if $r = jh$, where $2 \leq j \leq \lfloor (k+1)/h \rfloor$. This implies that

$$h \in dV(r) \iff r = 2h,\, 3h,\, \ldots,\, h\lfloor (k+1)/h \rfloor$$

which gives exactly $\lfloor (k+1-h)/h \rfloor$ instances. Therefore the multiplicity of $h$ in $\Pi_k$ is $e_h = \lfloor (k+1-h)/h \rfloor$.

It follows that the general form of $\Pi_k$ is as stated. □

**Remark 3.** Besides (23) and (24), $\Pi_{k+1}$ may be obtained from $\Pi_k$ using the following algorithm (cf. (10)):

"Insert the proper divisors of $k+2$ into $\Pi_k$" .

For example, since the proper divisors of 6 are $1, 2, 3$, we have

$$\Pi_5 = \pi_4 \cup (1, 2, 3) = (1^4, 2) \cup (1, 2, 3) = (1^5, 2^2, 3).$$

$$\text{Similarly,} \quad \Pi_6 = \pi_5 \cup (1) = (1^5, 2^2, 3) \cup (1) = (1^6, 2^2, 3).$$

**Corollary 4.** *Let $\ell(\Pi_k)$ denote the length and $w(\Pi_k)$ the weight of $\Pi_k$. Then*

$$\text{(i)}\quad \ell(\Pi_k) = -k + \sum_{j=1}^{k} \tau(j+1). \tag{26}$$

$$\text{(ii)}\quad w(\Pi_k) = \sum_{j=1}^{k} \left(\sigma(j+1) - j - 1\right). \tag{27}$$

*Proof.* (i) We add the multiplicities in Equation (24) and obtain

$$\begin{aligned}
\ell(\Pi_k) &= \sum_{h=1}^{s_k} \left\lfloor \frac{k+1-h}{h} \right\rfloor, \quad s_k = \left\lfloor \frac{k+1}{2} \right\rfloor \\
&= -s_k + \sum_{h=1}^{s_k} \left\lfloor \frac{k+1}{h} \right\rfloor \\
&= -s_k + \sum_{h=1}^{k+1} \left\lfloor \frac{k+1}{h} \right\rfloor - \sum_{h=s_k+1}^{k+1} 1 \\
&= -(k+1) + \sum_{h=1}^{k+1} \left\lfloor \frac{k+1}{h} \right\rfloor .
\end{aligned}$$

Thus on applying the identity $\sum\limits_{j=1}^{k} \tau(j) = \sum\limits_{r=1}^{k} \left\lfloor \frac{k}{r} \right\rfloor$ (see Equation (9)) we obtain the desired formula.
(ii) Using Equation (24) the proof is similar to that of part (i). □

For example, let $k = 7$. Then

$$\begin{aligned}
\ell(\Pi_7) &= -7 + \tau(2) + \tau(3) + \tau(4) + \tau(5) + \tau(6) + \tau(7) + \tau(8) \\
&= -7 + 2 + 2 + 3 + 2 + 4 + 2 + 4 \\
&= 12.
\end{aligned}$$

The sequence $\{w(\Pi_k)\}_{k \geq 1}$ begins as follows:

$$1, 2, 5, 6, 12, 13, 20, 24, 32, 33, 49, 50, 60, 69, 84, \ldots$$

This agrees with sequence $A153485$ in Sloane [11]. A comment also describes the $k$th term as "The sum of all aliquot divisors of all positive integers $\leq k+1$".

Lastly, the following assertion is the result of applying the formula in Proposition 1 to the explicit form (24).

**Corollary 5.** *The cardinality of the set of derived partitions of $\Pi_k$ is given by*

$$|D(\Pi_k)| = \prod_{h=1}^{s_k} \left\lfloor \frac{k+1}{h} \right\rfloor, \quad s_k = \left\lfloor \frac{k+1}{2} \right\rfloor .$$

## References


[1] A. K. Agarwal and M.V Subbarao, Some properties of perfect partitions, *Indian J. Pure Appl. Math.*, **22** (1991), 737–743.

[2] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, 1984.

[3] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.

[4] V. E. Hoggatt and C. King, Problem E1424, *Amer. Math. Monthly* **67** (1960), p. 593.

[5] H. Lee, Double perfect partitions, *Discrete Math.* **306** (2006), 519–525.

[6] P. A. MacMahon, *Combinatory Analysis*, Volume 1, Cambridge University Press, 1915.

[7] P. A. MacMahon, Memoir on the theory of the compositions of numbers, *Philos. Trans. Roy. Soc. London Ser. A*, **184** (1893), 835–901.

[8] E. O'Shea, $M$-Partitions: optimal partitions of weight for one scale pan, *Discrete Math.* **289** (2004), 81–93.

[9] S. Park, Complete partitions, *Fibonacci Quart.* **36** (1998), 354–360.

[10] S. Park, The $r$-complete partitions, *Discrete Math.* **36** (1998), 354–360.

[11] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences.* Published electronically at https://oeis.org, 2019.