AVERAGE BEHAVIOR OF THE FOURIER COEFFICIENTS OF THE SYMMETRIC SQUARE L-FUNCTION OVER SOME SEQUENCE OF INTEGERS

Anubhav Sharma[1]
School of Mathematics and Statistics, University of Hyderabad Central University, Gachibowli, Hyderabad, India
19mmpp02@uohyd.ac.in

Ayyadurai Sankaranarayanan
School of Mathematics and Statistics, University of Hyderabad Central University, Gachibowli, Hyderabad, India
sank@uohyd.ac.in

Received: 1/18/22, Accepted: 7/14/22, Published: 8/3/22

Abstract

In this paper, we investigate the average behavior of the n^{th} normalized Fourier coefficients of the symmetric square L-function attached to a primitive holomorphic cusp form of weight k for the full modular group $SL(2,\mathbb{Z})$. We prove an asymptotic formula for

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{sym^2 f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2),$$

where x is sufficiently large and

$$L(s, sym^2 f) := \sum_{n=1}^{\infty} \frac{\lambda_{sym^2 f}(n)}{n^s}.$$

1. Introduction

Let $L(s,f)$ be the L-function associated with the primitive holomorphic cusp form f of weight k for the group $SL(2,\mathbb{Z})$. Let $\lambda_f(n)$ be the normalized n^{th} Fourier

[1]Corresponding author

coefficient of the Fourier expansion of $f(z)$ at the cusp ∞, i.e.,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z},$$

where $\Im(z) > 0$. Then the *L-function attached to* $\lambda_f(n)$ is defined as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$$

for $\Re(s) > 1$, where $\lambda_f(n)$ is the eigenvalue of all the Hecke operators T_n.

Let χ be the Dirichlet character modulo N. If

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ (upper half plane) and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, then f is known as a *modular form of weight k and level N with Nebentypus χ*. Here, $\Gamma_0(N)$ is the *congruence subgroup*, i.e.,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \colon c \equiv 0 \pmod{N} \right\}.$$

In 1974, P. Deligne [1] proved that for any prime p, there exist complex numbers $\alpha(p)$ and $\beta(p)$ such that

$$\alpha(p) + \beta(p) = \lambda_f(p),$$

and

$$|\alpha(p)| = |\beta(p)| = 1 = \alpha(p)\beta(p).$$

Then $L(s, f)$ can be written as

$$L(s, f) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1}.$$

Also, $|\lambda_f(n)| \leq d(n)$, where $d(n)$ is the divisor function.

The *symmetric square L-function* is defined as

$$\begin{aligned} L(s, sym^2 f) &:= \sum_{n=1}^{\infty} \frac{\lambda_{sym^2 f}(n)}{n^s} \\ &= \prod_p \left(1 - \frac{\alpha^2(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta^2(p)}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1}, \end{aligned}$$

for $\Re(s) > 1$, where $\lambda_{sym^2 f}(n)$ is multiplicative.

Several authors have studied the average behavior of these Fourier coefficients. In 2006, Fomenko [2] proved the following results for the symmetric square L-functions. He showed that

$$\sum_{n\leq x}\lambda_{sym^2 f}(n) \ll x^{\frac{1}{2}}\log^2 x,$$

and also established that

$$\sum_{n\leq x}\lambda^2_{sym^2 f}(n) = cx + O(x^{\theta}),$$

where $\theta < 1$. For more related results, see [12], [7], [6], and [9]. In 2013, Zhai [14] proved an asymptotic formula for

$$\sum_{\substack{a^2+b^2\leq x\\(a,b)\in\mathbb{Z}^2}} \lambda^l_f(a^2+b^2),$$

for $x \geq 1$, and $3 \leq l \leq 8$.

In an earlier paper [11], we considered

$$\sum_{\substack{a^2+b^2+c^2+d^2\leq x\\(a,b,c,d)\in\mathbb{Z}^4}} \lambda^2_{sym^2 f}(a^2+b^2+c^2+d^2),$$

for a sufficiently large x, and established an asymptotic formula with a specific error term. In this paper, we study the behavior of the sum

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{sym^2 f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2),$$

where x is sufficiently large. More precisely, we prove the following.

Theorem 1. *For sufficiently large x, and any $\epsilon > 0$, we have*

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{sym^2 f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2) = cx^3 + O(x^{\frac{14}{5}+\epsilon}).$$

Here, c is an effective constant defined as

$$c = \frac{16}{3}L(3,\chi)L(1,sym^2 f)L(3,sym^2 f\otimes\chi)L(1,sym^4 f)L(3,sym^4 f\otimes\chi)H_2(3),$$

and χ is the non-principal Dirichlet character modulo 4.

Remark 1. The main idea of the proof here is that the sum in Theorem 1 is being related to the sum involving $r_6(n)$. The main difference from our earlier result (see [11]) related to a sum involving $r_4(n)$ is that $r_6(n)$ is not multiplicative. However, Lemma 1 demonstrates that $r_6(n)$ can be written as a sum of two multiplicative functions. The sum in Theorem 1 is split into two sums involving the corresponding multiplicative functions, which are dealt with independently. The two sums are then combined suitably to obtain the result. There are two Dirichlet characters, principal and non-principal, when the modulus is 4. In this case, the Dirichlet character is non-principal and primitive.

2. Preliminaries and Some Important Lemmas

Let $r_k(n) := \#\{(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k \colon n_1^2 + n_2^2 + \cdots + n_k^2 = n\}$ (allowing zeros, distinguishing signs, and order). We will be concerned with the function $r_6(n)$.

Lemma 1. *For any positive integer n, we have*

$$r_6(n) = 16\sum_{d|n} \chi(d')d^2 - 4\sum_{d|n} \chi(d)d^2, \tag{1}$$

where $dd' = n$, and χ is the non-principal Dirichlet character modulo 4, i.e.,

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv -1 \pmod 4 \\ 0 & \text{if } n \equiv 0 \pmod 2 \end{cases}.$$

Proof. From [5, p. 313], we observe that

$$\chi(n) = \begin{cases} 0 & \text{if } 2|n \\ (-1)^{\frac{n-1}{2}} & \text{if } 2 \nmid n \end{cases}.$$

It is clear that $\chi(n)$ is the non-principal character modulo 4 and can be defined as stated. Also, we find that Equation (1) follows from [5, p. 415]. $\square$

We can reframe the Equation (1) as

$$\begin{aligned} r_6(n) &= 16\sum_{d|n} \chi(d)\frac{n^2}{d^2} - 4\sum_{d|n} \chi(d)d^2 \\ &=: 16l(n) - 4v(n). \end{aligned}$$

We write $l_1(n) = 16l(n)$, and $v_1(n) = 4v(n)$.

The functions $\chi(d)$ and $\dfrac{n^2}{d^2}$ are completely multiplicative functions. This implies that $\chi(d)\dfrac{n^2}{d^2}$ is multiplicative. If $g(d)$ is any multiplicative function, then $\sum_{d|n} g(d)$ is also multiplicative. Therefore, $l(n)$ is a multiplicative function. Similarly, $v(n)$ is also multiplicative.

Note that

$$l(p) = p^2 + \chi(p),$$
$$l(p^2) = p^4 + p^2\chi(p) + \chi(p^2),$$

and

$$v(p) = 1 + p^2\chi(p),$$
$$v(p^2) = 1 + p^2\chi(p) + p^4\chi(p^2).$$

We can also write

$$\begin{aligned}
&\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\le x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{sym^2 f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2)\\
&= \sum_{n\le x}\lambda^2_{sym^2 f}(n) \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} 1\\
&= \sum_{n\le x}\lambda^2_{sym^2 f}(n) r_6(n)\\
&= \sum_{n\le x}\lambda^2_{sym^2 f}(n)\left(l_1(n) - v_1(n)\right) \qquad (2)\\
&= 16\sum_{n\le x}\lambda^2_{sym^2 f}(n) l(n) - 4\sum_{n\le x}\lambda^2_{sym^2 f}(n) v(n),
\end{aligned}$$

where $l(n) = \sum_{d|n}\chi(d)\frac{n^2}{d^2}$, and $v(n) = \sum_{d|n}\chi(d)d^2$.

Lemma 2 ([3]). *For any $\epsilon > 0$, we have*

$$\int_1^T \left| L\left(\frac{1}{2}+it\right)\right|^2 dt \ll T\log T,$$

uniformly for $T \ge 1$.

Lemma 3. *For any $\epsilon > 0$, we have*

$$L(\sigma + it) \ll_\epsilon (1 + |t|)^{\frac{1}{3}(1+\epsilon-\sigma)+\epsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$, and $|t| \geq t_0$ (where t_0 is sufficiently large).

Proof. We get the result by using the maximum-modulus principle in a suitable rectangle. For instance, see [3]. □

Lemma 4 ([13, Theorem 7.2])**.** *For any $\epsilon > 0$, we have*

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \ll T^{1+\epsilon},$$

uniformly for $T \geq 1$.

Lemma 5. *For any $\epsilon > 0$, we have*

$$\zeta(\sigma + it) \ll_\epsilon (1 + |t|)^{\frac{1}{3}(1+\epsilon-\sigma)+\epsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$, and $|t| \geq t_0$ (where t_0 is sufficiently large).

Proof. We get the result when we apply the maximum-modulus principle to $F(w) = \zeta(w)e^{(w-s)^2}x^{w-s}$ in a suitable rectangle and by using Hardy's estimate $\zeta(\frac{1}{2} + it) \ll (|t| + 10)^{\frac{1}{6}}$. For instance, see [10]. □

Lemma 6. *Let $L(s, f)$ be a Dirichlet series with Euler product of degree $m \geq 2$, i.e.,*

$$L(s, f) = \prod_{p<\infty} \prod_{i=0}^{m} \left(1 - \frac{\alpha(p, i)}{p^s}\right)^{-1},$$

where $\alpha(p, i)$ are local parameters of $L(s, f)$ at prime p. If the Euler product converges absolutely for $\Re(s) > 1$, admits a meromorphic continuation to the whole complex plane $\mathbb{C}$, and satisfies a functional equation of Riemann-zeta type, then we have

$$\int_T^{2T} \left| L\left(\frac{1}{2} + \epsilon + it, f\right) \right|^2 dt \ll T^{\frac{m}{2}+\epsilon}, \tag{3}$$

for $T \geq 1$; and for $0 \leq \sigma \leq 1 + \epsilon$, we have

$$|L(\sigma + it, f)| \ll (1 + |t|)^{\frac{m}{2}(1+\epsilon-\sigma)+\epsilon}. \tag{4}$$

Proof. The proof of Equation (3) is derived in a similar fashion as in [8, Theorem 4.1]. Equation (4) follows by the maximum-modulus principle. □

We know that (for $0 \leq j \leq 4$ and $\Re(s) > 1$) for a prime p, the p^{th} *Fourier coefficient of the* j^{th} *symmetric power L-function of* f can be written as

$$\lambda_{sym^j f}(p) = \sum_{m=0}^{j} \alpha^{j-m}(p)\beta^m(p).$$

Lemma 7. *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2,\mathbb{Z})$. Let $\lambda_{sym^2 f}(n)$ be the n^{th} normalized Fourier coefficient of the symmetric square L-function associated with f. If*

$$F_2(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{sym^2 f}(n) l(n)}{n^s},$$

for $\Re(s) > 3$, then

$$F_2(s) = G_2(s)H_2(s),$$

where

$$\begin{aligned} G_2(s) :=&\zeta(s-2)L(s,\chi)L(s-2, sym^2 f)L(s, sym^2 f \otimes \chi) \\ &\times L(s-2, sym^4 f)L(s, sym^4 f \otimes \chi), \end{aligned}$$

and χ is the non-principal character modulo 4. Here, $H_2(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $H_2(s) \neq 0$ on $\Re(s) = 3$.

Proof. We observe that $\lambda^2_{sym^2 f}(n) l(n)$ is multiplicative, and hence

$$F_2(s) = \prod_p \left(1 + \frac{\lambda^2_{sym^2 f}(p) l(p)}{p^s} + \cdots + \frac{\lambda^2_{sym^2 f}(p^m) l(p^m)}{p^{ms}} + \cdots \right).$$

Note that

$$\begin{aligned} \lambda^2_{sym^2 f}(p) l(p) &= \left(\sum_{m=0}^{2} \alpha^{2-m}(p)\beta^m(p)\right)^2 \left(p^2 + \chi(p)\right) \\ &= (\alpha^4(p) + 2\alpha^2(p) + 3 + 2\beta^2(p) + \beta^4(p)) \left(p^2 + \chi(p)\right) \\ &= p^2 + \chi(p) + (\alpha^2(p) + 1 + \beta^2(p))(p^2 + \chi(p)) \\ &\quad + (\alpha^4(p) + \alpha^2(p) + 1 + \beta^2(p) + \beta^4(p)) \left(p^2 + \chi(p)\right) \\ &= p^2 + \chi(p) + \lambda_{sym^2 f}(p)(p^2 + \chi(p)) + \lambda_{sym^4 f}(p)(p^2 + \chi(p)) \\ &= p^2 + \chi(p) + p^2 \lambda_{sym^2 f}(p) + \chi(p)\lambda_{sym^2 f}(p) \\ &\quad + p^2 \lambda_{sym^4 f}(p) + \chi(p)\lambda_{sym^4 f}(p) \\ &=: b(p). \end{aligned}$$

From the structure of $b(p)$, we define the coefficients $b(n)$ as

$$\sum_{n=1}^{\infty}\frac{b(n)}{n^s} = \zeta(s-2)L(s,\chi)L(s-2,sym^2 f)L(s,sym^2 f\otimes\chi)L(s-2,sym^4 f) \times L(s,sym^4 f\otimes\chi),$$

which is absolutely convergent in $\Re(s)>3$. We also note that

$$\begin{aligned}
&\prod_p\left(1+\frac{b(p)}{p^s}+\cdots+\frac{b(p^m)}{p^{ms}}+\cdots\right)\\
&=\zeta(s-2)L(s,\chi)L(s-2,sym^2 f)L(s,sym^2 f\otimes\chi)\\
&\quad\times L(s-2,sym^4 f)L(s,sym^4 f\otimes\chi)\\
&=: G_2(s),
\end{aligned}$$

for $\Re(s)>3$. Observe that $b(n)\ll_\epsilon n^{2+\epsilon}$ for any small positive constant ϵ.

Now, we note that in the half plane $\Re(s)\geq 3+2\epsilon$, we have

$$\begin{aligned}
\left|\frac{b(p)}{p^s}+\frac{b(p^2)}{p^{2s}}+\cdots+\frac{b(p^m)}{p^{ms}}+\cdots\right| &\ll \sum_{m=1}^{\infty}\frac{p^{(2+\epsilon)m}}{p^{m\sigma}}\\
&\leq \sum_{m=1}^{\infty}\frac{p^{(2+\epsilon)m}}{p^{(3+2\epsilon)m}}\\
&= \sum_{m=1}^{\infty}\frac{1}{p^{(1+\epsilon)m}}\\
&= \frac{\frac{1}{p^{1+\epsilon}}}{1-\frac{1}{p^{1+\epsilon}}}\\
&= \frac{1}{p^{1+\epsilon}-1}\\
&< 1.
\end{aligned}$$

Let us write

$$A=\frac{\lambda^2_{sym^2 f}(p)l(p)}{p^s}+\cdots+\frac{\lambda^2_{sym^2 f}(p^m)l(p^m)}{p^{ms}}+\cdots,$$

and

$$B = \frac{b(p)}{p^s} + \cdots + \frac{b(p^m)}{p^{ms}} + \cdots .$$

From the above calculations, we observe that $|B| < 1$ in $\Re(s) \geq 3 + 2\epsilon$.

We note that in the half plane $\Re(s) \geq 3 + 2\epsilon$, we have

$$\begin{aligned} \frac{1+A}{1+B} &= (1+A)(1 - B + B^2 - B^3 + \cdots) \\ &= 1 + A - B - AB + \text{higher terms} \\ &= 1 + \frac{\lambda^2_{sym^2 f}(p^2) l(p^2) - b(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots , \end{aligned}$$

with $c_m(n) \ll_\epsilon n^{2+\epsilon}$. So, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned} \prod_p \left(\frac{1+A}{1+B} \right) &= \prod_p \left(1 + \frac{\lambda^2_{sym^2 f}(p^2) l(p^2) - b(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots \right) \\ &\ll_\epsilon 1. \end{aligned}$$

Thus, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned} H_2(s) :&= \frac{F_2(s)}{G_2(s)} \\ &= \prod_p \left(\frac{1+A}{1+B} \right) \\ &\ll_\epsilon 1, \end{aligned}$$

and also $H_2(s) \neq 0$ on $\Re(s) = 3$.

□

Lemma 8. *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2, \mathbb{Z})$. Let $\lambda_{sym^2 f}(n)$ be the n^{th} normalized Fourier coefficient of the symmetric square L-function associated with f. If*

$$\widetilde{F}_2(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{sym^2 f}(n) v(n)}{n^s},$$

for $\Re(s) > 3$, then

$$\widetilde{F}_2(s) = \widetilde{G}_2(s) \widetilde{H}_2(s),$$

where

$$\begin{aligned} \widetilde{G}_2(s) :=& \zeta(s) L(s-2, \chi) L(s, sym^2 f) L(s-2, sym^2 f \otimes \chi) \\ & \times L(s, sym^4 f) L(s-2, sym^4 f \otimes \chi), \end{aligned}$$

and χ is the non-principal character modulo 4. Here, $\widetilde{H}_2(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $\widetilde{H}_2(s) \neq 0$ on $\Re(s) = 3$.

Proof. We observe that $\lambda^2_{sym^2 f}(n)v(n)$ is multiplicative, and hence

$$\widetilde{F}_2(s) = \prod_p \left(1 + \frac{\lambda^2_{sym^2 f}(p)v(p)}{p^s} + \cdots + \frac{\lambda^2_{sym^2 f}(p^m)v(p^m)}{p^{ms}} + \cdots\right).$$

Note that

$$\begin{aligned}
\lambda^2_{sym^2 f}(p)v(p) &= \left(\sum_{m=0}^{2} \alpha^{2-m}(p)\beta^m(p)\right)^2 \left(1 + p^2\chi(p)\right) \\
&= (\alpha^4(p) + 2\alpha^2(p) + 3 + 2\beta^2(p) + \beta^4(p)) \left(1 + p^2\chi(p)\right) \\
&= 1 + p^2\chi(p) + (\alpha^2(p) + 1 + \beta^2(p))(1 + p^2\chi(p)) \\
&\quad + (\alpha^4(p) + \alpha^2(p) + 1 + \beta^2(p) + \beta^4(p)) \left(1 + p^2\chi(p)\right) \\
&= 1 + p^2\chi(p) + \lambda_{sym^2 f}(p)(1 + p^2\chi(p)) + \lambda_{sym^4 f}(p)(1 + p^2\chi(p)) \\
&= 1 + p^2\chi(p) + \lambda_{sym^2 f}(p) + p^2\chi(p)\lambda_{sym^2 f}(p) \\
&\quad + \lambda_{sym^4 f}(p) + p^2\chi(p)\lambda_{sym^4 f}(p) \\
&=: h(p).
\end{aligned}$$

From the structure of $h(p)$, we define the coefficients $h(n)$ as

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{h(n)}{n^s} =&\zeta(s)L(s-2,\chi)L(s, sym^2 f)L(s-2, sym^2 f \otimes \chi)L(s, sym^4 f) \\
&\times L(s-2, sym^4 f \otimes \chi),
\end{aligned}$$

which is absolutely convergent in $\Re(s) > 3$. We also note that

$$\begin{aligned}
&\prod_p \left(1 + \frac{h(p)}{p^s} + \cdots + \frac{h(p^m)}{p^{ms}} + \cdots\right) \\
&= \zeta(s)L(s-2,\chi)L(s, sym^2 f)L(s-2, sym^2 f \otimes \chi)L(s, sym^4 f)L(s-2, sym^4 f \otimes \chi) \\
&=: \widetilde{G}_2(s),
\end{aligned}$$

for $\Re(s) > 3$. Observe that $h(n) \ll_\epsilon n^{2+\epsilon}$ for any small positive constant ϵ. Now, we note that in the half plane $\Re(s) \geq 3 + 2\epsilon$, we have

$$\begin{aligned} \left| \frac{h(p)}{p^s} + \frac{h(p^2)}{p^{2s}} + \cdots + \frac{h(p^m)}{p^{ms}} + \cdots \right| &\ll \sum_{m=1}^{\infty} \frac{p^{(2+\epsilon)m}}{p^{m\sigma}} \\ &< 1. \end{aligned}$$

Let us write

$$\widetilde{A} = \frac{\lambda^2_{sym^2 f}(p)v(p)}{p^s} + \cdots + \frac{\lambda^2_{sym^2 f}(p^m)v(p^m)}{p^{ms}} + \cdots ,$$

and

$$\widetilde{B} = \frac{h(p)}{p^s} + \cdots + \frac{h(p^m)}{p^{ms}} + \cdots .$$

From the above calculations, we observe that $|\widetilde{B}| < 1$ in $\Re(s) \geq 3 + 2\epsilon$.

We note that in the half plane $\Re(s) \geq 3 + 2\epsilon$, we have

$$\begin{aligned} \frac{1+\widetilde{A}}{1+\widetilde{B}} &= (1+\widetilde{A})(1-\widetilde{B}+\widetilde{B}^2-\widetilde{B}^3+\cdots) \\ &= 1+\widetilde{A}-\widetilde{B}-\widetilde{A}\widetilde{B} + \text{higher terms} \\ &= 1 + \frac{\lambda^2_{sym^2 f}(p^2)v(p^2) - h(p^2)}{p^{2s}} + \cdots + \frac{\widetilde{c}_m(p^m)}{p^{ms}} + \cdots , \end{aligned}$$

with $\widetilde{c}_m(n) \ll_\epsilon n^{2+\epsilon}$. So, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned} \prod_p \left(\frac{1+\widetilde{A}}{1+\widetilde{B}} \right) &= \prod_p \left(1 + \frac{\lambda^2_{sym^2 f}(p^2)v(p^2) - h(p^2)}{p^{2s}} + \cdots + \frac{\widetilde{c}_m(p^m)}{p^{ms}} + \cdots \right) \\ &\ll_\epsilon 1. \end{aligned}$$

Thus, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned} \widetilde{H}_2(s) &:= \frac{\widetilde{F}_2(s)}{\widetilde{G}_2(s)} \\ &= \prod_p \left(\frac{1+\widetilde{A}}{1+\widetilde{B}} \right) \\ &\ll_\epsilon 1, \end{aligned}$$

and also $\widetilde{H}_2(s) \neq 0$ on $\Re(s) = 3$. $\square$

3. Proof of Theorem 1

From Equation (2), we can write

$$\sum_{n\le x}\lambda^2_{sym^2f}(n)r_6(n)=\sum_{n\le x}\lambda^2_{sym^2f}(n)l_1(n)-\sum_{n\le x}\lambda^2_{sym^2f}(n)v_1(n).$$

Firstly, we consider the sum $\sum_{n\le x}\lambda^2_{sym^2f}(n)l_1(n)$. We begin by applying Perron's formula (see [4, Chapter 2.4]) to $F_2(s)$ with $\eta=3+\epsilon$ and $10\le T\le x$. Thus, we have

$$\begin{aligned}\sum_{n\le x}\lambda^2_{sym^2f}(n)l_1(n)&=16\sum_{n\le x}\lambda^2_{sym^2f}(n)l(n)\\&=\frac{16}{2\pi i}\int_{\eta-iT}^{\eta+iT}F_2(s)\frac{x^s}{s}ds+O\left(\frac{x^{3+3\epsilon}}{T}\right).\end{aligned}$$

We move the line of integration to $\Re(s)=\frac{5}{2}+\epsilon$. By Cauchy's residue theorem there is only one simple pole at $s=3$, coming from the factor $\zeta(s-2)$. This contributes a residue, which is cx^3, where c is an effective constant depending on the values of various L-functions appearing in $G_2(s)$ at $s=3$.

More precisely,

$$\begin{aligned}c&=16\lim_{s\to 3}(s-3)\frac{F_2(s)}{s}\\&=\frac{16}{3}L(3,\chi)L(1,sym^2f)L(3,sym^2f\otimes\chi)\\&\quad\times L(1,sym^4f)L(3,sym^4f\otimes\chi)H_2(3).\end{aligned}$$

So, we obtain

$$\begin{aligned}\sum_{n\le x}\lambda^2_{sym^2f}(n)l_1(n)&=cx^3+\frac{16}{2\pi i}\left\{\int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT}+\int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT}+\int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT}\right\}F_2(s)\frac{x^s}{s}ds\\&\quad+O\left(\frac{x^{3+3\epsilon}}{T}\right)\\&=:cx^3+\frac{16}{2\pi i}(J_1+J_2+J_3)+O\left(\frac{x^{3+3\epsilon}}{T}\right).\end{aligned}$$

The contribution of the horizontal line integrals (using Lemma 5 and Lemma 6) is

$$J_2 + J_3 \ll \int_{\frac{5}{2}+\epsilon}^{3+\epsilon} \frac{|\zeta(\sigma - 2 + iT)L(\sigma - 2 + iT, sym^2 f)L(\sigma - 2 + iT, sym^4 f)|}{T} x^{\sigma} d\sigma$$

$$\ll \int_{\frac{1}{2}+\epsilon}^{1+\epsilon} \frac{|\zeta(\sigma + iT)L(\sigma + iT, sym^2 f)L(\sigma + iT, sym^4 f)|}{T} x^{\sigma+2} d\sigma$$

$$\ll \left(\frac{x^2}{T}\right) \max_{\frac{1}{2}+\epsilon \le \sigma \le 1+\epsilon} x^{\sigma} T^{\frac{1}{3}(1+\epsilon-\sigma)} T^{\frac{8}{2}(1+\epsilon-\sigma)}$$

$$\ll \left(\frac{x^{2+2\epsilon}}{T}\right) \max_{\frac{1}{2}+\epsilon \le \sigma \le 1+\epsilon} \left(\frac{x}{T^{\frac{13}{3}}}\right)^{\sigma} T^{\frac{13}{3}}.$$

Clearly, $\left(\frac{x}{T^{\frac{13}{3}}}\right)^{\sigma}$ is monotonic as a function of σ for $\frac{1}{2} + \epsilon \le \sigma \le 1 + \epsilon$, and hence the maximum is attained at the extremities of the interval $[\frac{1}{2} + \epsilon, 1 + \epsilon]$. Thus,

$$\begin{aligned} J_2 + J_3 &\ll x^{2+2\epsilon} \left(x^{\frac{1}{2}+\epsilon} T^{\frac{13}{6}-1+\epsilon} + \frac{x^{1+\epsilon}}{T} \right) \\ &\ll x^{\frac{5}{2}+3\epsilon} T^{\frac{7}{6}+\epsilon} + \frac{x^{3+3\epsilon}}{T}. \end{aligned}$$

The contribution of the left vertical line integral (using the Cauchy-Schwarz inequality, Lemma 4, and Lemma 6) is

$$J_1 \ll \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} \frac{|\zeta(\frac{1}{2} + \epsilon + it)L(\frac{1}{2} + \epsilon + it, sym^2 f)L(\frac{1}{2} + \epsilon + it, sym^4 f)|}{|\frac{5}{2} + \epsilon + it|} x^{\frac{5}{2}+\epsilon} dt$$

$$\begin{aligned} &\ll x^{\frac{5}{2}+\epsilon} + x^{\frac{5}{2}+\epsilon} \left(\int_{10 \le |t| \le T} \frac{|\zeta(\frac{1}{2} + \epsilon + it)|^2}{t} dt \right)^{\frac{1}{2}} \\ &\times \left(\int_{10 \le |t| \le T} \frac{|L(\frac{1}{2} + \epsilon + it, sym^2 f)L(\frac{1}{2} + \epsilon + it, sym^4 f)|^2}{t} dt \right)^{\frac{1}{2}} \end{aligned}$$

$$\ll x^{\frac{5}{2}+\epsilon} + x^{\frac{5}{2}+\epsilon} \left(T^{\frac{\epsilon}{2}} T^{\frac{3}{2}+\frac{\epsilon}{2}} \right)$$

$$\ll x^{\frac{5}{2}+2\epsilon} T^{\frac{3}{2}+2\epsilon}.$$

Note that $10 \leq T \leq x$. Thus, we obtain

$$\sum_{n \leq x} \lambda^2_{sym^2 f}(n) l_1(n) = cx^3 + O(x^{\frac{5}{2}+2\epsilon} T^{\frac{3}{2}+2\epsilon}) + O\left(\frac{x^{3+3\epsilon}}{T}\right).$$

We choose T such that $x^{\frac{5}{2}} T^{\frac{3}{2}} \asymp \frac{x^3}{T}$ i.e., $T^{\frac{5}{2}} \asymp x^{\frac{1}{2}}$. Therefore, $T \asymp x^{\frac{1}{5}}$. Thus, we get

$$\sum_{n \leq x} \lambda^2_{sym^2 f}(n) l_1(n) = cx^3 + O(x^{\frac{14}{5}+\epsilon}). \tag{5}$$

Similarly, we apply Perron's formula (see [4, Chapter 2.4]) to $\widetilde{F}_2(s)$ with $\eta = 3+\epsilon$ and $10 \leq T \leq x$. Thus, we have

$$\begin{aligned}\sum_{n \leq x} \lambda^2_{sym^2 f}(n) v_1(n) &= 4 \sum_{n \leq x} \lambda^2_{sym^2 f}(n) v(n) \\ &= \frac{4}{2\pi i} \int_{\eta - iT}^{\eta + iT} \widetilde{F}_2(s) \frac{x^s}{s} ds + O\left(\frac{x^{3+3\epsilon}}{T}\right).\end{aligned}$$

We move the line of integration to $\Re(s) = \frac{5}{2} + \epsilon$. There is no singularity in the rectangle obtained and the function $\widetilde{F}_2(s) \dfrac{x^s}{s}$ is analytic in this region. Thus, using Cauchy's theorem for rectangles pertaining to analytic functions, we get

$$\begin{aligned}\sum_{n \leq x} \lambda^2_{sym^2 f}(n) v_1(n) &= \frac{4}{2\pi i} \left\{ \int_{\frac{5}{2}+\epsilon - iT}^{\frac{5}{2}+\epsilon+iT} + \int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT} + \int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT} \right\} \widetilde{F}_2(s) \frac{x^s}{s} ds \\ &\quad + O\left(\frac{x^{3+3\epsilon}}{T}\right) \\ &=: \frac{4}{2\pi i} (J'_1 + J'_2 + J'_3) + O\left(\frac{x^{3+3\epsilon}}{T}\right).\end{aligned}$$

The contribution of the horizontal line integrals (using Lemma 3 and Lemma 6) is

$$J'_2 + J'_3 \ll \int_{\frac{5}{2}+\epsilon}^{3+\epsilon} \frac{|L(\sigma - 2 + iT, \chi) L(\sigma - 2 + iT, sym^2 f \otimes \chi) L(\sigma - 2 + iT, sym^4 f) \otimes \chi|}{T} x^\sigma d\sigma$$

$$\ll \int_{\frac{1}{2}+\epsilon}^{1+\epsilon} \frac{|L(\sigma + iT, \chi) L(\sigma + iT, sym^2 f \otimes \chi) L(\sigma + iT, sym^4 f \otimes \chi)|}{T} x^{\sigma+2} d\sigma.$$

Thus,

$$J_2' + J_3' \ll \left(\frac{x^2}{T}\right) \max_{\frac{1}{2}+\epsilon \le \sigma \le 1+\epsilon} x^\sigma T^{\frac{1}{3}(1+\epsilon-\sigma)} T^{\frac{8}{2}(1+\epsilon-\sigma)}$$

$$\ll \left(\frac{x^{2+2\epsilon}}{T}\right) \max_{\frac{1}{2}+\epsilon \le \sigma \le 1+\epsilon} \left(\frac{x}{T^{\frac{13}{3}}}\right)^\sigma T^{\frac{13}{3}}.$$

Clearly, $\left(\frac{x}{T^{\frac{13}{3}}}\right)^\sigma$ is monotonic as a function of σ for $\frac{1}{2}+\epsilon \le \sigma \le 1+\epsilon$, and hence the maximum is attained at the extremities of the interval $\left[\frac{1}{2}+\epsilon, 1+\epsilon\right]$. Thus,

$$J_2' + J_3' \ll x^{2+2\epsilon}\left(x^{\frac{1}{2}+\epsilon} T^{\frac{13}{6}-1+\epsilon} + \frac{x^{1+\epsilon}}{T}\right)$$
$$\ll x^{\frac{5}{2}+3\epsilon} T^{\frac{7}{6}+\epsilon} + \frac{x^{3+3\epsilon}}{T}.$$

The contribution of the left vertical line integral (using the Cauchy-Schwarz inequality, Lemma 2, and Lemma 6) is

$$J_1' \ll \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} \frac{|L(\frac{1}{2}+\epsilon+it,\chi)L(\frac{1}{2}+\epsilon+it, sym^2 f\otimes\chi)L(\frac{1}{2}+\epsilon+it, sym^4 f\otimes\chi)|}{|\frac{5}{2}+\epsilon+it|} x^{\frac{5}{2}+\epsilon} dt$$

$$\ll x^{\frac{5}{2}+\epsilon} + x^{\frac{5}{2}+\epsilon}\left(\int_{10\le|t|\le T} \frac{|L(\frac{1}{2}+\epsilon+it,\chi)|^2}{t} dt\right)^{\frac{1}{2}}$$
$$\times \left(\int_{10\le|t|\le T} \frac{|L(\frac{1}{2}+\epsilon+it, sym^2 f\otimes\chi)L(\frac{1}{2}+\epsilon+it, sym^4 f\otimes\chi)|^2}{t} dt\right)^{\frac{1}{2}}$$

$$\ll x^{\frac{5}{2}+\epsilon} + x^{\frac{5}{2}+\epsilon}\left(T^{\frac{\epsilon}{2}} T^{\frac{3}{2}+\frac{\epsilon}{2}}\right)$$

$$\ll x^{\frac{5}{2}+2\epsilon} T^{\frac{3}{2}+2\epsilon}.$$

Note that $10 \le T \le x$. Thus, we obtain

$$\sum_{n\le x} \lambda^2_{sym^2 f}(n) v_1(n) = O(x^{\frac{5}{2}+2\epsilon} T^{\frac{3}{2}+2\epsilon}) + O\left(\frac{x^{3+3\epsilon}}{T}\right).$$

We choose T such that $x^{\frac{5}{2}} T^{\frac{3}{2}} \asymp \frac{x^3}{T}$ i.e., $T^{\frac{5}{2}} \asymp x^{\frac{1}{2}}$. Therefore, $T \asymp x^{\frac{1}{5}}$.

Thus, we get

$$\sum_{n\le x} \lambda^2_{sym^2 f}(n) v_1(n) = O(x^{\frac{14}{5}+\epsilon}). \tag{6}$$

Combining Equations (5) and (6), we get

$$\sum_{n\leq x}\lambda^2_{sym^2 f}(n)r_6(n) = cx^3 + O(x^{\frac{14}{5}+\epsilon}),$$

where c is an effective constant given by

$$c = \frac{16}{3}L(3,\chi)L(1, sym^2 f)L(3, sym^2 f\otimes\chi)L(1, sym^4 f)L(3, sym^4 f\otimes\chi)H_2(3),$$

and χ is the non-principal Dirichlet character modulo 4. This proves the theorem.

Acknowledgements. The first author, Anubhav Sharma, wishes to express his gratitude to the University of Hyderabad for its financial support towards his Ph.D. Program. The authors are also grateful to the anonymous referee for some fruitful comments.

References

[1] P. Deligne, La conjecture de Weil. I, *Publ. Math. Inst. Hautes Études Sci.* **43(1)** (1974), 273-307.

[2] O. M. Fomenko, Identities involving coefficients of automorphic L-functions, *J. Math. Sci. (N.Y.)* **133(6)** (2006).

[3] A. Good, The square mean of Dirichlet series associated with cusp forms, *Mathematika* **29(2)** (1982), 278-295.

[4] A. Granville and K. Soundararajan, Multiplicative number theory: The pretentious approach, available at https://dms.umontreal.ca/~andrew/PDF/BookChaps1n2.pdf.

[5] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, London, 1979.

[6] X. He, Integral power sums of Fourier coefficients of symmetric square L-functions, *Proc. Amer. Math. Soc.* **147(7)** (2019), 2847-2856.

[7] H. Lao, On the fourth moment of coefficients of symmetric square L-function, *Chinese Ann. Math. Ser. B* **33(6)** (2012), 877-888.

[8] H. Lao and A. Sankaranarayanan, On the mean-square of the error term related to $\sum_{n\leq x}\lambda^2(n^j)$, *Sci. China Math.* **5(54)** (2011), 855-864.

[9] Y. K. Lau and G. Lü, Sums of Fourier coefficients of cusp forms, *Q. J. Math.* **62(3)** (2011), 687-716.

[10] A. Sankaranarayanan, Zeros of quadratic zeta-functions on the critical line, *Acta Arith.* **69** (1995), 21-38.

[11] A. Sharma and A. Sankaranarayanan, Discrete mean square of the coefficients of symmetric square L-functions on certain sequence of positive numbers, *Res. Number Theory* **8(1)** (2022), 1-13.

[12] H. Tang, Estimates for the Fourier coefficients of symmetric square L-functions, *Arch. Math. (Basel)* **100(2)** (2013), 123-130.

[13] E. C. Titchmarsh and D. R. Heath-Brown, *The Theory of the Riemann Zeta-function*, Oxford University Press, New York, 1986.

[14] S. Zhai, Average behavior of Fourier coefficients of cusp forms over sum of two squares, *J. Number Theory* **133(11)** (2013), 3862-3876.