AVERAGE BEHAVIOR OF THE FOURIER COEFFICIENTS OF
THE SYMMETRIC SQUARE L-FUNCTION OVER SOME
SEQUENCE OF INTEGERS

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Abstract
In this paper, we investigate the average behavior of the $n^{th}$ normalized Fourier
coefficients of the symmetric square $L$-function attached to a primitive holomorphic
cusp form of weight $k$ for the full modular group $SL(2,\mathbb{Z})$. We prove an asymptotic
formula for

$$\sum_{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 \leq x, (a_1,a_2,a_3,a_4,a_5,a_6) \in \mathbb{Z}^6} \lambda_{sym^2 f}(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2),$$

where $x$ is sufficiently large and

$$L(s, sym^2 f) := \sum_{n=1}^{\infty} \frac{\lambda_{sym^2 f}(n)}{n^s}.$$

1. Introduction
Let $L(s, f)$ be the $L$-function associated with the primitive holomorphic cusp form $f$ of weight $k$ for the group $SL(2,\mathbb{Z})$. Let $\lambda_f(n)$ be the normalized $n^{th}$ Fourier
coefficient of the Fourier expansion of \( f(z) \) at the cusp \( \infty \), i.e.,

\[
f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi inz},
\]

where \( \Im(z) > 0 \). Then the \( L \)-function attached to \( \lambda_f(n) \) is defined as

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}
\]

for \( \Re(s) > 1 \), where \( \lambda_f(n) \) is the eigenvalue of all the Hecke operators \( T_n \).

Let \( \chi \) be the Dirichlet character modulo \( N \). If

\[
f \left( \frac{az+b}{cz+d} \right) = \chi(d)(cz+d)^k f(z)
\]

for all \( z \in \mathbb{H} \) (upper half plane) and \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N) \), then \( f \) is known as a \textit{modular form of weight k and level N with Nebentypus \( \chi \)}. Here, \( \Gamma_0(N) \) is the congruence subgroup, i.e.,

\[
\Gamma_0(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.
\]

In 1974, P. Deligne [1] proved that for any prime \( p \), there exist complex numbers \( \alpha(p) \) and \( \beta(p) \) such that

\[
\alpha(p) + \beta(p) = \lambda_f(p),
\]

and

\[
|\alpha(p)| = |\beta(p)| = 1 = \alpha(p)\beta(p).
\]

Then \( L(s, f) \) can be written as

\[
L(s, f) = \prod_p \left( 1 - \frac{\alpha(p)}{p^s} \right)^{-1} \left( 1 - \frac{\beta(p)}{p^s} \right)^{-1}.
\]

Also, \( |\lambda_f(n)| \leq d(n) \), where \( d(n) \) is the divisor function.

The \textit{symmetric square} \( L \)-function is defined as

\[
L(s, \text{sym}^2 f) := \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^2 f}(n)}{n^s}
\]

\[
= \prod_p \left( 1 - \frac{\alpha^2(p)}{p^s} \right)^{-1} \left( 1 - \frac{\beta^2(p)}{p^s} \right)^{-1} \left( 1 - \frac{1}{p^s} \right)^{-1},
\]
for $\Re(s) > 1$, where $\lambda_{sym^2 f}(n)$ is multiplicative.

Several authors have studied the average behavior of these Fourier coefficients. In 2006, Fomenko \[2\] proved the following results for the symmetric square $L$-functions. He showed that

$$\sum_{n \leq x} \lambda_{sym^2 f}(n) \ll x^{\frac{1}{2}} \log^2 x,$$

and also established that

$$\sum_{n \leq x} \lambda_{sym^2 f}^2(n) = cx + O(x^\theta),$$

where $\theta < 1$. For more related results, see \[12\], \[7\], \[6\], and \[9\]. In 2013, Zhai \[14\] proved an asymptotic formula for

$$\sum_{a^2 + b^2 \leq x} \lambda_{sym}^2 f(a^2 + b^2),$$

for $x \geq 1$, and $3 \leq l \leq 8$.

In an earlier paper \[11\], we considered

$$\sum_{a^2 + b^2 + c^2 + d^2 \leq x} \lambda_{sym^2 f}^2(a^2 + b^2 + c^2 + d^2),$$

for a sufficiently large $x$, and established an asymptotic formula with a specific error term. In this paper, we study the behavior of the sum

$$\sum_{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 \leq x} \lambda_{sym^2 f}^2(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2),$$

where $x$ is sufficiently large. More precisely, we prove the following.

**Theorem 1.** For sufficiently large $x$, and any $\epsilon > 0$, we have

$$\sum_{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 \leq x} \lambda_{sym^2 f}^2(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2) = cx^3 + O(x^{\frac{14}{5} + \epsilon}).$$

Here, $c$ is an effective constant defined as

$$c = \frac{16}{3} L(3, \chi)L(1, sym^2 f)L(3, sym^2 f \otimes \chi)L(1, sym^4 f)L(3, sym^4 f \otimes \chi)H_2(3),$$

and $\chi$ is the non-principal Dirichlet character modulo 4.
Remark 1. The main idea of the proof here is that the sum in Theorem 1 is being related to the sum involving $r_6(n)$. The main difference from our earlier result (see [11]) related to a sum involving $r_4(n)$ is that $r_6(n)$ is not multiplicative. However, Lemma 1 demonstrates that $r_6(n)$ can be written as a sum of two multiplicative functions. The sum in Theorem 1 is split into two sums involving the corresponding multiplicative functions, which are dealt with independently. The two sums are then combined suitably to obtain the result. There are two Dirichlet characters, principal and non-principal, when the modulus is 4. In this case, the Dirichlet character is non-principal and primitive.

2. Preliminaries and Some Important Lemmas

Let $r_k(n) := \#\{(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k: n_1^2 + n_2^2 + \cdots + n_k^2 = n\}$ (allowing zeros, distinguishing signs, and order). We will be concerned with the function $r_6(n)$.

Lemma 1. For any positive integer $n$, we have

$$r_6(n) = 16 \sum_{d|n} \chi(d')d^2 - 4 \sum_{d|n} \chi(d)d^2,$$

where $dd' = n$, and $\chi$ is the non-principal Dirichlet character modulo 4, i.e.,

$$\chi(n) = \begin{cases} 
1 & \text{if } n \equiv 1 \pmod{4} \\
-1 & \text{if } n \equiv -1 \pmod{4} \\
0 & \text{if } n \equiv 0 \pmod{2}
\end{cases}.$$

Proof. From [5, p. 313], we observe that

$$\chi(n) = \begin{cases} 
0 & \text{if } 2|n \\
(-1)^{\frac{n-1}{2}} & \text{if } 2 \nmid n.
\end{cases}$$

It is clear that $\chi(n)$ is the non-principal character modulo 4 and can be defined as stated. Also, we find that Equation (1) follows from [5, p. 415].

We can reframe the Equation (1) as

$$r_6(n) = 16 \sum_{d|n} \chi(d)\frac{n^2}{d^2} - 4 \sum_{d|n} \chi(d)d^2$$

$$=: 16l_1(n) - 4v(n).$$

We write $l_1(n) = 16l(n)$, and $v_1(n) = 4v(n)$. 

The functions \( \chi(d) \) and \( n^2 \frac{d^2}{n^2} \) are completely multiplicative functions. This implies that \( \chi(d) n^2 \frac{d^2}{n^2} \) is multiplicative. If \( g(d) \) is any multiplicative function, then \( \sum_{d|n} g(d) \) is also multiplicative. Therefore, \( l(n) \) is a multiplicative function. Similarly, \( v(n) \) is also multiplicative.

Note that
\[
\begin{align*}
l(p) &= p^2 + \chi(p), \\
l(p^2) &= p^4 + p^2 \chi(p) + \chi(p^2),
\end{align*}
\]
and
\[
\begin{align*}
v(p) &= 1 + p^2 \chi(p), \\
v(p^2) &= 1 + p^2 \chi(p) + p^4 \chi(p^2).
\end{align*}
\]

We can also write
\[
\begin{align*}
\sum_{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 \leq x} \lambda_{sym}^2 f(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2)
&= \sum_{n \leq x} \lambda_{sym}^2 f(n) \sum_{n = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 \leq x} 1 \\
&= \sum_{n \leq x} \lambda_{sym}^2 f(n) r_6(n) \\
&= \sum_{n \leq x} \lambda_{sym}^2 f(n) (l_1(n) - v_1(n)) \\
&= 16 \sum_{n \leq x} \lambda_{sym}^2 f(n) l(n) - 4 \sum_{n \leq x} \lambda_{sym}^2 f(n) v(n),
\end{align*}
\]

where \( l(n) = \sum_{d|n} \chi(d) \frac{n^2}{d^2} \), and \( v(n) = \sum_{d|n} \chi(d) d^2 \).

**Lemma 2 ([3]).** For any \( \epsilon > 0 \), we have
\[
\int_{1}^{T} \left| L \left( \frac{1}{2} + it \right) \right|^2 dt \ll T \log T,
\]
uniformly for \( T \geq 1 \).
Lemma 3. For any $\epsilon > 0$, we have
\[ L(\sigma + it) \ll_{\epsilon} (1 + |t|)^{\frac{1}{2}(1+\epsilon-\sigma)+\epsilon}, \]
uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$, and $|t| \geq t_0$ (where $t_0$ is sufficiently large).

Proof. We get the result by using the maximum-modulus principle in a suitable rectangle. For instance, see [3].

Lemma 4 ([13, Theorem 7.2]). For any $\epsilon > 0$, we have
\[ \int_1^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt \ll T^{1+\epsilon}, \]
uniformly for $T \geq 1$.

Lemma 5. For any $\epsilon > 0$, we have
\[ \zeta(\sigma + it) \ll_{\epsilon} (1 + |t|)^{\frac{1}{2}(1+\epsilon-\sigma)+\epsilon}, \]
uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$, and $|t| \geq t_0$ (where $t_0$ is sufficiently large).

Proof. We get the result when we apply the maximum-modulus principle to $F(w) = \zeta(w)e^{(w-s)x}x^{w-s}$ in a suitable rectangle and by using Hardy’s estimate $\zeta(\frac{1}{2} + it) \ll (|t|+10)^{\frac{1}{2}}$. For instance, see [10].

Lemma 6. Let $L(s, f)$ be a Dirichlet series with Euler product of degree $m \geq 2$, i.e.,
\[ L(s, f) = \prod_{p<\infty} \prod_{i=0}^{m} \left( 1 - \frac{\alpha(p, i)}{p^s} \right)^{-1}, \]
where $\alpha(p, i)$ are local parameters of $L(s, f)$ at prime $p$. If the Euler product converges absolutely for $\Re(s) > 1$, admits a meromorphic continuation to the whole complex plane $\mathbb{C}$, and satisfies a functional equation of Riemann-zeta type, then we have
\[ \int_T^{2T} \left| L \left( \frac{1}{2} + \epsilon + it, f \right) \right|^2 \, dt \ll T^{\frac{m}{2}+\epsilon}, \quad (3) \]
for $T \geq 1$; and for $0 \leq \sigma \leq 1 + \epsilon$, we have
\[ |L(\sigma + it, f)| \ll (1 + |t|)^{\frac{m}{2}(1+\epsilon-\sigma)+\epsilon}. \quad (4) \]

Proof. The proof of Equation (3) is derived in a similar fashion as in [8, Theorem 4.1]. Equation (4) follows by the maximum-modulus principle. 

\[ \Box \]
We know that (for 0 ≤ j ≤ 4 and ℜ(\(s\)) > 1) for a prime \(p\), the \(p^{th}\) Fourier coefficient of the \(j^{th}\) symmetric power \(L\)-function of \(f\) can be written as

\[
\lambda_{\text{sym}^j f}(p) = \sum_{m=0}^{j} \alpha^{j-m}(p) \beta^m(p).
\]

**Lemma 7.** Let \(f\) be a normalized primitive holomorphic cusp form of weight \(k\) for \(SL(2, \mathbb{Z})\). Let \(\lambda_{\text{sym}^2 f}(n)\) be the \(n^{th}\) normalized Fourier coefficient of the symmetric square \(L\)-function associated with \(f\). If

\[
F_2(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^2 f}(n)L(n)}{n^s},
\]

for \(\Re(s) > 3\), then

\[
F_2(s) = G_2(s)H_2(s),
\]

where

\[
G_2(s) := \zeta(s-2)L(s, \chi)L(s-2, \text{sym}^2 f)L(s, \text{sym}^2 f \otimes \chi)
\]

\[
\times L(s-2, \text{sym}^4 f)L(s, \text{sym}^4 f \otimes \chi),
\]

and \(\chi\) is the non-principal character modulo 4. Here, \(H_2(s)\) is a Dirichlet series which converges uniformly and absolutely in the half plane \(\Re(s) > \frac{5}{2}\), and \(H_2(s) \neq 0\) on \(\Re(s) = 3\).

**Proof.** We observe that \(\lambda_{\text{sym}^2 f}(n)l(n)\) is multiplicative, and hence

\[
F_2(s) = \prod_p \left(1 + \frac{\lambda_{\text{sym}^2 f}(p)l(p)}{p^s} + \cdots + \frac{\lambda_{\text{sym}^2 f}(p^m)l(p^m)}{p^{ms}} + \cdots\right).
\]

Note that

\[
\lambda_{\text{sym}^2 f}(p)l(p) = \left(\sum_{m=0}^{2} \alpha^{2-m}(p) \beta^m(p)\right)^2 (p^2 + \chi(p))
\]

\[
= (\alpha^4(p) + 2\alpha^2(p) + 3 + 2\beta^2(p) + \beta^4(p)) (p^2 + \chi(p))
\]

\[
= p^2 + \chi(p) + (\alpha^2(p) + 1 + \beta^2(p))(p^2 + \chi(p))
\]

\[
+ (\alpha^4(p) + \alpha^2(p) + 1 + \beta^2(p) + \beta^4(p)) (p^2 + \chi(p))
\]

\[
= p^2 + \chi(p) + \lambda_{\text{sym}^2 f}(p)(p^2 + \chi(p)) + \lambda_{\text{sym}^2 f}(p)(p^2 + \chi(p))
\]

\[
= p^2 + \chi(p) + p^2 \lambda_{\text{sym}^2 f}(p) + p\lambda_{\text{sym}^2 f}(p)
\]

\[
+ p^2 \lambda_{\text{sym}^2 f}(p) + \chi(p)\lambda_{\text{sym}^2 f}(p)
\]

\[
=: b(p).
\]
From the structure of $b(p)$, we define the coefficients $b(n)$ as

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta(s-2)L(s, \chi)L(s-2, \text{symb}^2 f)L(s, \text{symb}^2 f \otimes \chi)L(s-2, \text{symb}^4 f)$$

$$\times L(s, \text{symb}^4 f \otimes \chi),$$

which is absolutely convergent in $\Re(s) > 3$. We also note that

$$\prod_p \left(1 + \frac{b(p)}{p^s} + \cdots + \frac{b(p^m)}{p^{ms}} + \cdots \right)$$

$$= \zeta(s-2)L(s, \chi)L(s-2, \text{symb}^2 f)L(s, \text{symb}^2 f \otimes \chi)$$

$$\times L(s-2, \text{symb}^4 f)L(s, \text{symb}^4 f \otimes \chi)$$

$$=: G_2(s),$$

for $\Re(s) > 3$. Observe that $b(n) \ll_{\varepsilon} n^{2+\varepsilon}$ for any small positive constant $\varepsilon$.

Now, we note that in the half plane $\Re(s) \geq 3 + 2\varepsilon$, we have

$$\left| \frac{b(p)}{p^s} + \frac{b(p^2)}{p^{2s}} + \cdots + \frac{b(p^m)}{p^{ms}} + \cdots \right| \leq \sum_{m=1}^{\infty} \frac{p^{(2+\varepsilon)m}}{p^{ms}}$$

$$\leq \sum_{m=1}^{\infty} \frac{p^{(2+\varepsilon)m}}{p^{(3+2\varepsilon)m}}$$

$$= \sum_{m=1}^{\infty} \frac{1}{p^{(1+\varepsilon)m}}$$

$$= \frac{1}{p^{1+\varepsilon}} - \frac{1}{p^{1+\varepsilon}}$$

$$= \frac{1}{p^{1+\varepsilon}} - 1$$

$$< 1.$$

Let us write

$$A = \frac{\lambda_{\text{symb}^2 f}(p)l(p)}{p^s} + \cdots + \frac{\lambda_{\text{symb}^2 f}(p^m)l(p^m)}{p^{ms}} + \cdots,$$
and
\[ B = \frac{b(p)}{p^s} + \cdots + \frac{b(p^m)}{p^{ms}} + \cdots. \]

From the above calculations, we observe that \(|B| < 1\) in \(\Re(s) \geq 3 + 2\epsilon\).

We note that in the half plane \(\Re(s) \geq 3 + 2\epsilon\), we have
\[
\frac{1 + A}{1 + B} = (1 + A)(1 - B + B^2 - B^3 + \cdots)
= 1 + A - B - AB + \text{higher terms}
= 1 + \frac{\lambda_{\text{sym}^2 f}(p^2) l(p^2) - b(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots,
\]
with \(c_m(n) \ll n^{2+\epsilon}\). So, we have (in the half plane \(\Re(s) > \frac{5}{2}\))
\[
\prod_p \left( \frac{1 + A}{1 + B} \right) = \prod_p \left( 1 + \frac{\lambda_{\text{sym}^2 f}(p^2) l(p^2) - b(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots \right) \ll \epsilon 1.
\]

Thus, we have (in the half plane \(\Re(s) > \frac{5}{2}\))
\[ H_2(s) := \frac{F_2(s)}{G_2(s)} \]
\[ = \prod_p \left( \frac{1 + A}{1 + B} \right) \ll \epsilon 1, \]
and also \(H_2(s) \neq 0\) on \(\Re(s) = 3\).

\[ \square \]

**Lemma 8.** Let \(f\) be a normalized primitive holomorphic cusp form of weight \(k\) for \(SL(2, \mathbb{Z})\). Let \(\lambda_{\text{sym}^2 f}(n)\) be the \(n^{th}\) normalized Fourier coefficient of the symmetric square \(L\)-function associated with \(f\). If
\[
\tilde{F}_2(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^2 f}(n) v(n)}{n^s},
\]
for \(\Re(s) > 3\), then
\[ \tilde{F}_2(s) = \tilde{G}_2(s) \tilde{H}_2(s), \]
where
\[
\tilde{G}_2(s) := \zeta(s)L(s - 2, \chi)L(s, \text{sym}^2 f)L(s - 2, \text{sym}^2 f \otimes \chi)
\times L(s, \text{sym}^4 f)L(s - 2, \text{sym}^4 f \otimes \chi),
\]
and
and $\chi$ is the non-principal character modulo 4. Here, $\tilde{H}_2(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $\tilde{H}_2(s) \neq 0$ on $\Re(s) = 3$.

**Proof.** We observe that $\lambda^2_{sym^2 f}(n)v(n)$ is multiplicative, and hence

$$\tilde{F}_2(s) = \prod_p \left( 1 + \frac{\lambda^2_{sym^2 f}(p)v(p)}{p^s} + \cdots + \frac{\lambda^2_{sym^2 f}(p^m)v(p^m)}{p^{ms}} + \cdots \right).$$

Note that

$$\lambda^2_{sym^2 f}(p)v(p) = \left( \sum_{m=0}^{2} a^{2-m}(p)\beta^m(p) \right)^2 (1 + p^2\chi(p))$$

$$= (\alpha^4(p) + 2\alpha^2(p) + 3 + 2\beta^2(p) + \beta^4(p))(1 + p^2\chi(p))$$

$$= 1 + p^2\chi(p) + (\alpha^2(p) + 1 + \beta^2(p))(1 + p^2\chi(p))$$

$$+ (\alpha^4(p) + \alpha^2(p) + 1 + \beta^2(p) + \beta^4(p))(1 + p^2\chi(p))$$

$$= 1 + p^2\chi(p) + \lambda_{sym^2 f}(p)(1 + p^2\chi(p)) + \lambda_{sym^4 f}(p)(1 + p^2\chi(p))$$

$$= 1 + p^2\chi(p) + \lambda_{sym^2 f}(p) + p^2\chi(p)\lambda_{sym^2 f}(p)$$

$$+ \lambda_{sym^4 f}(p)(1 + p^2\chi(p))$$

$$=: h(p).$$

From the structure of $h(p)$, we define the coefficients $h(n)$ as

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \zeta(s)L(s - 2, \chi)L(s, sym^2 f)L(s - 2, sym^2 f \otimes \chi)L(s, sym^4 f)$$

$$\times L(s - 2, sym^4 f \otimes \chi),$$

which is absolutely convergent in $\Re(s) > 3$. We also note that

$$\prod_p \left( 1 + \frac{h(p)}{p^s} + \cdots + \frac{h(p^m)}{p^{ms}} + \cdots \right)$$

$$= \zeta(s)L(s - 2, \chi)L(s, sym^2 f)L(s - 2, sym^2 f \otimes \chi)L(s, sym^4 f)L(s - 2, sym^4 f \otimes \chi)$$

$$=: \tilde{G}_2(s),$$

for $\Re(s) > 3$. Observe that $h(n) \ll n^{2+\epsilon}$ for any small positive constant $\epsilon$. Now,
we note that in the half plane $\Re(s) \geq 3 + 2\epsilon$, we have

$$\left| \frac{h(p)}{p^s} + \frac{h(p^2)}{p^{2s}} + \cdots + \frac{h(p^m)}{p^{ms}} + \cdots \right| \ll \sum_{m=1}^{\infty} \frac{p^{(2+\epsilon)m}}{p^{m\sigma}}$$

$$< 1.$$

Let us write

$$\tilde{A} = \frac{\lambda^2_{sym^2}(p)v(p)}{p^s} + \cdots + \frac{\lambda^2_{sym^2}(p^m)v(p^m)}{p^{ms}} + \cdots,$$

and

$$\tilde{B} = \frac{h(p)}{p^s} + \cdots + \frac{h(p^m)}{p^{ms}} + \cdots.$$

From the above calculations, we observe that $|\tilde{B}| < 1$ in $\Re(s) \geq 3 + 2\epsilon$.

We note that in the half plane $\Re(s) \geq 3 + 2\epsilon$, we have

$$\frac{1 + \tilde{A}}{1 + \tilde{B}} = (1 + \tilde{A})(1 + \tilde{B}) - \tilde{A}\tilde{B} + \text{higher terms}$$

$$= 1 + \tilde{A} - \tilde{B} - \tilde{A}\tilde{B} + \text{higher terms}$$

$$= 1 + \frac{\lambda^2_{sym^2}(p^2)v(p^2) - h(p^2)}{p^{2s}} + \cdots + \frac{\tilde{c}_m(p^m)}{p^{ms}} + \cdots,$$

with $\tilde{c}_m(n) \ll n^{2+\epsilon}$. So, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\prod_p \left( \frac{1 + \tilde{A}}{1 + \tilde{B}} \right) = \prod_p \left( 1 + \frac{\lambda^2_{sym^2}(p^2)v(p^2) - h(p^2)}{p^{2s}} + \cdots + \frac{\tilde{c}_m(p^m)}{p^{ms}} + \cdots \right)$$

$$\ll \epsilon 1.$$

Thus, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\widetilde{H}_2(s) := \frac{\widetilde{F}_2(s)}{\widetilde{G}_2(s)}$$

$$= \prod_p \left( \frac{1 + \tilde{A}}{1 + \tilde{B}} \right)$$

$$\ll \epsilon 1,$$

and also $\widetilde{H}_2(s) \neq 0$ on $\Re(s) = 3$. $\square$
3. Proof of Theorem 1

From Equation (2), we can write

$$\sum_{n \leq x} \lambda^2_{\text{sym}^2f}(n)r_6(n) = \sum_{n \leq x} \lambda^2_{\text{sym}^2f}(n)l_1(n) - \sum_{n \leq x} \lambda^2_{\text{sym}^2f}(n)v_1(n).$$

Firstly, we consider the sum $\sum_{n \leq x} \lambda^2_{\text{sym}^2f}(n)l_1(n)$. We begin by applying Perron’s formula (see [4, Chapter 2.4]) to $F_2(s)$ with $\eta = 3 + \epsilon$ and $10 \leq T \leq x$. Thus, we have

$$\sum_{n \leq x} \lambda^2_{\text{sym}^2f}(n)l_1(n) = 16 \sum_{n \leq x} \lambda^2_{\text{sym}^2f}(n)l(n)$$

$$= \frac{16}{2\pi i} \int_{\eta-iT}^{\eta+iT} F_2(s) \frac{x^s}{s} ds + O \left( \frac{x^{3+3\epsilon}}{T} \right).$$

We move the line of integration to $\Re(s) = \frac{5}{2} + \epsilon$. By Cauchy’s residue theorem there is only one simple pole at $s = 3$, coming from the factor $\zeta(s-2)$. This contributes a residue, which is $cx^3$, where $c$ is an effective constant depending on the values of various $L$-functions appearing in $G_2(s)$ at $s = 3$.

More precisely,

$$c = 16 \lim_{s \to 3} (s-3) \frac{F_2(s)}{s}$$

$$= \frac{16}{3} L(3, \chi)L(1, \text{sym}^2 f)L(3, \text{sym}^2 f \otimes \chi)$$

$$\times L(1, \text{sym}^4 f)L(3, \text{sym}^4 f \otimes \chi)H_2(3).$$

So, we obtain

$$\sum_{n \leq x} \lambda^2_{\text{sym}^2f}(n)l_1(n) = cx^3 + \frac{16}{2\pi i} \left\{ \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} + \int_{\frac{3}{2}+\epsilon-iT}^{\frac{3}{2}+\epsilon+iT} \right\} F_2(s) \frac{x^s}{s} ds$$

$$+ O \left( \frac{x^{3+3\epsilon}}{T} \right)$$

$$=: cx^3 + \frac{16}{2\pi i} (J_1 + J_2 + J_3) + O \left( \frac{x^{3+3\epsilon}}{T} \right).$$

The contribution of the horizontal line integrals (using Lemma 5 and Lemma 6)
is
\[
J_2 + J_3 \ll \int_{\frac{1}{2} + \epsilon}^{3 + \epsilon} \left| \zeta(\sigma - 2 + iT) L(\sigma - 2 + iT, \text{sym}^2 f) L(\sigma - 2 + iT, \text{sym}^4 f) \right| x^\sigma d\sigma
\]
\[
\ll \int_{\frac{1}{2} + \epsilon}^{1 + \epsilon} \left| \zeta(\sigma + iT) L(\sigma + iT, \text{sym}^2 f) L(\sigma + iT, \text{sym}^4 f) \right| x^{\sigma + 2} d\sigma
\]
\[
\ll \left( \frac{x^2}{T} \right) \max_{\frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon} x^\sigma T^{\frac{1}{2}(1 + \epsilon - \sigma)} T^{\frac{1}{2}(1 + \epsilon - \sigma)}
\]
\[
\ll \left( \frac{x^{2+2\epsilon}}{T} \right) \max_{\frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon} \left( \frac{x}{T^{\frac{1}{2}}} \right)^\sigma T^{\frac{1}{2}}
\]
Clearly, \( \left( \frac{x}{T^{\frac{1}{2}}} \right)^\sigma \) is monotonic as a function of \( \sigma \) for \( \frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon \), and hence the maximum is attained at the extremities of the interval \( \left[ \frac{1}{2} + \epsilon, 1 + \epsilon \right] \). Thus,
\[
J_2 + J_3 \ll x^{2+2\epsilon} \left( x^{\frac{1}{2} + \epsilon} T^{\frac{1}{2}(1 + \epsilon - \sigma)} + \frac{x^{1+\epsilon}}{T} \right)
\]
\[
\ll x^{\frac{5}{2} + 3\epsilon} T^\epsilon + \frac{x^{3+3\epsilon}}{T}.
\]

The contribution of the left vertical line integral (using the Cauchy-Schwarz inequality, Lemma 4, and Lemma 6) is
\[
J_1 \ll \int_{\frac{1}{2} + \epsilon - iT}^{\frac{1}{2} + \epsilon + iT} \left| \zeta(\frac{1}{2} + \epsilon + it) L(\frac{1}{2} + \epsilon + it, \text{sym}^2 f) L(\frac{1}{2} + \epsilon + it, \text{sym}^4 f) \right| x^{2+\epsilon} dt
\]
\[
\ll x^{\frac{5}{2} + \epsilon} + x^{\frac{5}{2} + \epsilon} \left( \int_{10 \leq |t| \leq T} \frac{|\zeta(\frac{1}{2} + \epsilon + it)|^2}{t^2} dt \right)^\frac{1}{2}
\]
\[
\times \left( \int_{10 \leq |t| \leq T} \frac{|L(\frac{1}{2} + \epsilon + it, \text{sym}^2 f) L(\frac{1}{2} + \epsilon + it, \text{sym}^4 f)|^2}{t^2} dt \right)^\frac{1}{2}
\]
\[
\ll x^{\frac{5}{2} + \epsilon} + x^{\frac{5}{2} + \epsilon} \left( T^\epsilon T^\frac{1}{2} T^{\frac{3}{2}} \right)
\]
\[
\ll x^{\frac{5}{2} + 2\epsilon} T^\frac{1}{2} + 2\epsilon.
\]
Note that $10 \leq T \leq x$. Thus, we obtain
\[
\sum_{n \leq x} \lambda_{2 \text{sym}f}(n) l_1(n) = cx^3 + O(x^{\frac{3}{2} + 2\epsilon} T^{\frac{3}{2} + 2\epsilon}) + O\left(\frac{x^{3+3\epsilon}}{T}\right).
\]

We choose $T$ such that $x^{\frac{3}{2}} T^\frac{3}{2} \approx x$ i.e., $T^\frac{3}{2} \approx x^\frac{1}{2}$. Therefore, $T \approx x^\frac{1}{2}$. Thus, we get
\[
\sum_{n \leq x} \lambda_{2 \text{sym}f}(n) l_1(n) = cx^3 + O(x^{\frac{44}{3}+\epsilon}).
\] (5)

Similarly, we apply Perron’s formula (see [4, Chapter 2.4]) to $\tilde{F}_2(s)$ with $\eta = 3 + \epsilon$ and $10 \leq T \leq x$. Thus, we have
\[
\sum_{n \leq x} \lambda_{2 \text{sym}f}(n) v_1(n) = 4 \sum_{n \leq x} \lambda_{2 \text{sym}f}(n) v(n)
= \frac{4}{2\pi i} \int_{\eta-iT}^{\eta+iT} \tilde{F}_2(s) \frac{x^s}{s} ds + O\left(\frac{x^{3+3\epsilon}}{T}\right).
\]

We move the line of integration to $\Re(s) = \frac{5}{2} + \epsilon$. There is no singularity in the rectangle obtained and the function $\tilde{F}_2(s) x^s$ is analytic in this region. Thus, using Cauchy’s theorem for rectangles pertaining to analytic functions, we get
\[
\sum_{n \leq x} \lambda_{2 \text{sym}f}(n) v_1(n) = \frac{4}{2\pi i} \left\{ \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} \tilde{F}_2(s) \frac{x^s}{s} ds + \int_{3+\epsilon-iT}^{3+\epsilon+iT} \tilde{F}_2(s) \frac{x^s}{s} ds \right\} + O\left(\frac{x^{3+3\epsilon}}{T}\right)
= \frac{4}{2\pi i} (J'_1 + J'_2 + J'_3) + O\left(\frac{x^{3+3\epsilon}}{T}\right).
\]

The contribution of the horizontal line integrals (using Lemma 3 and Lemma 6) is
\[
J'_2 + J'_3 \ll \int_{\frac{5}{2}+\epsilon}^{3+\epsilon} \left| L(\sigma - 2 + iT, \chi)L(\sigma - 2 + iT, \text{sym}^2 f \otimes \chi) L(\sigma - 2 + iT, \text{sym}^3 f \otimes \chi) \right| \frac{x^\sigma}{\sigma} d\sigma
\ll \int_{\frac{1}{2}+\epsilon}^{1+\epsilon} \left| L(\sigma + iT, \chi)L(\sigma + iT, \text{sym}^2 f \otimes \chi) L(\sigma + iT, \text{sym}^3 f \otimes \chi) \right| \frac{x^{\sigma + 2}}{\sigma} d\sigma.
\]
Thus,

\[ J'_2 + J'_3 \ll \left( \frac{x^2}{T} \right) \max_{\frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon} x^{\sigma} T^{\frac{1}{2}(1+\epsilon-\sigma)} T^{\frac{1}{2}(1+\epsilon-\sigma)} \]

\[ \ll \left( \frac{x^{2+2\epsilon}}{T} \right) \max_{\frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon} \left( \frac{x}{T^{1/2}} \right)^{\sigma} T^{13/2}. \]

Clearly, \( \left( \frac{x}{T^{1/2}} \right)^{\sigma} \) is monotonic as a function of \( \sigma \) for \( \frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon \), and hence the maximum is attained at the extremities of the interval \( \left[ \frac{1}{2} + \epsilon, 1 + \epsilon \right] \). Thus,

\[ J'_2 + J'_3 \ll x^{2+2\epsilon} \left( x^{1+\epsilon} T^{\frac{13}{2}-1+\epsilon} + \frac{x^{1+\epsilon}}{T} \right) \]

\[ \ll x^{2+3\epsilon} T^{2+\epsilon} + \frac{x^{3+3\epsilon}}{T}. \]

The contribution of the left vertical line integral (using the Cauchy-Schwarz inequality, Lemma 2, and Lemma 6) is

\[ J'_1 \ll \int_{\frac{1}{2} + \epsilon - i T}^{\frac{1}{2} + \epsilon + i T} \left| L\left( \frac{1}{2} + \epsilon + it, \chi \right) L\left( \frac{1}{2} + \epsilon + it, \text{sym}^2 f \otimes \chi \right) \right|^2 \frac{dt}{\left| \frac{1}{2} + \epsilon + it \right|} \]

\[ \ll \left( \int_{10 \leq |t| \leq T} \left| L\left( \frac{1}{2} + \epsilon + it, \chi \right) \right|^2 \frac{dt}{t} \right)^{1/2} \times \left( \int_{10 \leq |t| \leq T} \left| L\left( \frac{1}{2} + \epsilon + it, \text{sym}^2 f \otimes \chi \right) \right|^2 \frac{dt}{t} \right)^{1/2} \]

\[ \ll x^{2+\epsilon} + x^{2+\epsilon} \left( T^2 T^{2+\frac{1}{2}} \right) \]

\[ \ll x^{2+2\epsilon} T^{3+2\epsilon}. \]

Note that \( 10 \leq T \leq x \). Thus, we obtain

\[ \sum_{n \leq x} \lambda_{\text{sym}^2 f}^2(n) v_1(n) = O(x^{2+2\epsilon} T^{3+2\epsilon}) + O \left( \frac{x^{3+3\epsilon}}{T} \right). \]

We choose \( T \) such that \( x^{3/2} \ll \frac{x^3}{T} \) i.e., \( T^{2} \ll x^{3/2} \). Therefore, \( T \ll x^{2/3} \).

Thus, we get

\[ \sum_{n \leq x} \lambda_{\text{sym}^2 f}^2(n) v_1(n) = O(x^{14/3+\epsilon}). \]
Combining Equations (5) and (6), we get
\[ \sum_{n \leq x} \lambda_{sym^2 f}^2(n) r_6(n) = cx^3 + O(x^{14+\epsilon}), \]
where \( c \) is an effective constant given by
\[ c = \frac{16}{3} L(3, \chi)L(1, sym^2 f)L(3, sym^2 f \otimes \chi)L(1, sym^4 f)L(3, sym^4 f \otimes \chi)H_2(3), \]
and \( \chi \) is the non-principal Dirichlet character modulo 4. This proves the theorem.

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References


