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ON THE COMPUTATION OF FUNDAMENTAL PERIODS OF v-PALINDROMIC NUMBERS

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Abstract

A v-palindrome, recently introduced by the author, is a natural number n, neither a multiple of 10 nor a decimal palindrome, such that the sum of the prime factors of n and their corresponding exponents larger than 1 is equal to that of the number formed by reversing the decimal digits of n. If $n(k)$ denotes the k-copy repeated concatenation of the decimal digits of a natural number n, then whether $n(k)$ is a v-palindrome depends only on k modulo a natural number ω. The indicator function of n is defined to be the periodic function $I^n\colon \mathbb{Z} \to \{0,1\}$ such that for $k \geq 1$, $I^n(k) = 1$ if $n(k)$ is a v-palindrome and $I^n(k) = 0$ if not. We give a procedure to express I^n as a kind of linear combination with some "subscripts", which can be implemented on a computer. We then show that the smallest possible ω, the fundamental period, is simply the least common multiple of those "subscripts".

1. Introduction

In [5], natural numbers satisfying an unusual property are defined and their infinitude proved. Consider the natural number 56056. The number formed by reversing its decimal digits is 65065. Their prime factorizations are

$$56056 = 2^3 \cdot 7^2 \cdot 11 \cdot 13, \tag{1}$$

$$65065 = 5 \cdot 7 \cdot 11 \cdot 13^2. \tag{2}$$

Notice that

$$(2+3) + (7+2) + 11 + 13 = 5 + 7 + 11 + (13+2),$$

which is a bit surprising. In other words, the sum of the prime factors and exponents larger than 1 on the right-hand side of Equation (1) is equal to that of Equation (2). Such numbers are called v-palindromic numbers in [4], of which we shall give a formal definition.

Definition 1. If n is a natural number, then its *reverse* is the number formed by reversing its decimal digits, denoted $r(n)$. Therefore $r(n)$ has the same number of digits as n if $10 \nmid n$ and fewer digits if $10 \mid n$.

Definition 2. For a natural number $n > 1$, its *factorization sum* is the sum of the prime factors and exponents larger than 1 in its prime factorization, denoted $v(n)$. Also, by convention, $v(1) = 1$.

The notation $v(n)$ is the one used in [5, 4], so we continue to use it here. The "v" actually came from "value". The quantity $v(n)$ is thought of as the "value" of n. We obviously have the following.

Theorem 1. *The function $v\colon \mathbb{N} \to \mathbb{N} \cup \{0\}$ is additive. That is, $v(mn) = v(m) + v(n)$ whenever m and n are coprime natural numbers.*

Definition 3. A natural number n is *v-palindromic* (or a *v-palindrome*) if n is not a multiple of 10, $n \neq r(n)$ (i.e., n is not palindromic), and $v(n) = v(r(n))$.

We explain the choice of the name *v-palindrome*. A natural number n is a *palindrome* if $n = r(n)$. The relation $v(n) = v(r(n))$ differs by having a v in front. The condition that n is not a multiple of 10 is for ensuring that $r(n)$ does not have fewer digits than n. If we do not impose this condition, then 560 would be v-palindromic. However, we insist on imposing that $10 \nmid n$ and therefore do not consider 560 as v-palindromic. The condition $n \neq r(n)$ is included because if $n = r(n)$, then obviously $v(n) = v(r(n))$, which would not be surprising at all. Alternatively, we can discard $n \neq r(n)$ in the definition of v-palindromes and regard the palindromes as *trivial* v-palindromes. We do not adopt this alternative viewpoint though.

The smallest v-palindrome is 18, because $18 = 2 \cdot 3^2$, $81 = 3^4$, and both factorization sums are 7. The first natural question is then whether there are infinitely many of them, and the answer is affirmative. As proved in [5], all the numbers

$$18, 198, 1998, 19998, \ldots$$

are v-palindromes. Also mentioned in [5] is another sequence of v-palindromes

$$18, 1818, 181818, \ldots, \tag{3}$$

where we simply continue to concatenate 18. It was this sequence which inspired the content of [4], which investigated which of the repeated concatenations of a number are v-palindromes and found a periodic phenomenon. It then posed three problems of further investigation pertaining to this periodic phenomenon. This paper addresses the first two of these problems. In the next section, we shall recall the main theorem in [4].

2. Repeated Concatenations and v-Palindromicity

We first give the following notation.

Definition 4. If n is a natural number, then the number formed by repeatedly concatenating its decimal digits k times is denoted by $n(k)$.

For example, $18(3) = 181818$ and $56056(4) = 56056560565605656056$. The main theorem in [4] can now be stated.

Theorem 2 ([4, Theorem 1])**.** *Let n be a natural number such that $10 \nmid n$ and $n \neq r(n)$. There exists an integer $\omega > 0$ such that for all integers $k \geq 1$, $n(k)$ is v-palindromic if and only if $n(k+\omega)$ is v-palindromic. In other words, whether $n(k)$ is v-palindromic depends only on k modulo ω.*

Definition 5. For n as in Theorem 2, a permissible ω will be called a *period* of n. The smallest period of n will be called the *fundamental period* of n, denoted $\omega_0(n)$.

Regarding periods and the fundamental period, we have the following, which follows from [1, Exercise 17(a) on p. 145].

Theorem 3 ([4, Theorem 2])**.** *Let n be as in Theorem 2. Then the set of all periods of n is the set of all positive integral multiples of $\omega_0(n)$.*

For example, since all the numbers in the sequence (3) are v-palindromes, we have $\omega_0(18) = 1$. In fact, $\omega_0(56056) = 1$ too. In [4, Theorem 1], the author give a constructive proof and find a particular period. In order to state this period, we need to define certain numbers which are introduced in [4, Lemma 1].

Definition 6. Let p^α be a prime power, where $p \neq 2, 5$, and let d be a natural number. Denote by $h_{p^\alpha,d}$ the order of 10^d regarded as an element of the group $(\mathbb{Z}/p^{\alpha+\mathrm{ord}_p(10^d-1)}\mathbb{Z})^\times$. (Here, $\mathrm{ord}_p(a)$ denotes the exponent of the prime p in the prime factorization of a.) In other words, $h_{p^\alpha,d}$ is the smallest positive integer such that

$$(10^d)^{h_{p^\alpha,d}} \equiv 1 \pmod{p^{\alpha+\mathrm{ord}_p(10^d-1)}}.$$

By [4, Lemma 1], $h_{p^\alpha,d} > 1$.

Definition 7. Let n be as in Theorem 2. A *crucial prime* of n is a prime p for which $\mathrm{ord}_p(n) \neq \mathrm{ord}_p(r(n))$. The set of all crucial primes of n will be denoted $K(n)$.

The constructed period of n in [4] is the following.

Theorem 4 ([4])**.** *Let n be as in Theorem 2 and let d denote the number of decimal digits that n has. Then*

$$\omega_f(n) = \mathrm{lcm}\{h_{p^2,d} \mid p \in K(n) \setminus \{2, 5\}\} \tag{4}$$

is a period of n.

Equation (4) is originally written as

$$\omega_f(n) = \operatorname{lcm}\{h_{p,d}, h_{p^2,d} \mid p \in K(n) \setminus \{2,5\}\} \tag{5}$$

in [4]. However, in fact we always have $h_{p,d} \mid h_{p^2,d}$, and thus Equation (5) can be written more succinctly as Equation (4). We show that in fact $h_{p,d} \mid h_{p^2,d}$. Since

$$(10^d)^{h_{p^2,d}} \equiv 1 \pmod{p^{2+\operatorname{ord}_p(10^d-1)}},$$

obviously

$$(10^d)^{h_{p^2,d}} \equiv 1 \pmod{p^{1+\operatorname{ord}_p(10^d-1)}}.$$

Now $h_{p,d}$ is the order of 10^d regarded as an element of $(\mathbb{Z}/p^{1+\operatorname{ord}_p(10^d-1)}\mathbb{Z})^\times$, thus $h_{p,d} \mid h_{p^2,d}$ follows from the structure of cyclic groups.

After calculating $\omega_0(n)$ and $\omega_f(n)$ for small n, the following is conjectured in [4].

Conjecture 1. Let n be as in Theorem 2. Then either $\omega_0(n) = 1$ or $\omega_0(n) = \omega_f(n)$.

One purpose of this paper is to provide a counterexample to Conjecture 1, thereby disproving it.

Another issue raised in [4] is whether given an n, there exists a repeated concatenation of n which is a v-palindrome. For $n = 12$, no such repeated concatenation exists, i.e., all the numbers

$$12, 1212, 121212, \ldots$$

are not v-palindromic. However, for $n = 13$, the first fourteen repeated concatenations are not v-palindromic, but the fifteenth is. That is, $13(k)$ is not v-palindromic for $1 \leq k \leq 14$, but $13(15)$ is v-palindromic. Based on this phenomenon, the following definition is also given in [4].

Definition 8. Let n be as in Theorem 2. If there exists an integer $k \geq 1$ such that $n(k)$ is v-palindromic, the least such integer will be called the *order* of n and denoted $c(n)$. If no such k exists, then we write $c(n) = \infty$.

In [4], the following question is posed for further consideration: is there a simple way to determine, for a given n, whether or not $c(n) = \infty$? In this paper, we provide a general procedure, starting with a given n as in Theorem 2, i.e., n is a natural number, not a multiple of 10, and not a palindrome. This procedure will determine whether $c(n) = \infty$, and if not, determine both $\omega_0(n)$ and the precise conditions on $k \geq 1$ such that $n(k)$ is v-palindromic. This procedure is mostly a realization of [4, proof of Theorem 1] into a more algorithmic nature.

For any given n as in Theorem 2, we shall construct in Section 5 a function $I^n : \mathbb{Z} \to \{0, 1\}$, which for positive integers k, evaluates to 1 if $n(k)$ is a v-palindrome and evaluates to 0 otherwise. The superscript n is only for specifying n and does not denote composition of functions. Then, both $c(n)$ and $\omega_0(n)$ can be directly "read off" from I^n when it is expressed in a certain form. An important part of this paper is the proof of Theorem 15 using [6, Theorem 12] (Theorem 14 in this paper). As a corollary of Theorem 15, $\omega_0(n)$ can be easily found from I^n (Corollary 4).

We provide an appendix on the more general topic of periodic arithmetical functions. There, a formula for the fundamental period of an arbitrary periodic arithmetical function from $\mathbb{Z}$ to $\mathbb{C}$ is given (Theorem 20). This formula is actually equivalent to the formula given in [6, Theorem 9]. We prove their equivalence in Section 11.2. Although [6] contains a proof of Theorem 14, in Section 11.3 we provide a proof using Theorem 20. In this way, our paper is more self-contained.

In the field of signal processing, an arithmetical function from $\mathbb{Z}$ to $\mathbb{C}$ is called a *discrete signal* or *discrete-time signal* (see [3, 6, 8]). We feel that it is better to include the appendix because our presentation differs from that in the signal processing context and might be interesting in its own right, and perhaps in a language more familiar to number theorists.

3. The Functions $\varphi_{p,\delta}$

In this section we define certain functions which will be used later.

Definition 9. For a prime p and integer $\delta \geq 2$, define the function

$$\varphi_{p,\delta}(\alpha) = \begin{cases} p+\delta & \text{if } \alpha = 0, \\ 1+\delta & \text{if } \alpha = 1, \\ \delta & \text{if } \alpha \geq 2. \end{cases} \tag{6}$$

For a prime $p \neq 2$, define the function

$$\varphi_{p,1}(\alpha) = \begin{cases} p & \text{if } \alpha = 0, \\ 2 & \text{if } \alpha = 1, \\ 1 & \text{if } \alpha \geq 2. \end{cases} \tag{7}$$

Finally, define

$$\varphi_{2,1}(\alpha) = \begin{cases} 2 & \text{if } \alpha = 0,1, \\ 1 & \text{if } \alpha \geq 2. \end{cases} \tag{8}$$

Hence we have, for any pair (p, δ) of a prime p and natural number δ, defined a function $\varphi_{p,\delta} \colon \mathbb{N} \cup \{0\} \to \mathbb{N}$. We give notation for their ranges as follows.

Definition 10. For p a prime and δ a natural number, define $R_{p,\delta} = \varphi_{p,\delta}(\mathbb{N} \cup \{0\})$.

Hence $|R_{p,\delta}| \in \{2, 3\}$, and equals 2 if and only if $(p, \delta) = (2, 1)$. We have the following lemma.

Lemma 1. *For an ordered quadruple (p, δ, u, μ), where p is a prime, δ a natural number, $u \in R_{p,\delta}$, and $\mu \geq 0$ an integer, exactly one of the following is the case.*

[i] $\varphi_{p,\delta}^{-1}(u) = \{0\}$ *and* $\mu = 0$*, or* $\varphi_{p,\delta}^{-1}(u) = \{1\}$ *and* $\mu = 1$*, or* $\varphi_{p,\delta}^{-1}(u) = \{0,1\}$ *and* $\mu = 1$*,*

[ii] $\varphi_{p,\delta}^{-1}(u) = \{1\}$ *and* $\mu = 0$*,*

[iii] $\varphi_{p,\delta}^{-1}(u) = \{0,1\}$ *and* $\mu = 0$*,*

[iv] $\varphi_{p,\delta}^{-1}(u) = \mathbb{N} \setminus \{1\}$ *and* $\mu = 1$*,*

[v] $\varphi_{p,\delta}^{-1}(u) = \mathbb{N} \setminus \{1\}$ *and* $\mu = 0$*,*

[vi] $\varphi_{p,\delta}^{-1}(u) = \mathbb{N} \setminus \{1\}$ *and* $\mu \geq 2$*,*

[vii] *otherwise.*

Proof. In view of Equations (6), (7), and (8), $\varphi_{p,\delta}^{-1}(u)$ is one of $\{0\}$, $\{1\}$, $\{0,1\}$, and $\mathbb{N} \setminus \{1\}$. Then we see that the first six cases are mutually exclusive. That each case is possible is obvious. □

Definition 11. For each quadruple (p, δ, u, μ) as in Lemma 1, denote by $D(p, \delta, u, \mu)$ the case number (in lower case Roman numerals in brackets as in the lemma). That is, $D(p, \delta, u, \mu) =$ [ii] if and only if $\varphi_{p,\delta}^{-1}(u) = \{1\}$ and $\mu = 0$, $D(p, \delta, u, \mu) =$ [iii] if and only if $\varphi_{p,\delta}^{-1}(u) = \{0,1\}$ and $\mu = 0$, etc.

4. The Conditions When a Repeated Concatenation is a v-Palindrome

Throughout this section we let n be a fixed natural number as in Theorem 2, i.e., n is not a multiple of 10 and is not a palindrome. We find the precise conditions when the number formed by repeatedly concatenating k times the decimal digits of n, i.e., $n(k)$, is a v-palindrome. Almost no proofs are given because they all follow from [4].

Suppose that n and $r(n)$ have the prime factorizations

$$n = \prod_p p^{a_p},$$
$$r(n) = \prod_p p^{b_p},$$

where the products are over the primes, the $a_p, b_p \geq 0$ are integers, and $a_p = b_p = 0$ for all but finitely many primes p. Let the number of decimal digits of n be denoted d. We give the following notation.

Definition 12. For $k \geq 1$, put

$$\rho_k = \overbrace{1\underbrace{0\ldots0}_{d-1}1\underbrace{0\ldots0}_{d-1}1\ldots1\underbrace{0\ldots0}_{d-1}1}^{k}.$$

That is, we have k ones and in between any two consecutive ones, $d-1$ zeros.

We consider n as fixed and therefore d is also fixed. The k is considered as a variable and denotes the number of times we repeatedly concatenate the digits of n. We shall describe the necessary and sufficient condition on k such that $n(k)$ is a v-palindrome. Because of the way ρ_k is defined, we obviously have the following.

Lemma 2. *For every $k \geq 1$, we have $n(k) = n\rho_k$.*

We shall determine a complete set of mutually exclusive conditions on k such that $n(k)$ is a v-palindrome if and only if k satisfies one of those conditions.

For each crucial prime p of n (Definition 7), put

$$\begin{aligned}
\delta_p &= a_p - b_p \neq 0,\\
\mu_p &= \min(a_p, b_p) \geq 0,\\
g_p &= \operatorname{ord}_p(\rho_k),\\
\alpha_p &= \mu_p + g_p,
\end{aligned}$$

where $\delta_p \neq 0$ by the definition of crucial prime. The δ_p and μ_p depend only on n, thus can be considered as fixed. The g_p clearly depend on not only p but also on k. However, we omit k from the notation for simplicity, keeping in mind that g_p depends also on the variable k. Consequently, α_p also depends on k. In describing the conditions for which k must satisfy for $n(k)$ to be a v-palindrome, α_p would occur. We shall simply denote the set of crucial primes of n as K, which is a nonempty finite set of prime numbers.

Definition 13. The equation

$$\sum_{p\in K} \operatorname{sgn}(\delta_p) u_p = 0, \tag{9}$$

where sgn is the sign function with $\operatorname{sgn}(\delta_p) = 1$ if $\delta_p > 0$ and $\operatorname{sgn}(\delta_p) = -1$ if $\delta_p < 0$, will be called the *characteristic equation* for n, where the u_p are variables.

We want to solve Equation (9) for the u_p but with certain restrictions.

Definition 14. A solution $(u_p)_{p\in K}$ to Equation (9) with $u_p \in R_{p,|\delta_p|}$ for all $p \in K$ will be called a *characteristic solution* for n. The set of all characteristic solutions will be denoted by U.

Since the number of digits of n is denoted d, we have the numbers $h_{q,d}$ defined in Definition 6 for every prime power q coprime to 10. We shall omit the d and simply write h_q. Let $\mathbf{u} = (u_p)_{p\in K}$ be a characteristic solution for n. We denote, for $p \in K \setminus \{2,5\}$,

$$T_{p,\mathbf{u}} = (A_{p,\mathbf{u}}, B_{p,\mathbf{u}}) = \begin{cases} (\varnothing, \{h_p\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{i}], \\ (\{h_p\}, \{h_{p^2}\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{ii}], \\ (\varnothing, \{h_{p^2}\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{iii}], \\ (\{h_p\}, \varnothing) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{iv}], \\ (\{h_{p^2}\}, \varnothing) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{v}]. \end{cases} \tag{10}$$

For $p \in \{2,5\}$, denote

$$T_{p,\mathbf{u}} = (A_{p,\mathbf{u}}, B_{p,\mathbf{u}}) = \begin{cases} (\varnothing, \varnothing) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{i}], \\ (\varnothing, \{1\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{ii}], \\ (\varnothing, \varnothing) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{iii}], \\ (\varnothing, \{1\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{iv}], \\ (\varnothing, \{1\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{v}]. \end{cases} \tag{11}$$

Also, we denote, for any $p \in K$,

$$T_{p,\mathbf{u}} = (A_{p,\mathbf{u}}, B_{p,\mathbf{u}}) = \begin{cases} (\varnothing, \varnothing) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{vi}], \\ (\varnothing, \{1\}) & \text{if } D(p, |\delta_p|, u_p, \mu_p) = [\text{vii}]. \end{cases} \tag{12}$$

Therefore $T_{p,\mathbf{u}}$ is an ordered pair of sets of at most one positive integer. The "T" comes from "table", because in practice these ordered pairs are arranged into the form of a table of p versus $\mathbf{u}$, with entries $T_{p,\mathbf{u}}$. We give the following general notation.

Definition 15. Let A and B be finite sets of positive integers. Then denote

$$S(A,B) = \{x \in \mathbb{Z} \mid (\text{for all } a \in A, a \mid x) \text{ and } (\text{for all } b \in B, b \nmid x)\}.$$

That is, $S(A,B)$ is the set of all integers divisible by every element of A, but indivisible by every element of B.

Definition 16. For each characteristic solution $\mathbf{u}$ for n, put

$$A_{\mathbf{u}} = \bigcup_{p\in K} A_{p,\mathbf{u}}, \quad B_{\mathbf{u}} = \bigcup_{p\in K} B_{p,\mathbf{u}},$$

and $S_{\mathbf{u}} = S(A_{\mathbf{u}}, B_{\mathbf{u}})$.

The first sentence in the following theorem is [4, Lemma 4], and the second sentence follows from [4, arguments following Lemma 4].

Theorem 5 ([4])**.** *For $k \geq 1$, the number $n(k)$ is a v-palindrome if and only if for some characteristic solution $\mathbf{u} = (u_p)_{p \in K}$ for n,*

$$\varphi_{p,|\delta_p|}(\alpha_p) = u_p \quad \textit{for all } p \in K. \tag{13}$$

Moreover, given a characteristic solution $\mathbf{u} = (u_p)_{p \in K}$ for n, Equations (13) *hold if and only if*

$$k \in S_{\mathbf{u}}. \tag{14}$$

The condition that Equations (13) hold might seem to be independent of k, but recall that the α_p actually depend on k. To write Equations (13) out so that the dependence on k is more visible, we can recover Equations (13) into

$$\varphi_{p,|\delta_p|}(\mu_p + \mathrm{ord}_p(\rho_k)) = u_p \quad \text{for all } p \in K. \tag{15}$$

Since the condition that Equations (13) (or equivalently, Equations (15)) hold cannot be true, for the same k, for two distinct characteristic solutions, the conditions that Equations (13) hold are mutually exclusive over $\mathbf{u}$. Consequently, the conditions (14) are also mutually exclusive over $\mathbf{u}$. Therefore the sets $S_{\mathbf{u}} \cap \mathbb{N}$ are pairwise disjoint and we write it into a corollary as follows.

Corollary 1. *The sets $S_{\mathbf{u}} \cap \mathbb{N}$ are pairwise disjoint over $\mathbf{u} \in U$.*

In fact, we have the following which says that not only are the intersections $S_{\mathbf{u}} \cap \mathbb{N}$ of the sets $S_{\mathbf{u}}$ with $\mathbb{N}$ pairwise disjoint, but the sets $S_{\mathbf{u}}$ themselves are already pairwise disjoint as subsets of $\mathbb{Z}$.

Theorem 6. *The sets $S_{\mathbf{u}}$ are pairwise disjoint over $\mathbf{u} \in U$.*

Proof. Suppose on the contrary that for some distinct $\mathbf{u}, \mathbf{v} \in U$ that there exists an integer

$$x \in S_{\mathbf{u}} \cap S_{\mathbf{v}} = S(A_{\mathbf{u}}, B_{\mathbf{u}}) \cap S(A_{\mathbf{v}}, B_{\mathbf{v}}).$$

If we let

$$\omega = \mathrm{lcm}(A_{\mathbf{u}} \cup B_{\mathbf{u}} \cup A_{\mathbf{v}} \cup B_{\mathbf{v}}),$$

then we see that $x + \omega \in S_{\mathbf{u}} \cap S_{\mathbf{v}}$ too. Therefore adding ω as many times as necessary to x, we obtain a natural number in $S_{\mathbf{u}} \cap S_{\mathbf{v}}$. This contradicts Corollary 1. □

Corollary 2. *The set of all $k \geq 1$ such that $n(k)$ is a v-palindrome is*

$$\bigsqcup_{\mathbf{u} \in U} (S_{\mathbf{u}} \cap \mathbb{N}) = \left(\bigsqcup_{\mathbf{u} \in U} S_{\mathbf{u}} \right) \cap \mathbb{N}.$$

Proof. This follows directly from Theorem 5, Corollary 1, and Theorem 6. □

Thus k can be categorized as to which $S_{\mathbf{u}}$ it belongs to. However, it could happen that $S_{\mathbf{u}} = \varnothing$, therefore we give the following definition.

Definition 17. If $S_{\mathbf{u}}$ is empty, then we call $\mathbf{u}$ a *degenerate* characteristic solution for n; otherwise it is *nondegenerate*. The set of all nondegenerate characteristic solutions will be denoted by U^*. For an $\mathbf{u} \in U^*$, an $n(k)$ which is a v-palindrome will be said to be of *type* $\mathbf{u}$ (with respect to n) if $k \in S_{\mathbf{u}}$. We also denote

$$S = \bigsqcup_{\mathbf{u} \in U^*} S_{\mathbf{u}}. \tag{16}$$

We have included "with respect to n" in our definition of type above because the same v-palindrome m might be $m = n_1(k_1) = n_2(k_2)$ for $n_1 \neq n_2$, and therefore the type of m can be considered with respect to n_1 as well as with respect to n_2. Whether the notion of type defined above is really dependent on n is still unclear. For instance, if we consider the v-palindrome $m = 13(15)$, then $m = 13(3)(5) = 13(5)(3) = 13(15)(1)$ also. Perhaps a bit surprisingly, in all four cases the type of m is $(2, 2)$, or more precisely, the $(u_p)_{p \in \{13,31\}}$ with $u_{13} = u_{31} = 2$. Thus we propose the following problem.

Problem 1. Let m be a v-palindrome such that $m = n_1(k_1) = n_2(k_2)$. Then is it necessarily true that the type of m with respect to n_1 be the same as the type of m with respect to n_2?

We shall omit saying "with respect to n" hereafter, it being understood implicitly, though if the answer to Problem 1 is affirmative, then omitting "with respect to n" would be completely appropriate. In Section 9.1, we describe an attempt at confirming Problem 1 in the affirmative by a computer but also how it gets infeasible.

Notice that if $\mathbf{u}$ is nondegenerate, then there exists a v-palindrome $n(k)$ of type $\mathbf{u}$ because $S_{\mathbf{u}}$ contains positive integers. We have thus categorized the v-palindromes $n(k)$ into various kinds, the number of which equals the number of nondegenerate characteristic solutions. For the characteristic equation Equation (9) for n, it could happen that there are no characteristic solutions at all, or that there are characteristic solutions but unfortunately all are degenerate, or that there are nondegenerate solutions. In the former two cases $n(k)$ is not a v-palindrome for any $k \geq 1$, i.e., $c(n) = \infty$. In the third case only, there will exist a $k \geq 1$ for which $n(k)$ is a v-palindrome.

We summarize this section. We started with a natural number n, not a multiple of 10, and not a palindrome. We have the sequence of repeated concatenations of the decimal digits of n, namely $n(k)$ ($k \geq 1$). We would like to know which of them are v-palindromes. In order to do this, we first solve for the characteristic solutions for n. Then, each nondegenerate characteristic solution $\mathbf{u}$ gives rise to a nonempty infinite subset $S_{\mathbf{u}} \cap \mathbb{N}$ of integers $k \geq 1$ for which $n(k)$ is a v-palindrome. The sets $S_{\mathbf{u}} \cap \mathbb{N}$ are pairwise disjoint over the nondegenerate solutions $\mathbf{u}$ and their union gives the set of all $k \geq 1$ for which $n(k)$ is a v-palindrome. This section is of a more theoretical and abstract nature, Section 8 puts these ideas into a more algorithmic description.

5. The Indicator Function

Again, throughout this section we let n be a fixed natural number n as in Theorem 2, i.e., n is a natural number, not a multiple of 10, and not a palindrome. In Section 4, the set of all $k \geq 1$ for which $n(k)$ is a v-palindrome is represented in Corollary 2 as a disjoint union of sets. In this section, we construct a function $I(k)$ which evaluates to 1 if $n(k)$ is a v-palindrome and 0 if not.

5.1. Definition of the Indicator Function

Recall that an indicator function is defined as follows.

Definition 18. If $A \subseteq \Omega$, then the *indicator function* of A in Ω is the function $I_A \colon \Omega \to \{0, 1\}$ defined by

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \Omega \setminus A. \end{cases}$$

In particular, for an integer $a \geq 1$, we denote the indicator function of $a\mathbb{Z} \subseteq \mathbb{Z}$ by I_a. That is, for $x \in \mathbb{Z}$,

$$I_a(x) = \begin{cases} 1 & \text{if } a \mid x, \\ 0 & \text{if } a \nmid x. \end{cases}$$

We have the following representation of the indicator function of the set $S(A, B)$ defined in Definition 15.

Lemma 3. *Let $A, B \subseteq \mathbb{N}$ be finite sets. Then for all $x \in \mathbb{Z}$,*

$$I_{S(A,B)}(x) = I_{\mathrm{lcm}(A)}(x) \prod_{b \in B} (1 - I_b(x)). \tag{17}$$

Hence $I_{S(A,B)}$ is periodic modulo $\mathrm{lcm}(A \cup B)$.

Proof. If $x \in S(A, B)$, then $a \mid x$ for all $a \in A$, and so $\mathrm{lcm}(A) \mid x$. Thus $I_{\mathrm{lcm}(A)}(x) = 1$. Moreover, for every $b \in B$, $b \nmid x$, and so $I_b(x) = 0$. Hence we see that the right-hand side of Equation (17) is 1.

On the other hand, assume that x is an integer with $x \notin S(A, B)$. Then either x is not divisible by some particular $a \in A$, or is divisible by some particular $b \in B$. In the first case, x cannot be a multiple of $\mathrm{lcm}(A)$, thus $I_{\mathrm{lcm}(A)}(x) = 0$ and we see that the right-hand side of Equation (17) is 0. In the second case, $I_b(x) = 1$ for some $b \in B$, hence one of the factors in the product on the right-hand side of Equation (17) becomes 0, and we see again that the right-hand side of Equation (17) is 0. This proves Equation (17).

To prove the periodicity, we see that if we add to x the quantity $\mathrm{lcm}(A \cup B)$, the values of $I_{\mathrm{lcm}(A)}$ and all the I_b ($b \in B$) do not change. Hence $I_{S(A,B)}$ is periodic modulo $\mathrm{lcm}(A \cup B)$. □

Consequently, we directly have the following.

Corollary 3. *Let* $\mathbf{u}$ *be a nondegenerate characteristic solution for* n*. Then for all* $x \in \mathbb{Z}$*,*

$$I_{S_{\mathbf{u}}}(x) = I_{\mathrm{lcm}(A_{\mathbf{u}})}(x) \prod_{b \in B_{\mathbf{u}}} (1 - I_b(x)). \tag{18}$$

Hence $I_{S_{\mathbf{u}}}$ *is periodic modulo* $\mathrm{lcm}(A_{\mathbf{u}} \cup B_{\mathbf{u}})$*. Moreover, for* $k \geq 1$*,* $n(k)$ *is* v*-palindromic of type* $\mathbf{u}$ *if and only if* $I_{S_{\mathbf{u}}}(k) = 1$*.*

Since we have the disjoint union Equation (16), we have the following.

Theorem 7. *We have that for all* $x \in \mathbb{Z}$*,*

$$I_S(x) = \sum_{\mathbf{u} \in U^*} I_{S_{\mathbf{u}}}(x). \tag{19}$$

Hence I_S *is periodic modulo*

$$\mathrm{lcm}\left(\bigcup_{\mathbf{u} \in U^*} (A_{\mathbf{u}} \cup B_{\mathbf{u}}) \right). \tag{20}$$

Moreover, for $k \geq 1$*,* $n(k)$ *is* v*-palindromic if and only if* $I_S(k) = 1$*.*

Proof. If $x \in S$, then $x \in S_{\mathbf{u}}$ for exactly one $\mathbf{u} \in U^*$, so the right-hand side of Equation (19) is 1. If x is an integer with $x \notin S$, then $I_{S_{\mathbf{u}}}(x) = 0$ for all $\mathbf{u} \in U^*$, so the right-hand side of Equation (19) is 0. Let the quantity (20) be denoted by ω. For each $\mathbf{u} \in U^*$, $I_{S_{\mathbf{u}}} = I_{S(A_{\mathbf{u}}, B_{\mathbf{u}})}$ is periodic modulo $\mathrm{lcm}(A_{\mathbf{u}} \cup B_{\mathbf{u}})$ by Corollary 3. Since ω is a multiple of $\mathrm{lcm}(A_{\mathbf{u}} \cup B_{\mathbf{u}})$ for every $\mathbf{u} \in U^*$, we see that I_S is periodic modulo ω. □

Definition 19. The function I_S in Theorem 7 will be called the *indicator function* for n and denoted simply by I, or if we want to indicate the n, denoted by I^n.

Theorem 8. *Let* $\omega > 0$ *be an integer. Then* ω *is a period of* I *if and only if it is a period of* $I|_{\mathbb{N}}$ *if and only if it is a period of* n *(in the sense of Definition 5). Hence* $\omega_0(n)$ *is the fundamental period of* I*.*

Proof. This follows from Theorems 17 and 7, and Definition 5. □

Because of the above theorem, to find $\omega_0(n)$ we just have to find the fundamental period of I. If we try to do this by using Theorem 20, then we will have to first express I in the form of Equation (25). This is doable but conceivably tedious. Instead, there is a much easier way, accomplished by writing I as a linear combination of functions of the form I_a with integer coefficients, which we discuss in the next subsection.

5.2. The Indicator Function as a Linear Combination

The following lemma is what is used to write the indicator function I as a linear combination of functions of the form I_a with integer coefficients.

Lemma 4. *For any integers $a, b \geq 1$, we have $I_a I_b = I_{\mathrm{lcm}(a,b)}$.*

Proof. We need to prove that for all $x \in \mathbb{Z}$,

$$I_a(x) I_b(x) = I_{\mathrm{lcm}(a,b)}(x).$$

If $\mathrm{lcm}(a, b) \mid x$, then both $a \mid x$ and $b \mid x$, and thus both sides of the above equation equal 1. If $\mathrm{lcm}(a, b) \nmid x$, then either $a \nmid x$ or $b \nmid x$, and thus, in the above equation, one of the factors on the left-hand side is 0, and the right-hand side is also 0. This completes the proof. □

Theorem 9. *Let $\mathbf{u}$ be a nondegenerate characteristic solution for n. Then*

$$I_{S_\mathbf{u}} = \sum_{B \subseteq B_\mathbf{u}} (-1)^{|B|} I_{\mathrm{lcm}(A_\mathbf{u} \cup B)}.$$

Proof. This follows by expanding Equation (18) in Corollary 3 and then simplifying using Lemma 4. □

Similarly, we have the following.

Theorem 10. *We have the expression*

$$I = \sum_{\mathbf{u} \in U^*} \sum_{B \subseteq B_\mathbf{u}} (-1)^{|B|} I_{\mathrm{lcm}(A_\mathbf{u} \cup B)} \tag{21}$$

for the indicator function for n.

Proof. This follows from Theorems 7 and 9. □

We consequently have the following.

Theorem 11. *There exist integers $0 < c_1 < c_2 < \cdots < c_q$ and nonzero integers $\lambda_1, \lambda_2, \ldots, \lambda_q$ such that*

$$I = \sum_{j=1}^{q} \lambda_j I_{c_j}, \tag{22}$$

possibly with $q = 0$ (i.e., we have an empty sum).

Proof. We simply collect like terms in Equation (21) in Theorem 10. □

We state without proof the following theorem, which can be proved by induction.

Theorem 12. *Let $f\colon \mathbb{Z} \to \mathbb{C}$ be a function of the form*

$$f = \sum_{j=1}^{q} \lambda_j I_{c_j}, \tag{23}$$

where $c_1 < \cdots < c_q$ are positive integers (possibly $q = 0$) and $\lambda_1, \ldots, \lambda_q \neq 0$ are any integers. Then this representation is unique, in the sense that if $c'_1 < \cdots < c'_{q'}$ are positive integers and $\lambda'_1, \ldots, \lambda'_{q'} \neq 0$ integers which satisfy

$$f = \sum_{j=1}^{q'} \lambda'_j I_{c'_j},$$

then $q = q'$ and for all $1 \leq j \leq q$, $c_j = c'_j$ and $\lambda_j = \lambda'_j$.

According to the above theorem, we have in particular that the indicator function I for n can be expressed in the form of Equation (22) uniquely. Examples of some indicator functions are given in Table 5.

6. Finding the Fundamental Period

Again we let n be a fixed natural number as in Theorem 2. Our goal is to find the fundamental period $\omega_0(n)$. If we try to do this by using Theorem 20, then we will have to first express the indicator function I for n in the form of Equation (25). We explicitly express I in this form in Section 6.1. However, this is not a smart way to find $\omega_0(n)$. Instead, in Section 6.2, we show that $\omega_0(n)$ is simply the least common multiple of the c_j's in Theorem 11, using a theorem in [6].

6.1. The Indicator Function in the Form of Equation (25)

According to [1, Theorem 8.1 on p. 158], we have the following representation of I_a in terms of the a-th roots of unity in $\mathbb{C}$. Let the set of all a-th roots of unity in $\mathbb{C}$ be denoted by $R(a)$.

Lemma 5 ([1, Theorem 8.1 on p. 158])**.** *For $a \geq 1$, we have that for all $x \in \mathbb{Z}$,*

$$I_a(x) = \frac{1}{a} \sum_{\zeta \in R(a)} \zeta^x.$$

If we use the above lemma in Equation (22) in Theorem 11, we can express the indicator function in the form of Equation (25) (see also the theorem below). In principle, we can use Theorem 20 to calculate the fundamental period of I, which will then be $\omega_0(n)$ in view of Theorem 8. However, as aforementioned, this is not a smart way.

Theorem 13. *The indicator function is*

$$\begin{aligned} I(x) &= \sum_{\mathbf{u}\in U^*} \sum_{B\subseteq B_{\mathbf{u}}} \frac{(-1)^{|B|}}{\operatorname{lcm}(A_{\mathbf{u}}\cup B)} \sum_{\zeta\in R(\operatorname{lcm}(A_{\mathbf{u}}\cup B))} \zeta^x \\ &= \sum_{\zeta\in R(\omega)} \left(\sum_{\mathbf{u}\in U^*, B\subseteq B_{\mathbf{u}}, \zeta\in R(\operatorname{lcm}(A_{\mathbf{u}}\cup B))} \frac{(-1)^{|B|}}{\operatorname{lcm}(A_{\mathbf{u}}\cup B)} \right) \zeta^x, \end{aligned}$$

where ω is the quantity (20) *in Theorem 7.*

Proof. The first equality follows by using Lemma 5 in Equation (21) in Theorem 10. The second equality is obtained simply by changing the order of summation, to that of summing over ζ first. □

6.2. Finding the Fundamental Period from Equation (22)

We first give some definitions, more or less equivalent to some definitions given in [6]. The set of all functions $f\colon \mathbb{Z}\to\mathbb{C}$, which can be denoted by $\mathbb{C}^{\mathbb{Z}}$, is obviously a vector space over $\mathbb{C}$ with both vector addition and scalar multiplication defined pointwise. The set, denoted by $\mathcal{F}$ in Section 11.1, of all periodic arithmetical functions, is a subspace of $\mathbb{C}^{\mathbb{Z}}$. The so-called Ramanujan spaces can be defined as follows.

Definition 20. Let $\omega \geq 1$ be an integer. The set of all functions

$$f(x) = \sum_{\zeta\in R^*(\omega)} g(\zeta)\zeta^x \quad \text{for } x\in\mathbb{Z},$$

where the $g(\zeta)$'s are complex numbers and $R^*(\omega)$ denotes the set of primitive ω-th roots of unity in $\mathbb{C}$, is a subspace of $\mathcal{F}$ called a *Ramanujan space* and is denoted by S_ω.

Then, [6, Theorem 12] can be stated as follows.

Theorem 14 ([6, Theorem 12])**.** *Let $\omega_1,\dots,\omega_m \geq 1$ be distinct integers and let $0\neq f_j\in S_{\omega_j}$ for each $1\leq j\leq m$ (the 0 here denoting the zero function). Then the fundamental period of $f = f_1+\cdots+f_m$ is $\operatorname{lcm}(\omega_1,\dots,\omega_m)$.*

We use the above theorem to show that the fundamental period of a function in the form of Equation (23) is the least common multiple of the c_j's.

Theorem 15. *For a function of the form*

$$f = \sum_{j=1}^{q} \lambda_j I_{c_j},$$

where the $c_1 < \cdots < c_q$ *are positive integers (possibly* $q = 0$*) and* $\lambda_1, \ldots, \lambda_q \neq 0$ *are any integers, its fundamental period is* $\operatorname{lcm}(c_1, \ldots, c_q)$*.*

Proof. Define the set

$$D = \{d \in \mathbb{N} \mid d \mid c_j \text{ for some } 1 \leq j \leq q\}.$$

That is, D is the union of the divisors of $c_1, \ldots, c_q$. In view of Lemma 5, the function f can be written as

$$f(x) = \sum_{d \in D} \sum_{\zeta \in R^*(d)} \left(\sum_{1 \leq j \leq q, d \mid c_j} \frac{\lambda_j}{c_j} \right) \zeta^x.$$

Let us denote, for $d \in D$,

$$f_d(x) = \sum_{\zeta \in R^*(d)} \left(\sum_{1 \leq j \leq q, d \mid c_j} \frac{\lambda_j}{c_j} \right) \zeta^x,$$

so that $f_d \in S_d$. Then $f = \sum_{d \in D} f_d$. In view of Theorem 14, the fundamental period of f is

$$\operatorname{lcm}\{d \in D \mid f_d \neq 0\} = \operatorname{lcm}\left\{ d \in D \;\middle|\; \sum_{1 \leq j \leq q, d \mid c_j} \frac{\lambda_j}{c_j} \neq 0 \right\}.$$

We have to show that

$$\operatorname{lcm}\left\{ d \in D \;\middle|\; \sum_{1 \leq j \leq q, d \mid c_j} \frac{\lambda_j}{c_j} \neq 0 \right\} = \operatorname{lcm}(c_1, \ldots, c_q). \tag{24}$$

That the right-hand side above, denote it by R (not the set of roots of unity defined in Section 11.1), is a multiple of the left-hand side, denote it by L, is obvious. For each $d \in D$, we can write

$$\sum_{1 \leq j \leq q, d \mid c_j} \frac{\lambda_j}{c_j} = \frac{\lambda_1}{c_1}[d \mid c_1] + \cdots + \frac{\lambda_q}{c_q}[d \mid c_q],$$

where $[\cdot]$ is the Iverson bracket with $[P] = 1$ if P is true and $[P] = 0$ if P is false. Now suppose that p^α is any prime power with $p^\alpha \mid R$ but $p^{\alpha+1} \nmid R$. Let j_0 be the largest integer with $1 \le j_0 \le q$ and $p^\alpha \mid c_{j_0}$. Then

$$\sum_{1\le j\le q, c_{j_0}\mid c_j} \frac{\lambda_j}{c_j} = \frac{\lambda_1}{c_1}[c_{j_0} \mid c_1] + \cdots + \frac{\lambda_{j_0}}{c_{j_0}}[c_{j_0} \mid c_{j_0}] + \cdots + \frac{\lambda_q}{c_q}[c_{j_0} \mid c_q] = \frac{\lambda_{j_0}}{c_{j_0}} \neq 0.$$

This holds because of the following: for $1 \le j < j_0$, as $c_j < c_{j_0}$, obviously $[c_{j_0} \mid c_j] = 0$; for $j_0 < j \le q$, if $c_{j_0} \mid c_j$, then $p^\alpha \mid c_j$, which contradicts our choice of j_0, thus $[c_{j_0} \mid c_j] = 0$. Therefore as L is a multiple of c_{j_0}, it is also a multiple of p^α. Consequently, as L is a multiple of every prime power divisor of R, $R \mid L$. Since both $L \mid R$ and $R \mid L$, Equation (24) holds. □

As a consequence of the above theorem, we have the following corollary.

Corollary 4. *Suppose that the indicator function for n is expressed as*

$$I = \sum_{j=1}^{q} \lambda_j I_{c_j},$$

where $q \ge 0$, $0 < c_1 < \ldots < c_q$, and $\lambda_1, \ldots, \lambda_q \neq 0$ are integers. Then the fundamental period of n is $\omega_0(n) = \operatorname{lcm}(c_1, \ldots, c_q)$.

7. Finding the Order

We have defined the order $c(n)$ of a number n in Definition 8. It is the smallest integer $k \ge 1$ such that $n(k)$ is a v-palindrome if such a k exists, and is ∞ otherwise. After expressing the indicator function for n in the form of Equation (22), it is easy to find $c(n)$. The following is obvious.

Theorem 16. *Let*

$$f = \sum_{j=1}^{q} \lambda_j I_{c_j}$$

be a function, where the $c_1 < \cdots < c_q$ are positive integers (possibly $q = 0$) and $\lambda_1, \ldots, \lambda_q \neq 0$ are any integers. If $q > 0$, then the smallest positive integer k such that $f(k) \neq 0$ is c_1. If $q = 0$, then for all integers $k \ge 1$, $f(k) = 0$.

As a consequence of the above theorem, we have the following corollary.

Corollary 5. *Suppose that the indicator function for n is expressed as*

$$I = \sum_{j=1}^{q} \lambda_j I_{c_j},$$

where $q \ge 0$, $0 < c_1 < \ldots < c_q$, and $\lambda_1, \ldots, \lambda_q \neq 0$ are integers. Then $c(n) = c_1$ when $q > 0$, and $c(n) = \infty$ when $q = 0$.

In this way, once we have expressed the indicator function of a number n in the form of Equation (22), it will be straightforward to determine both $\omega_0(n)$ and $c(n)$, using Corollaries 4 and 5 respectively. In the next section, we describe the general procedure, starting from a given n as in Theorem 2, to eventually express its indicator function in the form of Equation (22).

8. General Procedure

Throughout this section, we let n be a fixed natural number as in Theorem 2, i.e., n is not a multiple of 10, and not a palindrome. The following describes a general procedure, consisting of a few steps, to express the indicator function I for n in the form of Equation (22), which can be used to determine both $\omega_0(n)$ and $c(n)$. This procedure works due to the previous discussions.

Step 1. Factorize n and $r(n)$ into

$$n = p_1^{a_1} \cdots p_m^{a_m},$$
$$r(n) = p_1^{b_1} \cdots p_m^{b_m},$$

where $p_1 < \cdots < p_m$ are primes, and $a_i, b_i \geq 0$ are integers, not both 0.

Step 2. Look for those primes p_i for which $a_i \neq b_i$, i.e., the crucial primes. Since we are only going to focus on these primes, we denote them again by $p_1 < \cdots < p_m$, and the exponents are a_i, b_i. Define for $1 \leq i \leq m$ the numbers $\delta_i = a_i - b_i$ and $\mu_i = \min(a_i, b_i)$.

Step 3. The characteristic equation for n is

$$\operatorname{sgn}(\delta_1)u_1 + \operatorname{sgn}(\delta_2)u_2 + \cdots + \operatorname{sgn}(\delta_m)u_m = 0.$$

We want to solve it for $u_i \in R_{p_i,|\delta_i|}$, i.e., to find the characteristic solutions. The least efficient way to do this would be to try all tuples. If there are no solutions, then conclude that $c(n) = \infty$ and $\omega_0(n) = 1$. Otherwise, let the solutions be $\mathbf{u}_1, \ldots, \mathbf{u}_t$, in any order.

Step 4. For each characteristic solution $\mathbf{u}$, we have the sets $A_{\mathbf{u}}$ and $B_{\mathbf{u}}$ of Definition 16. The solution $\mathbf{u}$ is nondegenerate if and only if $S(A_{\mathbf{u}}, B_{\mathbf{u}}) \neq \varnothing$. Now $S(A_{\mathbf{u}}, B_{\mathbf{u}}) \neq \varnothing$ if and only if $b \nmid \operatorname{lcm}(A_{\mathbf{u}})$ for all $b \in B_{\mathbf{u}}$. Use this to rule out those characteristic solutions $\mathbf{u}$ which are degenerate. If no characteristic solutions remain, conclude that $c(n) = \infty$ and $\omega_0(n) = 1$. Otherwise, let the nondegenerate characteristic solutions be $\mathbf{u}_1^*, \ldots, \mathbf{u}_s^*$, in any order.

Step 5. The indicator function I^n for n is then given by Theorem 7 as

$$I^n = \sum_{i=1}^{s} I_{S_{\mathbf{u}_i^*}}.$$

By Corollary 3 this can be written as

$$I^n = \sum_{i=1}^{s} I_{\mathrm{lcm}(A_{\mathbf{u}_i^*})} \prod_{b \in B_{\mathbf{u}_i^*}} (1 - I_b).$$

Multiplying everything out on the right-hand side above with the help of Lemma 4 and collecting like terms, I^n can be expressed in the form of Equation (22), i.e.,

$$I^n = \sum_{j=1}^{q} \lambda_j I_{c_j},$$

where $q \geq 1$, $0 < c_1 < \cdots < c_q$, and $\lambda_1, \ldots, \lambda_q \neq 0$ are integers (how this is actually done is illustrated in the example of $n = 126$ in Section 9). Finally, conclude that

$$c(n) = c_1, \quad \omega_0(n) = \mathrm{lcm}(c_1, \ldots, c_q).$$

8.1. Some Remarks about the Procedure

We have described the general procedure in five steps as above. Whether $c(n) = \infty$ can be ascertained at certain points during the procedure. Namely, in Step 3, if there are no characteristic solutions at all, we immediately conclude that $c(n) = \infty$ and the procedure ends; and in Step 4, if all the characteristic solutions are degenerate, we immediately conclude that $c(n) = \infty$ and the procedure ends. Otherwise, $c(n)$ (which is going to be finite), $\omega_0(n)$, and the indicator function I^n are found in Step 5.

9. Counterexample to Conjecture 1

The smallest counterexample to Conjecture 1 is found by PARI/GP [2] to be $n = 126$. We perform the general procedure of Section 8 to $n = 126$ as follows.

Step 1. We factorize

$$126 = 2 \cdot 3^2 \cdot 7,$$
$$621 = 3^3 \cdot 23.$$

Step 2. The crucial primes are $2, 3, 7, 23$. We arrange the numbers $p_i, a_i, b_i, \delta_i, \mu_i$ into a table.

i	p_i	a_i	b_i	δ_i	μ_i
1	2	1	0	1	0
2	3	2	3	-1	2
3	7	1	0	1	0
4	23	0	1	-1	0

Table 1: The $p_i, a_i, b_i, \delta_i, \mu_i$ for $n = 126$.

Step 3. The characteristic equation is

$$u_1 - u_2 + u_3 - u_4 = 0,$$

where we want to solve for $u_1 \in \{1, 2\}$, $u_2 \in \{1, 2, 3\}$, $u_3 \in \{1, 2, 7\}$, and $u_4 \in \{1, 2, 23\}$. The characteristic solutions, found by trying all 54 tuples, are

$$\mathbf{u}_1 = (1, 1, 1, 1), \quad \mathbf{u}_2 = (1, 1, 2, 2), \quad \mathbf{u}_3 = (1, 2, 2, 1), \quad \mathbf{u}_4 = (2, 1, 1, 2),$$
$$\mathbf{u}_5 = (2, 2, 1, 1), \quad \mathbf{u}_6 = (2, 2, 2, 2), \quad \mathbf{u}_7 = (2, 3, 2, 1).$$

For each characteristic solution $\mathbf{u}_l$ $(1 \leq l \leq 7)$, also write

$$\mathbf{u}_l = (u_{l1}, u_{l2}, u_{l3}, u_{l4}).$$

Step 4. We make two tables of the crucial primes p_i $(1 \leq i \leq 4)$ versus the characteristic solutions $\mathbf{u}_l$ $(1 \leq l \leq 7)$ as follows.

The first is where in the $(p_i, \mathbf{u}_l)$-entry we have the $D(p_i, |\delta_i|, u_{li}, \mu_i)$ of Definition 11. The second is where in the $(p_i, \mathbf{u}_l)$-entry we have the $T_{p_i,\mathbf{u}_l}$ defined in Equations (10), (11), and (12), and also at the bottom, the sets $A_{\mathbf{u}_l}, B_{\mathbf{u}_l}, S_{\mathbf{u}_l}$. The first table helps us to construct the second table because the definition of $T_{p_i,\mathbf{u}_l}$ depends on $D(p_i, |\delta_i|, u_{li}, \mu_i)$.

	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{u}_4$	$\mathbf{u}_5$	$\mathbf{u}_6$	$\mathbf{u}_7$
2	[v]	[v]	[v]	[iii]	[iii]	[iii]	[iii]
3	[vi]	[vi]	[vii]	[vi]	[vii]	[vii]	[vii]
7	[v]	[ii]	[ii]	[v]	[v]	[ii]	[ii]
23	[v]	[ii]	[v]	[ii]	[v]	[ii]	[v]

Table 2: Table of $D(p_i, |\delta_i|, u_{li}, \mu_i)$.

	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{u}_4$	$\mathbf{u}_5$	$\mathbf{u}_6$	$\mathbf{u}_7$
2	$(\varnothing,\{1\})$	$(\varnothing,\{1\})$	$(\varnothing,\{1\})$	$(\varnothing,\varnothing)$	$(\varnothing,\varnothing)$	$(\varnothing,\varnothing)$	$(\varnothing,\varnothing)$
3	$(\varnothing,\varnothing)$	$(\varnothing,\varnothing)$	$(\varnothing,\{1\})$	$(\varnothing,\varnothing)$	$(\varnothing,\{1\})$	$(\varnothing,\{1\})$	$(\varnothing,\{1\})$
7	$(\{14\},\varnothing)$	$(\{2\},\{14\})$	$(\{2\},\{14\})$	$(\{14\},\varnothing)$	$(\{14\},\varnothing)$	$(\{2\},\{14\})$	$(\{2\},\{14\})$
23	$(\{506\},\varnothing)$	$(\{22\},\{506\})$	$(\{506\},\varnothing)$	$(\{22\},\{506\})$	$(\{506\},\varnothing)$	$(\{22\},\{506\})$	$(\{506\},\varnothing)$
$A_{\mathbf{u}_l}$	$\{14,506\}$	$\{2,22\}$	$\{2,506\}$	$\{14,22\}$	$\{14,506\}$	$\{2,22\}$	$\{2,506\}$
$B_{\mathbf{u}_l}$	$\{1\}$	$\{1,14,506\}$	$\{1,14\}$	$\{506\}$	$\{1\}$	$\{1,14,506\}$	$\{1,14\}$
$S_{\mathbf{u}_l}$	$\varnothing$	$\varnothing$	$\varnothing$	$S(\{14,22\},\{506\})$	$\varnothing$	$\varnothing$	$\varnothing$

Table 3: Table of $T_{p_i,\mathbf{u}_l}$ and $A_{\mathbf{u}_l}, B_{\mathbf{u}_l}, S_{\mathbf{u}_l}$.

We see immediately from the above table that the only nondegenerate solution is $\mathbf{u}_4$.

Step 5. The indicator function for 126 is then

$$I = I_{14}I_{22}(1 - I_{506}) = I_{154} - I_{3542}.$$

We conclude that $c(126) = 154$ and $\omega_0(126) = \mathrm{lcm}(154, 3542) = 3542$. Since $\omega_f(126) = 31878$ (calculation omitted), we see that $n = 126$ is a counterexample to Conjecture 1.

So it was not difficult to find a counterexample to Conjecture 1, because there is a counterexample as small as 126. One can try to improve Conjecture 1. One possibility is to look at Theorem 7, where the quantity (20), namely

$$\omega_b(n) = \mathrm{lcm}\left(\bigcup_{\mathbf{u}\in U^*}(A_{\mathbf{u}} \cup B_{\mathbf{u}})\right),$$

is a better candidate for a small period. Thus it would be natural, to mimic Conjecture 1, to think that $\omega_0(n)$ is always 1 or $\omega_b(n)$. But this too, is false, as the smallest counterexample found by PARI/GP [2] is $n = 5957$.

9.1. About Problem 1

We describe an attempt at confirming Problem 1 in the affirmative by a computer but also how it gets infeasible. Let m be a v-palindrome and write $m = n_0(k_0)$, where $n_0, k_0 \geq 1$ and n_0 is minimal. Then clearly

$$\{(n,k) \in \mathbb{N}^2 \mid m = n(k)\} = \{(n_0(d), k_0/d) \mid d \in D(k_0)\},$$

where $D(k_0)$ is the set of positive divisors of k_0. If we denote the type of m with respect to n by $\mathbf{Type}(m,n)$, then we have to confirm that

$$\mathbf{Type}(m, n_0(d)),$$

for $d \in D(k_0)$, are all equal.

Suppose that $m = n(k)$. We compute $\mathbf{Type}(m,n) = \mathbf{Type}(n(k),n)$ using a computer by writing the following user-defined functions.

`table(`n`)`. Takes as input an integer $n \geq 1$ with $10 \nmid n$ and $n \neq r(n)$. If there is no characteristic solution for n, outputs that this is so. Otherwise, outputs a table like Table 3 in Step 4 on the previous page. The procedure is just like Steps 1 to 4 earlier in this section (which was for $n = 126$), first prime factorizing n and $r(n)$. The nondegenerate characteristic solutions $\mathbf{u}$ for n and the corresponding sets $S_{\mathbf{u}}$ are found.

`type(`n`,`k`)`. Takes as input integers $n, k \geq 1$ with $10 \nmid n$ and $n \neq r(n)$. If $n(k)$ is a v-palindrome, outputs $\mathbf{Type}(n(k), n)$. Otherwise, outputs that $n(k)$ is not a v-palindrome. First perform `table(`n`)` to find the nondegenerate solutions $\mathbf{u}$ for n and the corresponding sets $S_{\mathbf{u}}$. The number $n(k)$ is a v-palindrome if and only if $k \in S_{\mathbf{u}}$ for some nondegenerate solution $\mathbf{u}$, in which case $\mathbf{Type}(n(k), n) = \mathbf{u}$; whether $k \in S_{\mathbf{u}}$ is checked by checking whether

$$\operatorname{lcm}(A_{\mathbf{u}}) \mid k \quad \text{and} \quad (\text{for all } b \in B_{\mathbf{u}}, b \nmid k).$$

Then, we can compute $\mathbf{Type}(n(k), n)$ by performing `type(`n`,`k`)`.

We perform `type(`$n_0(d)$`,`k_0/d`)` for each $d \in D(k_0)$ and see if the outputs are all equal. Performing `type(`$n_0(d)$`,`k_0/d`)` requires first performing `table(`$n_0(d)$`)`, which requires first prime factorizing $n_0(d)$ and $r(n_0(d))$. Hence prime factorization of $n_0(d)$ and $r(n_0(d))$ is carried out for each $d \in D(k_0)$. As prime factorization is generally time-consuming for large numbers, as k_0 gets large, this method gets increasingly time-consuming. The following table shows some actual running times, where h is hours, min is minutes, and ms is milliseconds.

m	running time	m	running time
13(15)	5 ms	37(12)	6 ms
13(30)	21 ms	37(36)	69 ms
13(45)	95 ms	37(60)	888 ms
13(60)	795 ms	37(84)	1 min, 17827 ms
13(75)	32 min, 46144 ms	37(108)	4 h, 14 min, 15810 ms

Table 4: Table of running times.

We see that the running time gets longer very fast as k_0, the number of repeated concatenations, gets larger. Notice that for each $d \in D(k_0)$, the prime factorization of $n_0(d)$ and $r(n_0(d))$ are carried out to find the crucial primes of $n_0(d)$. However, it can be shown that the crucial primes of any repeated concatenation of n_0 are the same, i.e.,

$$K(n_0) = K(n_0(2)) = K(n_0(3)) = \cdots .$$

It can also be shown that the characteristic solutions for any repeated concatenation of n_0 are the same, i.e.,

$$U(n_0) = U(n_0(2)) = U(n_0(3)) = \cdots .$$

Therefore the tables outputted by `table(`$n_0(d)$`)`, for $d \in D(k_0)$, all have the same "axes" but possibly different entries. It might be unnecessary to prime factorize $n_0(d)$ and $r(n_0(d))$ for each $d \in D(k_0)$. If we can reduce the number of prime factorizations carried out by examining more closely how the entries of the tables outputted by `table(`$n_0(j)$`)` varies as $j \geq 1$ varies, then it might be possible to save some running time. With this current method, a wide ranged computer search for counterexamples gets infeasible.

10. Table of Indicator Functions

We provide a table of the indicator functions, thereby fundamental periods and orders, for some numbers n. These are calculated by using PARI/GP [2].

n	I^n	$c(n)$	$\omega_0(n)$
13	$I_{15} - I_{195} - I_{465} + 2I_{6045}$	15	6045
17	$I_{280} - I_{4760} - I_{19880} + 2I_{337960}$	280	337960
18	I_1	1	1
19	$I_{819} - I_{15561}$	819	15561
26	$I_{15} - I_{195} - I_{465} + 2I_{6045}$	15	6045
37	$I_{12} - I_{444} - I_{876} + 2I_{32412}$	12	32412
39	$I_{15} - I_{195} - I_{465} + 2I_{6045}$	15	6045
48	$I_3 - I_{21}$	3	21
49	$I_{3243} - I_{22701}$	3243	22701
56	$I_3 - I_{21} - I_{39} + 2I_{273}$	3	273
79	$I_{624} - I_{49296} - I_{60528} + 2I_{4781712}$	624	4781712
103	$I_{10234} - I_{1054102}$	10234	1054102
107	$I_{37100} - I_{3969700} - I_{26007100} + 2I_{2782759700}$	37100	2782759700
109	$I_{1686672} - I_{183847248}$	1686672	183847248
113	$I_{17360} - I_{1961680} - I_{5398960} + 2I_{610082480}$	17360	610082480
117	I_{2054}	2054	2054
119	$I_{123760} - I_{112745360}$	123760	112745360
122	$I_{80} - I_{1040} - I_{1360} - I_{4880} + I_{17680} + 2I_{63440} + 2I_{82960} - 3I_{1078480}$	80	1078480

Table 5: Table of indicator functions.

Each indicator function above is computed within 3 milliseconds. We see that the indicator functions for 13, 26, and 39 are identical. There is another curiosity in the above table. We see that for all these indicator functions, the largest subscript is a multiple of all smaller subscripts. This is not always true and the smallest counterexample, found by PARI/GP [2], is $n = 21726$. We have

$$I^{21726} = I_{816} - I_{5712} - I_{8976} - I_{10608} + I_{16401} - I_{32802} + I_{62832} + I_{74256} + I_{116688} - I_{816816} - I_{1098867} + I_{2197734},$$

where $816 \nmid 2197734$.

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11. Appendix on Periodic Arithmetical Functions

In Section 11.1, we first recall some basic properties of periodic arithmetical functions. There, we prove a formula for the fundamental period of an arbitrary periodic arithmetical function $\mathbb{Z} \to \mathbb{C}$ (Theorem 20). This formula is actually equivalent to the formula given in [6, Theorem 9], and we prove their equivalence in Section 11.2.

Therefore Theorem 20 is not a new result. In Section 11.3, we prove [6, Theorem 12] (Theorem 14 in this paper) using Theorem 20. In this way, our paper becomes more self-contained, with the logical dependencies as follows.

- Though equivalent to [6, Theorem 9], in this paper Theorem 20 is proved from first principles.
- Though the same as [6, Theorem 12], in this paper Theorem 14 is proved by using Theorem 20.
- Theorem 15 is proved by using Theorem 14. As a consequence, we have Corollary 4.

The paper [6] contains proofs of both its Theorems 9 and 12. However, [6, proof of Theorem 12] does not seem to be a direct application of [6, Theorem 9].

11.1. Basic Properties and a Formula for the Fundamental Period

Let the function $e\colon \mathbb{R} \to \mathbb{C}$ be defined by

$$e(t) = e^{2\pi i t}.$$

Then the set of all roots of unity in $\mathbb{C}$ is

$$R = \{e(\alpha) \mid \alpha \in \mathbb{Q}\}.$$

For a $\zeta = e(\alpha) \in R$, where $0 \leq \alpha < 1$ is rational, write $\alpha = a/b$ in lowest terms, i.e., $a, b \in \mathbb{Z}$, $b > 0$, and $(a, b) = 1$. Denote $\nu(\zeta) = a$ and $\delta(\zeta) = b$. Thus ζ is a primitive $\delta(\zeta)$-th root of unity. For each integer $m \geq 1$, we denote by ζ_m the primitive m-th root of unity $e(1/m)$.

Consider functions $g\colon R \to \mathbb{C}$ with $g(\zeta) = 0$ outside a finite set. Let the set of all such functions be denoted $\mathcal{G}$. For a $g \in \mathcal{G}$, define an arithmetical function $f\colon \mathbb{Z} \to \mathbb{C}$ by

$$f(x) = \sum_{\zeta \in R} g(\zeta)\zeta^x. \tag{25}$$

The sum is actually finite because $g(\zeta) = 0$ for all but finitely many ζ's. We denote this f by $\Phi(g)$. We prove in Theorem 18 that f is a periodic function. Let us recall some definitions.

Definition 21. A function $f\colon \mathbb{Z} \to \mathbb{C}$ (respectively $f\colon \mathbb{N} \to \mathbb{C}$) is *periodic* if there is an integer $\omega > 0$ such that for all $x \in \mathbb{Z}$ (respectively $x \in \mathbb{N}$),

$$f(x + \omega) = f(x).$$

Such an ω is called a *period* of f, and we also say that f is *periodic modulo* ω. When f is periodic, the smallest period of f is called its *fundamental period*.

We have the following characterization of periods.

Theorem 17. *Let $f\colon \mathbb{Z} \to \mathbb{C}$ be a periodic function. Then we have the following.*

(i) *The restriction $f|_{\mathbb{N}}$ of f to $\mathbb{N}$ is periodic. Moreover, an integer $\omega > 0$ is a period of f if and only if it is a period of $f|_{\mathbb{N}}$.*

(ii) *The periods of f are precisely the positive integral multiples of its fundamental period ω_0.*

Proof. (i) Clearly any period of f is also a period of its restriction $f|_{\mathbb{N}}$. We only need to prove that any period of $f|_{\mathbb{N}}$ is conversely a period of f. So let ω be a period of $f|_{\mathbb{N}}$. Choose a period μ of f. Then μ is also a period of $f|_{\mathbb{N}}$. For any $x \in \mathbb{Z}$, there exists an integer $q > 0$ such that $x+q\mu > 0$. Also, $x+\omega+q\mu > 0$. Since ω is a period of $f|_{\mathbb{N}}$, we have $f(x+q\mu) = f(x+q\mu+\omega)$. Now since μ is a period of f,

$$f(x) = f(x+q\mu) = f(x+q\mu+\omega) = f(x+\omega).$$

Since the above holds for all $x \in \mathbb{Z}$, ω is a period of f.

(ii) Let ω be a period of f. Use the division algorithm to write $\omega = q\omega_0 + r$, where $q, r \in \mathbb{Z}$ are such that $0 \le r < \omega_0$ and $q > 0$. Assume that $r = \omega - q\omega_0 > 0$. For any $x \in \mathbb{Z}$,

$$f(x) = f(x+\omega) = f(x+\omega-q\omega_0) = f(x+r),$$

because both ω and ω_0 are periods of f. Hence r is a period of f smaller than ω_0, which is a contradiction. Hence $r = 0$ and so $\omega_0 \mid \omega$. The converse, that any positive integral multiple of ω_0 is a period of f, is obvious.

□

For the rest of this section, we shall deal only with arithmetical functions $f\colon \mathbb{Z} \to \mathbb{C}$ defined for every integer.

Theorem 18. *Let $g \in \mathcal{G}$. Then $\Phi(g)$ is periodic modulo*

$$\operatorname{lcm}\{\delta(\zeta) \mid \zeta \in R, g(\zeta) \neq 0\}. \tag{26}$$

Proof. Let the quantity (26) be denoted by ω and let $f = \Phi(g)$. For each $\zeta \in R$ with $g(\zeta) \neq 0$, ζ is a $\delta(\zeta)$-th root of unity, and therefore $\zeta^{x+\delta(\zeta)} = \zeta^x$ for every $x \in \mathbb{Z}$. As ω is a multiple of $\delta(\zeta)$, $\zeta^{x+\omega} = \zeta^x$ for every $x \in \mathbb{Z}$. Consequently, for every $x \in \mathbb{Z}$,

$$f(x+\omega) = \sum_{\zeta \in R, g(\zeta)\neq 0} g(\zeta)\zeta^{x+\omega} = \sum_{\zeta \in R, g(\zeta)\neq 0} g(\zeta)\zeta^{x} = f(x).$$

□

Let the set of all periodic arithmetical functions $f\colon \mathbb{Z} \to \mathbb{C}$ be denoted by $\mathcal{F}$. The above established a mapping $\Phi\colon \mathcal{G} \to \mathcal{F}$. We shall prove that it is bijective.

Theorem 19. *The mapping $\Phi\colon \mathcal{G} \to \mathcal{F}$ is bijective.*

Proof. Let $f \in \mathcal{F}$ be periodic modulo ω. By [1, Theorem 8.4 on p. 160], there exist unique coefficients $h_r \in \mathbb{C}$ for $0 \le r < \omega$ such that for all $x \in \mathbb{Z}$,

$$f(x) = \sum_{r=0}^{\omega-1} h_r \zeta_\omega^{xr}.$$

If we define the function $g\colon R \to \mathbb{C}$ by setting $g(\zeta_\omega^r) = h_r$ for $0 \le r < \omega$ and $g(\zeta) = 0$ for all other $\zeta \in R$, it is easy to see that $g \in \mathcal{G}$ and that $\Phi(g) = f$. Whence Φ is surjective.

We now prove injectivity. Assume that $\Phi(g_1) = \Phi(g_2) = f \in \mathcal{F}$. Then for all $x \in \mathbb{Z}$,

$$f(x) = \sum_{\zeta \in R} g_1(\zeta)\zeta^x = \sum_{\zeta \in R} g_2(\zeta)\zeta^x.$$

Let $S = \{\zeta \in R \mid (g_1(\zeta), g_2(\zeta)) \ne (0,0)\}$. Then S is finite and the above sums can be written as

$$\sum_{\zeta \in S} g_1(\zeta)\zeta^x = \sum_{\zeta \in S} g_2(\zeta)\zeta^x.$$

Consequently, for all $x \in \mathbb{Z}$,

$$\sum_{\zeta \in S} (g_1(\zeta) - g_2(\zeta))\zeta^x = 0. \tag{27}$$

If $S = \varnothing$, clearly $g_1 = g_2 = 0$ is identically zero. Thus assume otherwise and list the elements of S as $\{\xi_1, \ldots, \xi_m\}$. Put $x_j = g_1(\xi_j) - g_2(\xi_j)$ for $1 \le j \le m$. Then Equation (27) becomes

$$\sum_{j=1}^{m} x_j \xi_j^x = 0.$$

Since this holds for all $x \in \mathbb{Z}$, in particular it holds for all $0 \le x < m$, and we have a homogeneous system of linear equations. Since the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_m \\ \cdots & \cdots & \cdots & \cdots \\ \xi_1^{m-1} & \xi_2^{m-1} & \cdots & \xi_m^{m-1} \end{vmatrix}$$

is nonzero as the ξ_j's are distinct, $x_j = 0$ for all $1 \le j \le m$. That is, $g_1(\xi_j) = g_2(\xi_j)$ for all $1 \le j \le m$. In other words, $g_1(\zeta) = g_2(\zeta)$ for all $\zeta \in S$. Since $g_1(\zeta) = g_2(\zeta) = 0$ for all $\zeta \in R \setminus S$, we have shown that $g_1 = g_2$. Whence Φ is injective. $\square$

Theorem 20. *Let $g \in \mathcal{G}$. Then the fundamental period of $\Phi(g)$ is indeed given by the quantity* (26).

Proof. Let $f = \Phi(g)$. Theorem 18 already showed that

$$\omega = \operatorname{lcm}\{\delta(\zeta) \mid \zeta \in R, g(\zeta) \neq 0\} \tag{28}$$

is a period of f. We need to show that it is the smallest period. Assume that the smallest period is actually ω_0, where $0 < \omega_0 < \omega$. By part (ii) of Theorem 17, $\omega_0 \mid \omega$. By [1, Theorem 8.4 on p. 160], there exist unique coefficients h_r for $0 \leq r < \omega_0$ such that for all $x \in \mathbb{Z}$,

$$f(x) = \sum_{r=0}^{\omega_0 - 1} h_r \zeta_{\omega_0}^{xr}.$$

Hence we see that $g(\zeta_{\omega_0}^r) = h_r$ for $0 \leq r < \omega_0$ and $g(\zeta) = 0$ for all other $\zeta \in R$.

Now $\omega_0 < \omega$ and so in view of Equation (28), there exists some $\xi \in R$ with $g(\xi) \neq 0$ such that $\delta(\xi) \nmid \omega_0$. We have

$$\xi^{\omega_0} = \left(e\left(\frac{\nu(\xi)}{\delta(\xi)} \right) \right)^{\omega_0} = e\left(\frac{\nu(\xi)\omega_0}{\delta(\xi)} \right).$$

Now the argument on the right above is not an integer. For if it is, then $\delta(\xi) \mid \nu(\xi)\omega_0$. Since $(\delta(\xi), \nu(\xi)) = 1$, $\delta(\xi) \mid \omega_0$, which is a contradiction. Therefore $\nu(\xi)\omega_0/\delta(\xi)$ is not an integer, and so $\xi^{\omega_0} \neq 1$. That is, ξ is not an ω_0-th root of unity. But g vanishes at all $\zeta \in R$ which is not an ω_0-th root of unity. This is a contradiction. Hence ω is indeed the fundamental period of f. □

11.2. Equivalence of Theorem 20 and [6, Theorem 9]

Let $f \colon \mathbb{Z} \to \mathbb{C}$ be a periodic arithmetical function of period ω. By [1, Theorem 8.4 on p. 160], there exist unique coefficients h_r for $0 \leq r < \omega$ such that for all $x \in \mathbb{Z}$,

$$f(x) = \sum_{r=0}^{\omega - 1} h_r \zeta_{\omega}^{xr}.$$

We can write $f(x)$ as

$$f(x) = \sum_{k=1}^{\omega} h_{\omega - k} \zeta_{\omega}^{-xk},$$

or if we rename $h_{\omega-k}$ as h_k,

$$f(x) = \sum_{k=1}^{\omega} h_k \zeta_{\omega}^{-xk}.$$

Let the set of $1 \le k \le \omega$ such that $h_k \ne 0$ be $\{k_1, \ldots, k_l\}$. According to [6, Theorem 9], the fundamental period of f is

$$\omega_0 = \frac{\omega}{(k_1, \ldots, k_l, \omega)}, \tag{29}$$

where the parentheses denote the greatest common divisor. On the other hand, according to Theorem 20,

$$\omega_0 = \operatorname{lcm}(\delta(\zeta_\omega^{-k_1}), \ldots, \delta(\zeta_\omega^{-k_l})). \tag{30}$$

We show that the right-hand sides of Equations (29) and (30) are equal, i.e.,

$$\frac{\omega}{(k_1, \ldots, k_l, \omega)} = \operatorname{lcm}(\delta(\zeta_\omega^{-k_1}), \ldots, \delta(\zeta_\omega^{-k_l})). \tag{31}$$

When there are no $1 \le k \le \omega$ such that $h_k \ne 0$, i.e., when $l = 0$, this is obvious. Thus assume that $l > 0$. Notice that for $1 \le j \le l$,

$$\delta(\zeta_\omega^{-k_j}) = \delta\left(e\left(\frac{-k_j}{\omega}\right)\right) = \frac{\omega}{(k_j, \omega)}, \tag{32}$$

which is easily seen to divide the left-hand side of Equation (31). Hence in Equation (31), the left-hand side is a multiple of the right-hand side.

Conversely, denote the right-hand side of Equation (31) by M. Notice that for $1 \le j \le l$, because of Equation (32),

$$\frac{\omega}{(k_j, \omega)} \mid M, \quad \text{and therefore} \quad \omega \mid M(k_j, \omega). \tag{33}$$

Since

$$(k_1, \ldots, k_l, \omega) = ((k_1, \omega), \ldots, (k_l, \omega)),$$

we have a linear combination

$$(k_1, \ldots, k_l, \omega) = \sum_{j=1}^{l} y_j (k_j, \omega), \tag{34}$$

where the y_j's are integers. We prove that the right-hand side M of Equation (31) is a multiple of the left-hand side, or equivalently,

$$\omega \mid M(k_1, \ldots, k_l, \omega).$$

Since the divisibility relation (33) holds for all $1 \le j \le l$, using also Equation (34),

$$\omega \mid \sum_{j=1}^{l} y_j M(k_j, \omega) = M \sum_{j=1}^{l} y_j (k_j, \omega) = M(k_1, \ldots, k_l, \omega).$$

Since we have proved that each side of Equation (31) is a multiple of the other side, equality holds.

In summary, Theorem 20 and [6, Theorem 9] both give a formula for the fundamental period of a periodic arithmetical function. These formulae might look different on the surface, but indeed give the same fundamental period.

11.3. Proof of Theorem 14 Using Theorem 20

We have borrowed [6, Theorem 12] (Theorem 14 in this paper) in our proof of Theorem 15. To make this paper self-contained, we provide a proof of Theorem 14 using Theorem 20.

Proof. For each $1 \leq j \leq m$, we have

$$f_j(x) = \sum_{\zeta \in R^*(\omega_j)} g_j(\zeta)\zeta^x,$$

where $g_j = \Phi^{-1}(f_j)$. Since the $R^*(\omega_j)$ are pairwise disjoint, $\Phi^{-1}(f) = g$, where

$$g(\zeta) = \begin{cases} g_j(\zeta) & \text{if } \zeta \in R^*(\omega_j) \text{ for some } 1 \leq j \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 20, the fundamental period of f is

$$L = \operatorname{lcm}\{\delta(\zeta) \mid g(\zeta) \neq 0\},$$

whereas the fundamental period asserted by Theorem 14 is

$$T = \operatorname{lcm}\{\omega_1, \ldots, \omega_m\}.$$

So we have to prove that $L = T$.

Let ζ be a root of unity such that $g(\zeta) \neq 0$. Then $\zeta \in R^*(\omega_j)$ for some $1 \leq j \leq m$. Since $\delta(\zeta) = \omega_j$ and $\omega_j \mid T$, we have $\delta(\zeta) \mid T$. Since $\delta(\zeta) \mid T$ for any root of unity ζ such that $g(\zeta) \neq 0$, we have $L \mid T$. On the other hand, let $1 \leq j \leq m$. Since $f_j \neq 0$, $g(\zeta) = g_j(\zeta) \neq 0$ for some $\zeta \in R^*(\omega_j)$. Since $\delta(\zeta) = \omega_j$ and $\delta(\zeta) \mid L$, we have $\omega_j \mid L$. Since $\omega_j \mid L$ for any $1 \leq j \leq m$, we have $T \mid L$. Since both $L \mid T$ and $T \mid L$, we have $L = T$. □