FAULHABER'S FORMULA, ODD BERNOULLI NUMBERS, AND THE METHOD OF PARTIAL SUMS

Ryan Zielinski
Clifton, New Jersey
ryan_zielinski@fastmail.com

Received: 6/22/21, Revised: 7/5/22, Accepted: 7/23/22, Published: 8/3/22

Abstract

Let "Faulhaber's formula" refer to an expression for the sum of powers of integers written with terms in $n(n+1)/2$. Initially, the author used Faulhaber's formula to explain why odd Bernoulli numbers are equal to zero. Next, Cereceda gave alternate proofs of that result and then proved the converse, if odd Bernoulli numbers are equal to zero then we can derive Faulhaber's formula. Here, the original author will give a new proof of the converse using the method of partial sums and mathematical induction.

1. Motivation

If we knew nothing of the history of the problem and tried to discover for ourselves a general expression for

$$\sum_{k=1}^{n} k^m = 1^m + 2^m + \cdots + n^m,$$

where n, m are positive integers, we might notice there appear to be two ways to write such sums. For example,

$$\begin{aligned} \sum k =& \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n, \\ \sum k^2 =& \frac{2n+1}{3} \cdot \frac{n(n+1)}{2} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n, \\ \sum k^3 =& \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2. \end{aligned} \tag{1}$$

The next two cases are

$$\begin{aligned} \sum k^4 =& \frac{1}{5}\left[6 \cdot \frac{n(n+1)}{2} - 1\right] \cdot \sum k^2 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n, \\ \sum k^5 =& \frac{1}{3}\left[4 \cdot \frac{n(n+1)}{2} - 1\right]\left(\sum k\right)^2 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2. \end{aligned} \tag{2}$$

(When no confusion will arise, we will abbreviate $\sum_{k=1}^{n} k^m$ by $\sum k^m$.)

We can write each sum using terms of $\frac{n(n+1)}{2}$ or n. Of course, if we have the former then we always can expand it into the latter. Do we always have the former?

At a later point, reading up on the matter we would learn that writing an expression for $\sum k^m$ using terms in n is associated with the name of Jakob Bernoulli (1654-1705), and writing the same expression using terms in $\frac{n(n+1)}{2}$ is associated with that of Johann Faulhaber (1580-1635). Bernoulli's contribution has long overshadowed Faulhaber's, but now we know the two are linked inextricably.

2. Background

In order to write an expression for $\sum k^m$ in n, we introduce the Bernoulli numbers. Set $B_0 = 1$ and define B_n by

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0,$$

where $n \geq 1$. Then we can write

$$\sum k^m = \frac{1}{m+1} \sum_{j=0}^{m} (-1)^j \binom{m+1}{j} B_j n^{m+1-j}. \tag{3}$$

For a sum in $\frac{n(n+1)}{2}$, for even powers we have

$$\sum k^{2m} = \left[c_0 \left(\frac{n(n+1)}{2} \right)^{m-1} + \cdots + c_{m-2} \cdot \frac{n(n+1)}{2} + c_{m-1} \right] \cdot \sum k^2, \tag{4}$$

and for odd powers we have

$$\sum k^{2m+1} = \left[a_0 \left(\frac{n(n+1)}{2} \right)^{m-1} + \cdots + a_{m-2} \cdot \frac{n(n+1)}{2} + a_{m-1} \right] \left(\sum k \right)^2, \tag{5}$$

where the c_i, a_i are rational numbers and $m \geq 1$. We will refer to these two expressions as "Faulhaber's formula."

Regarding earlier work on the topic, Edwards took a matrix-based approach and looked for recurrence relations amongst all such sums (4) and (5) ([6, 7]). The coefficients c_i, a_i then followed as entries in the inverses of such matrices. Gessel and Viennot investigated sums of powers and alternating sums of powers ([8, section 12]). Explicit expressions for both sets of coefficients were derived and their combinatorial properties were discussed. Knuth looked at a lot of material ([12]). Perhaps the most important part was examining Faulhaber's original work and placing it into a modern perspective.

Now we begin with the present contribution, which aimed to unravel just what allows for writing $\sum k^m$ with terms in $\frac{n(n+1)}{2}$. But, we started in the opposite direction.

If we look at the expressions in (1) and (2), we notice a few powers of n are missing. The reason is because odd Bernoulli numbers are equal to zero: $B_1 = -\frac{1}{2}$, but for all $m \geq 1$, $B_{2m+1} = 0$. When we write such sums using (3), the powers of n which have odd Bernoulli numbers for coefficients drop out.

There are established ways to prove such a result (Rademacher [13, chapters 1-2]). The new insight of Zielinski [15] was that, with the different expressions for $\sum k^m$, we already have enough information to justify such an outcome.

If we write $\sum k^{2m+1}$ in the two forms of (3) and (5), the coefficients for the terms in n must agree. Since (5) contains a factor of $\left(\sum k\right)^2 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$, there is no term of n. That means the last term of (3), $-B_{2m+1} \cdot n$, must be equal to zero. In other words, $B_{2m+1} = 0$ for all $m \geq 1$.[1]

To prove the converse, Cereceda chose a different line of attack and introduced Bernoulli polynomials:

$$B_m\left(x\right) = \sum_{j=0}^{m} \binom{m}{j} B_j x^{m-j}, \tag{6}$$

where x is a real variable and $B_m\left(0\right) = B_m$ ([5]). This allows an expression for the sum of powers to be written as

$$\sum k^m = \frac{1}{m+1}\left(B_{m+1}(n+1) - B_{m+1}\right). \tag{7}$$

In the new context, the property $B_{2m+1} = 0$ is related to Bernoulli polynomials being evaluated at $x = \frac{1}{2}$. The notion of symmetry allows such polynomials to be rewritten with terms of $\left(x - \frac{1}{2}\right)$, which can be rewritten once more with terms of $\frac{x(x-1)}{2}$. Once this is done, (7) leads immediately to Faulhaber's formula. (Again, alternate proofs of the main result of Zielinski [15] are contained within the paper as well.)

Together, the papers lead to a surprising revelation, one which has been a long time in the making. Denote $\sum_{k=1}^{n} k^m$ by S_m. Then we have

Theorem 1 (Zielinski [15] and Cereceda [5]). *For positive integers m,*

$$B_{2m+1} = 0 \text{ if and only if } \begin{cases} S_{2m} = & S_2 \cdot F_{2m}\left(S_1\right), \\ S_{2m+1} = & S_1^2 \cdot F_{2m+1}\left(S_1\right), \end{cases}$$

where $F_{2m}\left(S_1\right)$ and $F_{2m+1}\left(S_1\right)$ are polynomials in $S_1 = \frac{n(n+1)}{2}$.

[1] If we write (4) in a manner analogous to (2.5) of Zielinski [15], we can give the same type of argument for even powers.

In this paper we will give a different proof of the converse. Our approach will be based on what commonly is referred to as the method of partial sums:

$$\sum_{k=1}^{n} k^{m+1} = (n+1)\sum_{k=1}^{n} k^m - \sum_{k=1}^{n}\sum_{l=1}^{k} l^m.$$

We state it in a pointed fashion to illustrate that it serves as a generator for sums of powers (Zielinski [14, section 2]).[2] In this context, when we use (3) to write expressions for $\sum\sum l^m$, the property $B_{2m+1} = 0$ will cause the bulk of the terms to be of even or odd parity like $\sum k^{m+1}$.

3. Main Result

We start with a proof of the method of partial sums.

Proposition 1. *For integers n, m, where $n \geq 1$ and $m \geq 0$,*

$$\sum_{k=1}^{n} k^{m+1} + \sum_{k=1}^{n}\sum_{l=1}^{k} l^m = (n+1)\sum_{k=1}^{n} k^m.$$

Proof. A way to prove $\frac{n(n+1)}{2} = \sum_{k=1}^{n} k$ is to write

$$n(n+1) = 2\,(1+2+\cdots+n)$$

and then to interpret the left side as a rectangle of area $n(n+1)$ and the right side as two pieces of $(1+2+\cdots+n)$ squares. We will do something analogous in this case.

Let us write $n+1$ rows of

$$\begin{array}{c} 1^m + 2^m + \cdots + (n-1)^m + n^m \\ 1^m + 2^m + \cdots + (n-1)^m + n^m \\ \vdots \\ 1^m + 2^m + \cdots + (n-1)^m + n^m \\ 1^m + 2^m + \cdots + (n-1)^m + n^m \end{array}$$

[2] This exact form of the method of partial sums is said to go back 1,000 years to ibn al-Haytham (965-1039). Some of the history can be found in Katz [11] and Boudreaux [2].

and then divide the sum into

$$
\begin{array}{lcr}
 & & 1^m + 2^m + \cdots + (n-1)^m + n^m \\
1^m & & 2^m + \cdots + (n-1)^m + n^m \\
1^m + 2^m & + & \vdots \\
\vdots & & (n-1)^m + n^m \\
1^m + 2^m + \cdots + (n-1)^m & & n^m \\
1^m + 2^m + \cdots + (n-1)^m + n^m & &
\end{array}
$$

This gives us

$$(n+1)\sum_{k=1}^{n} k^m = \sum_{k=1}^{n}\sum_{l=1}^{k} l^m + \sum_{k=1}^{n} k^{m+1}. \qquad \square$$

Before we prove the converse we state, without proof, a lemma which points out critical, intermediate relationships.

Lemma 1. *For positive integers n,*

$$\left(n+\frac{1}{2}\right)\left(\sum k\right)^2 = \frac{3}{2}\cdot\frac{n(n+1)}{2}\cdot\sum k^2,$$
$$\left(n+\frac{1}{2}\right)\sum k^2 = \left(\frac{4}{3}\cdot\frac{n(n+1)}{2}+\frac{1}{6}\right)\sum k.$$

A consequence of the second relationship is that $\left(\sum k^2\right)^2$ can be rewritten using only terms of $\sum k$, which then implies, through (4), that $\left(\sum k^{2m}\right)^2$ can be rewritten in $\sum k$ as well (Beardon [1, section 3]). We have

$$\left(\sum k^2\right)^2 = G\left(S_1\right) \text{ and } \left(\sum k^{2m}\right)^2 = H\left(S_1\right), \tag{8}$$

where $G\left(S_1\right)$ and $H\left(S_1\right)$ are polynomials in $S_1 = \frac{n(n+1)}{2}$.

Now we give a new proof of the converse.

Proposition 2. *If odd Bernoulli numbers are equal to zero, we can derive Faulhaber's formula.*

Proof. We proceed by mathematical induction. Expression (3), with $B_3 = 0$, allows us to write

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{2n+1}{3}\cdot\frac{n(n+1)}{2} = 1\cdot\sum k^2,$$
$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 - 0\cdot n = \left(\frac{n(n+1)}{2}\right)^2 = 1\cdot\left(\sum k\right)^2.$$

We will assume (4) and (5) are true for all $1, 2, ..., m$ and then establish the case of $m+1$.

For $m+1$, Proposition 1 tells us

$$\sum_{k=1}^{n} k^{2m+2} = (n+1)\sum_{k=1}^{n} k^{2m+1} - \sum_{k=1}^{n}\sum_{l=1}^{k} l^{2m+1}. \tag{9}$$

We want to write $\sum l^{2m+1}$ in k. Expression (3) tells us

$$\sum_{l=1}^{k} l^{2m+1} = \frac{1}{2m+2}\sum_{j=0}^{2m+1} (-1)^j \binom{2m+2}{j} B_j k^{2m+2-j}.$$

Since we are assuming $B_3 = B_5 = \cdots = 0$, we can rewrite the double sum as

$$\begin{aligned}\sum_{k=1}^{n}\sum_{l=1}^{k} l^{2m+1} &= \frac{1}{2m+2}\sum_{k=1}^{n}\Bigg(1\cdot B_0 k^{2m+2} - \binom{2m+2}{1} B_1 k^{2m+1} \\ &\quad + \binom{2m+2}{2} B_2 k^{2m} + \cdots + \binom{2m+2}{2m} B_{2m} k^2\Bigg) \\ &= \frac{1}{2m+2}\sum_{k=1}^{n} k^{2m+2} + \frac{1}{2}\sum_{k=1}^{n} k^{2m+1} + b_{2m}\sum_{k=1}^{n} k^{2m} + \cdots + b_2\sum_{k=1}^{n} k^2,\end{aligned}$$

where $b_{2m}, \ldots, b_2$ are rational numbers which do not interest us. Now we can rewrite (9) as

$$\begin{aligned}\sum k^{2m+2} &= (n+1)\sum k^{2m+1} - \frac{1}{2m+2}\sum k^{2m+2} - \frac{1}{2}\sum k^{2m+1} \\ &\quad - \left(b_{2m}\sum k^{2m} + \cdots + b_2\sum k^2\right). \end{aligned}\tag{10}$$

By the inductive hypothesis for $\sum k^{2m}$, we can rewrite the sum in the parentheses as

$$b_{2m}\sum k^2 \cdot F_{2m} + \cdots + b_2\sum k^2 \cdot F_2,$$

where $F_{2m}, \ldots, F_2$ are polynomials in $\frac{n(n+1)}{2}$. Together, this is just $F \cdot \sum k^2$ for another such polynomial F. Now (10) becomes

$$\frac{2m+3}{2m+2}\sum k^{2m+2} = \left(n+\frac{1}{2}\right)\sum k^{2m+1} - F\cdot\sum k^2. \tag{11}$$

For the next step of the proof, first we invoke the inductive hypothesis for $\sum k^{2m+1}$. This allows us to rewrite the right side of (11) as

$$\left(n+\frac{1}{2}\right)\left(\sum k\right)^2 G_{2m+1} - F\cdot\sum k^2, \tag{12}$$

where G_{2m+1} is a polynomial in $\frac{n(n+1)}{2}$. Then we use the lemma to rewrite the left side of (12) as

$$\frac{3}{2} \cdot \frac{n(n+1)}{2} \cdot \sum k^2 \cdot G_{2m+1} = G \cdot \sum k^2,$$

where G is another polynomial in $\frac{n(n+1)}{2}$. The final form of (11) becomes

$$\sum k^{2m+2} \quad = \quad \frac{2m+2}{2m+3}\left(G \cdot \sum k^2 - F \cdot \sum k^2\right) = H \cdot \sum k^2,$$

where H is a polynomial in $\frac{n(n+1)}{2}$.

The proof for $\sum k^{2m+3}$ proceeds along the same lines. We only wish to point out an important difference when rewriting the double sum using Bernoulli numbers. Starting with

$$\sum_{k=1}^{n} k^{2m+3} = (n+1)\sum_{k=1}^{n} k^{2m+2} - \sum_{k=1}^{n}\sum_{l=1}^{k} l^{2m+2},$$

the expression analogous to (11) will be

$$\frac{2m+4}{2m+3}\sum k^{2m+3} = \left(n+\frac{1}{2}\right)\sum k^{2m+2} - B_{2m+2}\sum k - F \cdot \left(\sum k\right)^2. \tag{13}$$

We need to eliminate the term of $-B_{2m+2}\sum k$, which we do as follows.

The coefficient of B_{2m+2} comes out of writing $\sum l^{2m+2}$ according to (3). If we write the same expression using (4), which we just established, we get

$$\sum k^{2m+2} = [G + c_m]\sum k^2 = [G + c_m] \cdot \frac{2n^3 + 3n^2 + n}{6},$$

where G is a polynomial in $\frac{n(n+1)}{2}$, of degree of at least one, and c_m is a rational number. The coefficient for the term of n is $\frac{c_m}{6}$. Since both coefficients must agree, we have $\frac{c_m}{6} = B_{2m+2}$.

When we invoke the lemma we get

$$\begin{aligned}\left(n+\frac{1}{2}\right)[G + c_m]\sum k^2 \quad &= \quad [G + c_m]\left(\frac{4}{3} \cdot \frac{n(n+1)}{2} + \frac{1}{6}\right)\sum k \\ &= \quad [G + c_m]\left(\frac{4}{3}\left(\sum k\right)^2 + \frac{1}{6}\sum k\right) \\ &= \quad [G + c_m] \cdot \frac{4}{3}\left(\sum k\right)^2 + G \cdot \frac{1}{6}\sum k + \frac{c_m}{6}\sum k.\end{aligned}$$

Since the polynomial G does not have a constant term, we can simplify the right side to

$$G' \cdot \left(\sum k\right)^2 + \frac{c_m}{6}\sum k, \tag{14}$$

where $G^{'}$ is another polynomial in $\frac{n(n+1)}{2}$. Now the term of $\frac{c_m}{6}\sum k$ cancels with that of $-B_{2m+2}\sum k$, and (13) and (14) become

$$\frac{2m+4}{2m+3}\sum k^{2m+3} = G^{'}\cdot\left(\sum k\right)^2 - F\cdot\left(\sum k\right)^2,$$

from which the desired result follows. □

4. Concluding Remarks

4.1. Sums in $\left(n+\frac{1}{2}\right)$

In the preceding pages there have been hints that there is a third way to write an expression for the sum of powers of integers, with terms in $\left(n+\frac{1}{2}\right)$. This in fact is true, and can be approached in a number of different ways (Beardon [1, section 3], Burrows and Talbot [3, section 2], Cereceda [4], Hersh [10]). Since we have developed the theory of writing such sums with terms in $\frac{n(n+1)}{2}$, we will proceed in that direction.

Let us start with even powers. We have the identity

$$\left(n+\frac{1}{2}\right)^2 = 2\cdot\frac{n(n+1)}{2}+\frac{1}{4}, \tag{15}$$

which allows us to write

$$\begin{aligned}\sum k &= \frac{n(n+1)}{2} = \frac{1}{2}\left(n+\frac{1}{2}\right)^2-\frac{1}{8},\\ \sum k^2 &= \frac{2n+1}{3}\cdot\frac{n(n+1)}{2} = \left(n+\frac{1}{2}\right)\left(\frac{1}{3}\left(n+\frac{1}{2}\right)^2-\frac{1}{12}\right).\end{aligned} \tag{16}$$

Using the shorthand $N = n+\frac{1}{2}$ we can rewrite (4) as

$$\begin{aligned}\sum k^{2m} &= \left[c_0\left(\frac{1}{2}N^2-\frac{1}{8}\right)^{m-1}+\cdots+c_{m-1}\right]\cdot N\left(\frac{1}{3}N^2-\frac{1}{12}\right)\\ &= \left[c_0^{'}N^{2(m-1)}+\cdots+c_{m-2}^{'}N^2+c_{m-1}^{'}\right]\cdot N\left(\frac{1}{3}N^2-\frac{1}{12}\right)\\ &= \left[c_0^{''}N^{2(m-1)+2}+\cdots+c_{m-2}^{''}N^{2+2}+c_{m-1}^{''}N^2+c_m^{''}\right]\cdot N\\ &= d_0\left(n+\frac{1}{2}\right)^{2m+1}+\cdots+d_{m-1}\left(n+\frac{1}{2}\right)^3+d_m\left(n+\frac{1}{2}\right),\end{aligned} \tag{17}$$

which is an odd polynomial in $\left(n+\frac{1}{2}\right)$ with rational coefficients d_i. For odd powers we can rewrite (5) as

$$\begin{aligned}\sum k^{2m+1} &= \left[a_0\left(\frac{1}{2}N^2-\frac{1}{8}\right)^{m-1}+\cdots+a_{m-1}\right]\left(\frac{1}{2}N^2-\frac{1}{8}\right)^2\\ &= \left[a_0^{'}N^{2(m-1)}+\cdots+a_{m-2}^{'}N^2+a_{m-1}^{'}\right]\left(\frac{1}{4}N^4-\frac{1}{8}N^2+\frac{1}{64}\right)\\ &= e_0\left(n+\frac{1}{2}\right)^{2m+2}+\cdots+e_m\left(n+\frac{1}{2}\right)^2+e_{m+1}, \qquad (18)\end{aligned}$$

which is an even polynomial in $\left(n+\frac{1}{2}\right)$ with rational coefficients e_i.

It is possible to proceed in the opposite direction: start with (17) and (18) and then derive (4) and (5). However, some care needs to be taken with the constant terms. Concerning the coefficients d_i, e_i, we can find explicit values in terms of Bernoulli numbers:

$$\begin{aligned}d_i &= \frac{1}{2\left(m-i\right)+1}\binom{2m}{2i}B_{2i}\left(\frac{1}{2}\right),\\ e_i &= \frac{1}{2\left(m-i\right)+2}\binom{2m+1}{2i}B_{2i}\left(\frac{1}{2}\right),\end{aligned} \qquad (19)$$

where $0 \le i \le m$. We evaluate the Bernoulli polynomial $B_r\left(x\right)$ using

$$B_r\left(\frac{1}{2}\right) = \left(2^{1-r}-1\right)B_r. \qquad (20)$$

For the lone coefficient of e_{m+1} we have

$$e_{m+1} = -\sum_{i=0}^{m}\frac{e_i}{4^{m-i+1}}. \qquad (21)$$

(These results were derived in a slightly different form in Cereceda [4].)

4.2. Partial Sums for Bernoulli Polynomials

Since the expressions for sums of powers of integers have the form of polynomials, it is common to interpret them as such. Previously we used the notation $S_m = \sum_{k=1}^{n} k^m$. We introduce $S_m\left(x\right)$ to designate the polynomial of degree $m+1$ in the real variable x that is obtained by replacing n by x in (3).

In [9, Theorem 2.2], He and Ricci derived the following expression:

$$B_m\left(x\right) = \left(x-\frac{1}{2}\right)B_{m-1}\left(x\right)-\frac{1}{m}\sum_{r=0}^{m-2}\binom{m}{r}B_{m-r}B_r\left(x\right), \qquad (22)$$

where $m \ge 1$, and $B_m\left(x\right)$ refers to the Bernoulli polynomials defined in (6). It followed as a corollary to more general results on Appell polynomials which were derived using differential operators. Here we will use the simpler approach of partial sums to derive an expression for $S_m(x)$ which is equivalent to (22).

Starting with Proposition 1, we can arrive at

$$S_{2m+2} = \frac{1}{2m+3}\left((2m+2)\left(n+\frac{1}{2}\right)S_{2m+1} - \sum_{r=1}^{2m}\binom{2m+2}{r}B_{2m+2-r}S_r\right),$$

which is analogous to (11). (Note: we do not assume $B_{2m+1} = 0$ and we write (3) without the alternating signs.) Likewise, an expression analogous to (13) is

$$S_{2m+3} = \frac{1}{2m+4}\left((2m+3)\left(n+\frac{1}{2}\right)S_{2m+2} - \sum_{r=1}^{2m+1}\binom{2m+3}{r}B_{2m+3-r}S_r\right).$$

Putting them together, we get

$$S_m = \frac{1}{m+1}\left(m\left(n+\frac{1}{2}\right)S_{m-1} - \sum_{r=1}^{m-2}\binom{m}{r}B_{m-r}S_r\right),$$

where $m \geq 2$, with the understanding that for $m = 2$, the sum on the right side is equal to zero. Generalizing to polynomials in x, we arrive at the final result of

$$S_m(x) = \frac{1}{m+1}\left(m\left(x+\frac{1}{2}\right)S_{m-1}(x) - \sum_{r=1}^{m-2}\binom{m}{r}B_{m-r}S_r(x)\right). \qquad (23)$$

Even though we will not give the details of the proof, we point out that (22) and (23) can be shown to be equivalent to each other.

Acknowledgements. The author thanks the referee for a careful reading of the paper, for suggesting the topics in the concluding section, and for pointing out additional references, in particular, the paper by He and Ricci. On a personal note, the author dedicates the paper to the victims of the Coronavirus pandemic.

References

[1] A. F. Beardon, Sums of powers of integers, *Amer. Math. Monthly* **103** (1996), 201–13.

[2] G. M. Boudreaux, What do you get when you cross a power sum with an Iraqi bank note? *Math. Mag.* **82** (2009), 289–293.

[3] B. L. Burrows and R. F. Talbot, Sums of powers of integers, *Amer. Math. Monthly* **91** (1984), 394–403.

[4] J. L. Cereceda, Explicit form of the Faulhaber polynomials, *College Math. J.* **46** (2015), 359–363.

[5] J. L. Cereceda, Bernoulli and Faulhaber, *Fibonacci Quart.* **59** (2021), 145–49.

[6] A. W. F. Edwards, Sums of powers of integers: a little of the history, *Math. Gaz.* **66** (1982), 22-28.

[7] A. W. F. Edwards, A quick route to sums of powers, *Amer. Math. Monthly* **93** (1986), 451–455.

[8] I. M. Gessel and X. G. Viennot, Determinants, paths, and plane partitions, http://people.brandeis.edu/~gessel/homepage/papers/pp.pdf.

[9] M. X. He and P. E. Ricci, Differential equation of Appell polynomials via the factorization method, *J. Comput. Appl. Math.* **139** (2002), 231–237.

[10] R. Hersh, Why the Faulhaber polynomials are sums of even or odd powers of (n + 1/2), *College Math. J.* **43** (2012), 322–324.

[11] V. J. Katz, Ideas of calculus in Islam and India, *Math. Mag.* **68** (1995), 163–174.

[12] D. E. Knuth, Johann Faulhaber and sums of powers, *Math. Comp.* **61** (1993), 277–294.

[13] H. Rademacher, *Topics in Analytic Number Theory*, Springer-Verlag, New York, 1973.

[14] R. Zielinski, Induction and analogy in a problem of finite sums, https://arxiv.org/abs/1608.04006.

[15] R. Zielinski, Faulhaber and Bernoulli, *Fibonacci Quart.* **57** (2019), 32–34.