UNFAIR DISTRIBUTIONS COUNTED BY THE GENERALIZED STIRLING NUMBERS

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Received: 10/2/21, Accepted: 7/28/22, Published: 8/24/22

Abstract

We investigate the generalized Stirling numbers $S(n,k;\alpha,\beta,\gamma)$ introduced by Hsu and Shiue from a combinatorial point of view. We present a combinatorial interpretation in terms of certain restricted distributions of labeled balls into unlabeled cells and a special cell where all cells are divided into distinct compartments. Using our interpretation, we find combinatorial proofs of several identities involving $S(n,k;\alpha,\beta,\gamma)$ and the associated generalized Bell numbers. Connections are made with some prior combinatorial models for the r-Lah numbers and other arrays, one via a sign-changing involution and another through a direct bijection. Finally, an additional parameter is introduced into our model which allows for further generalization.

1. Introduction

The literature on families of Stirling numbers is quite rich with many generalizations and related sequences being introduced and studied from various standpoints. One of the most far reaching of these generalizations is due to Hsu and Shiue [20] whose original motivation was algebraic in nature. This work is devoted to the study of certain combinatorial aspects of these generalized Stirling numbers.

Such a generalization is often an attempt at a unified approach to a topic that permits one to understand various connections. It places disparate items sometimes having arisen in seemingly unrelated studies under one roof, finding their correct place in a larger perspective. On the other hand, the more general a model is, then the more complicated it may become due to additional parameters. It is frequently not easy therefore to strike a balance between the level of generalization and the understandability of a model. Our aim here is to provide further combinatorial insight into the general Hsu-Shiue Stirling number model.

The Hsu-Shiue generalized Stirling numbers were introduced in [20] and are defined, for real or complex α, β, γ, as the pair

$$(z|\alpha)_n = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)(z-\gamma|\beta)_k, \tag{1}$$

$$(z|\beta)_n = \sum_{k=0}^{n} S(n,k;\beta,\alpha,-\gamma)(z+\gamma|\alpha)_k, \tag{2}$$

where $(z|\alpha)_n$ denotes the generalized factorial $(z|\alpha)_n = z(z-\alpha)(z-2\alpha)\cdots(z-n\alpha+\alpha)$ if $n \geq 1$, with $(z|\alpha)_0 = 1$. The following exponential generating function (egf) formula for the array $S(n,k;\alpha,\beta,\gamma)$ is given in [20]:

$$\sum_{n=0}^{\infty} S(n,k;\alpha,\beta,\gamma)\frac{t^n}{n!} = (1+\alpha t)^{\frac{\gamma}{\alpha}}\frac{1}{k!}\left[\frac{(1+\alpha t)^{\frac{\beta}{\alpha}}-1}{\beta}\right]^k. \tag{3}$$

The $S(n,k;\alpha,\beta,\gamma)$ offer a common generalization of several well-known sequences (see [20, 22]), including the r-Stirling, Lah, Carlitz and Howard degenerate Stirling, Gould-Hopper, Todorov and r-Whitney numbers. By an alternative model involving a pit and urn game, the Hsu-Shiue Stirling numbers have recently been afforded a combinatorial interpretation in [25] as the number of ways of placing balls into urns and a pit subject to certain restrictions.

The outline of the current paper is as follows. First, we provide a combinatorial interpretation of the numbers $S(n,k;\alpha,\beta,\gamma)$ in terms of certain restricted distributions of labeled balls in unlabeled cells with compartments that generalizes the formulation given in [14] (see also [13]). Our model applies to cases where β and γ are non-negative integers and where either α is positive and divides both β and γ or $\alpha \leq 0$. This interpretation may be extended to the generalized Bell polynomials $S_n(x;\alpha,\beta,\gamma)$ introduced in [20] and defined as

$$S_n(x;\alpha,\beta,\gamma) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)x^k. \tag{4}$$

Combinatorial proofs are found in terms of our model for several identities involving $S(n,k;\alpha,\beta,\gamma)$ and/or $S_n(x;\alpha,\beta,\gamma)$, among them a generalized version of Spivey's Bell number formula. Connections are made to some prior models, including ones related to extended Lah distributions, r-Lah numbers and multiparameter Stirling numbers. Finally, we introduce a new parameter h into our model and briefly discuss its combinatorial significance.

2. Unfair Distributions

In this section, we present the underlying combinatorial model. We describe this model in different languages and invite the reader to switch between these descriptions according to their own taste. The goal of these various descriptions is to reveal further the nature of the generalized Stirling numbers and to highlight the roles played by the different parameters. For these reasons, we describe the models in detail as such a preparation supports the proofs of the identities that follow.

2.1. Distribution of Balls into Cells with Compartments

The first description is based on a model of distributing balls into compartmentalized cells.

Suppose we wish to distribute one at a time some distinguishable balls into cells that are divided further into distinct compartments. A ball will be said to either *close* some consecutive compartments or *create* some new compartments. This property is fixed by the parameter α. If α is a positive integer, each ball that is inserted closes the compartment receiving the ball *and* the next $\alpha-1$ compartments in the cell, in the sense that no more balls can be placed in these compartments. If α is a negative integer, each ball creates (or opens) $-\alpha+1$ new compartments next to the compartment to which it belongs which is itself closed. If $\alpha = 0$, then compartments are allowed to contain more than one ball. The compartments within a cell may be thought of as being cyclically ordered.

Note that these three cases for α are in accordance with the combinatorial models of x^n, $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ and $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ as follows: x^n counts the number of ways of placing n distinguishable balls into x distinguishable cells; $x^{\underline{n}}$ counts the number of ways of placing n distinguishable balls into x distinguishable cells such that each cell contains at most one ball; $x^{\overline{n}}$ counts the number of ways of placing n distinguishable flags on x distinguishable flagpoles. See the beautiful paper of Joni et al. [21] for a detailed description. These classical cases correspond to $\alpha = 0$, $\alpha = 1$ and $\alpha = -1$, respectively.

For $\alpha \geq 1$, we will refer to α available consecutive compartments as an *α-block*. An inserted ball may then be thought of as closing an α-block which depends upon the point of insertion. For example, if the first ball in a cell with c compartments where $\alpha|c$ goes into compartment x, then the α-block comprising the compartments $x, x+1, \ldots x+\alpha-1$ would be closed where addition is done mod c (as compartments are arranged cyclicly). If the second ball were then to go in compartment $x-2$, then the α-block $x-2, x-1, x+\alpha, x+\alpha+1, \ldots, x+2\alpha-3$ would be closed and so on for additional balls.

Now say that in a cell we have a "favorite" α-block, denoted by α', and the first ball that is inserted into a cell always goes into α' in a particular compartment, i.e., the "favorite" compartment (this is an extension of the interpretation presented in

[14], where also the cells were assumed to be labeled). This justifies our calling such distributions "unfair".

Assume that we distribute in this way n labeled balls into a cell that has β compartments, where β is assumed to be a multiple of α. How many possibilities are there? The position of the first ball is fixed (our favorite compartment), for the next ball there are $\beta - \alpha$ compartments available, and so on, until the n-th ball is placed for which there are $\beta - (n-1)\alpha$ available compartments. Thus, we have $\frac{(\beta|\alpha)_n}{\beta} = (\beta-\alpha|\alpha)_{n-1}$ possibilities. Let a_n denote the number of ways of distributing n balls into a single cell in the manner described above and let $A(t) = \sum_{n=1}^{\infty} a_n \frac{t^n}{n!}$ be the corresponding egf. Then we have

$$A(t) = \frac{(1+\alpha t)^{\frac{\beta}{\alpha}} - 1}{\beta}.$$

Consider now $k+1$ cells with the following properties:

1. The first k cells each have β compartments and are nonempty. We call these cells *ordinary cells*.
2. The last cell has γ compartments where $\alpha|\gamma$ and may be empty. We call this cell a *special cell*.

Let $S(n, k; \alpha, \beta, \gamma)$ denote the number of ways of placing n labeled balls into these $k+1$ cells as described. For example, if $n = 3$, $k = 1$, $\alpha = 2$, $\beta = 8$ and $\gamma = 4$, then there are 24, 72 and 24 possibilities for $\ell = 0, 1, 2$, respectively, where ℓ denotes the number of balls that are placed in the special cell, whence $S(3, 1; 2, 8, 4) = 24 + 72 + 24 = 120$. This agrees with the value found by formula (12) below. For the sake of simplicity, we will refer to a distribution of balls into cells as described above as an (α, β, γ)-*partition*.

By the construction above, we have that $S(n, k; \alpha, \beta, \gamma)$ has egf formula given by

$$\sum_{n=0}^{\infty} S(n, k; \alpha, \beta, \gamma) \frac{t^n}{n!} = (1+\alpha t)^{\frac{\gamma}{\alpha}} \frac{1}{k!} \left[\frac{(1+\alpha t)^{\frac{\beta}{\alpha}} - 1}{\beta} \right]^k.$$

Remark 1. The special cell corresponds to the parameter r in models for the r-Stirling, r-Lah and r-Whitney numbers, among others. For instance, the r-Stirling numbers of the second kind enumerate partitions of $[n]$ into k nonempty blocks such that the elements of $[r]$ belong to distinct blocks. In terms of our model, this corresponds to a $(0, 1, r)$-partition of $n - r$ balls where the r compartments in the special cell can be regarded as each already containing a ball.

2.2. Reformulation: Boxes

To motivate this reformulation, note the following identity:

$$\frac{(\beta|\alpha)_n}{\beta} = \frac{n!}{\beta}\binom{\beta/\alpha}{n}\alpha^n. \tag{5}$$

The right-hand side of the formula (5) suggests combinatorially the following. Suppose we have $\frac{\beta}{\alpha}$ boxes with α compartments each and a favorite compartment (out of all the boxes). We then distribute n labeled balls into the boxes such that at most one ball can be placed in each box in any one of its compartments, where the favorite compartment is always occupied when at least one box is nonempty. As a consequence, we may conclude

$$S(n,k;\alpha,\beta,\gamma) = \sum \binom{n}{n_0, n_1, \ldots, n_k} \frac{1}{k!} \prod_{i=1}^{k} \frac{(\beta|\alpha)_{n_i}}{\beta} (\gamma|\alpha)_{n_0}, \tag{6}$$

where $\binom{n}{n_0,n_1,\ldots,n_k}$ denotes a multinomial coefficient and the sum is over all integers $n_1, n_2, \ldots, n_k \geq 1$ with $n_0 \geq 0$ such that $n_0 + n_1 + \cdots + n_k = n$.

In other words, $S(n,k;\alpha,\beta,\gamma)$ counts the partitions of n balls into k nonempty groups of $\frac{\beta}{\alpha}$ boxes each (called ordinary groups), and a special group with $\frac{\gamma}{\alpha}$ boxes that may be empty, in the manner described above. We take $S(n,k;\alpha,\beta,\gamma)$ to be zero if $k < 0$ or $k > n$, which is in accordance with our combinatorial interpretations of this array.

2.3. Reformulation: Colored Partitions

Assume we have $\alpha \geq 1$, with $\alpha|\beta$ and $\alpha|\gamma$. The notion of compartments in a cell may be replaced/exchanged with the notion of colored elements. Suppose we have a partition of n elements into k nonempty ordinary blocks and a special block. The elements are colored in the following way. For each color, there are α shades of the hue; for instance, instead of having a single color "red", we have a set of colors, e.g., $\{\text{pink}, \text{ruby}, \text{burgundy}, \text{crimson}\}$. Elements within the blocks are colored subject to the following rules.

1. In an ordinary block, there are β colors available altogether (counting all of the possible shades), while in the special block, there are γ colors altogether.

2. Only one shade from a given color set can be used to color an element in a block.

3. We have in each ordinary block a favorite color, with the first element in the block always receiving that color.

Let us call such a partition an (α,β,γ)-*colored partition.* It is clear that the number of (α,β,γ)-colored partitions of n elements into k ordinary blocks and a special block is given by $S(n,k;\alpha,\beta,\gamma)$.

3. Combinatorial Proofs of Identities

In this section, we demonstrate by providing explanations of several identities the potential usefulness of our combinatorial interpretation. The identities are known in the literature as noted, however the elementary proofs we present here are new.

First, we show the basic recurrence relation.

Theorem 1 ([20]). *For n non-negative and k, α, β, γ positive integers such that $\alpha|\beta$ and $\alpha|\gamma$, the following identity holds:*

$$S(n+1,k;\alpha,\beta,\gamma) = S(n,k-1;\alpha,\beta,\gamma) + (k\beta - n\alpha + \gamma)S(n,k;\alpha,\beta,\gamma). \tag{7}$$

Proof. The proof is based on the placement of the element $n+1$.

Case 1: The element $n+1$ is a singleton, i.e., in the cell containing it there are no other elements. The members of $[n]$ can be arranged in $S(n,k-1;\alpha,\beta,\gamma)$ ways. Suppose initially a cell having β compartments is empty. The unfairness property, i.e., the property that the first element in a cell be placed in the favorite compartment which then closes α compartments, implies that the position of $n+1$ is fixed in this case. So if $n+1$ is a singleton, we have $S(n,k-1;\alpha,\beta,\gamma)$ ways in which to construct a partition.

Case 2 : The element $n+1$ is in a cell together with at least one other element. The elements of $[n]$ can be partitioned in $S(n,k;\alpha,\beta,\gamma)$ ways into k ordinary cells and a special cell. After placing the elements of $[n]$ into these $k+1$ cells, how many free compartments are still available? There are $k\beta+\gamma$ in total, with each of the n prior elements having closed α compartments. Hence, the number of compartments available is $k\beta - n\alpha + \gamma$, where we may assume here $n\alpha \leq k\beta + \gamma$ (for otherwise, $S(n,k;\alpha,\beta,\gamma) = 0$ making the contribution from this case zero automatically). Thus, the element $n+1$ can be inserted in $k\beta - n\alpha + \gamma$ ways into the given partition, which completes the proof. □

Next, we present the proof of a convolution formula.

Theorem 2 ([20]). *The identity*

$$\binom{k_1+k_2}{k_1} S(n,k_1+k_2;\alpha,\beta,\gamma_1+\gamma_2) = \sum_{m=0}^{n} \binom{n}{m} S(m,k_1;\alpha,\beta,\gamma_1) S(n-m,k_2;\alpha,\beta,\gamma_2) \tag{8}$$

holds for n, k_1, k_2 non-negative and α, β, γ_1, γ_2 positive integers such that $\alpha|\beta$, $\alpha|\gamma_1$, and $\alpha|\gamma_2$.

Proof. Consider an (α,β,γ)-partition of $[n]$ having k $(k = k_1+k_2)$ ordinary cells such that the special cell has $\gamma_1+\gamma_2$ compartments. Mark some ordinary cells, say k_1 of them. Clearly, the left-hand side counts these marked partitions. On the other hand, one can also obtain such partitions as follows. Let $\mathcal{S}$ denote the set of elements that either occur in one of the first γ_1 compartments or are contained in one of the

k_1 marked cells. Let $|\mathcal{S}| = m$. Choose m members of $[n]$ and construct an $(\alpha, \beta, \gamma_1)$-partition into k_1 ordinary cells and a special cell with γ_1 compartments. Mark the ordinary cells in this partition. Using the remaining $n - m$ elements, construct an $(\alpha, \beta, \gamma_2)$-partition into k_2 ordinary cells and a special cell with γ_2 compartments. Considering all possible values of m, we obtain the right-hand side. □

We also can give a simple proof of the following recursion on the parameter n.

Theorem 3 ([11])**.** *For n, k, α, β, γ positive integers such that $\alpha|\beta$ and $\alpha|\gamma$, the following identity holds:*

$$kS(n, k; \alpha, \beta, \gamma) = \sum_{j=0}^{n-1} \binom{n}{j} (\beta - \alpha|\alpha)_{n-j-1} S(j, k-1; \alpha, \beta, \gamma). \tag{9}$$

Proof. Consider an (α, β, γ)-partition of $[n]$ into k ordinary cells and a special cell. Mark an ordinary cell. The number of ways of doing this is clearly counted by the left-hand side. On the other hand, let $n - j$ be the number of elements that are contained in the marked cell. We can construct this cell in $\binom{n}{n-j}(\beta - \alpha|\alpha)_{n-j-1}$ ways. We partition the remaining j elements into $k - 1$ ordinary cells and a special cell in $S(j, k-1; \alpha, \beta, \gamma)$ ways. □

Note that (9) corresponds to the special case of (8) when $k_1 = k - 1$, $k_2 = 1$, $\gamma_1 = \gamma$ and $\gamma_2 = 0$, as $S(n - m, 1; \alpha, \beta, 0) = (\beta - \alpha|\alpha)_{n-m-1}$ if $m \leq n - 1$.

In the combinatorial proof of the next identity, we use the reformulation in terms of boxes.

Theorem 4 ([11])**.** *For n, k non-negative and α, β, γ positive integers such that $\alpha|\beta$ and $\alpha|\gamma$, we have*

$$\sum_{j=0}^{k} k^{\underline{j}} \beta^j S(n, j; \alpha, \beta, \gamma) = (k\beta + \gamma|\alpha)_n. \tag{10}$$

Proof. The right-hand side gives the number of distributions of n labeled balls into $(\beta/\alpha)k + (\gamma/\alpha)$ labeled boxes each having α distinct compartments with each box restricted to having at most one ball. How do we get this from the left-hand side?

Imagine that there are k groups of β/α boxes and another group with γ/α boxes. Let j be the number out of the k groups of boxes that contain at least one ball (within all boxes in the group). First, choose these j groups in $k^{\underline{j}}$ ways and then designate one particular compartment out of each of the chosen groups of boxes to be the favorite compartment, which can be done in β^j ways.

After this preparation, consider a distribution of the n labeled balls into the j selected groups of β/α boxes, along with the special group of γ/α boxes, such that at least one ball goes into each of the j selected groups with the first ball always going into the favorite compartment for the group. Thus, there are $k^{\underline{j}}\beta^j S(n, j; \alpha, \beta, \gamma)$ possible distributions in which exactly j of the k groups of β/α boxes contain at least one ball for each j. Summing over all j, we obtain the identity. □

In order to provide a combinatorial proof of the explicit formula for $S(n, k; \alpha, \beta, \gamma)$ given in Theorem 5 below, we first express $(k\beta + \gamma|\alpha)_n$ in a different way.

Lemma 1. *For n non-negative and α, β, γ positive integers such that $\alpha|\beta$ and $\alpha|\gamma$, we have*

$$(k\beta + \gamma|\alpha)_n = n!\binom{(\beta/\alpha)k + (\gamma/\alpha)}{n}\alpha^n. \tag{11}$$

Proof. The left-hand side of (11) gives the number of ways of placing the elements of $[n]$ into k ordinary cells and a special cell, such that whenever a ball is placed in a cell into any one of the compartments $q\alpha + 1, q\alpha + 2, \ldots, (q + 1)\alpha$ for some $q \geq 0$, then all of the compartments within this group are closed (we will refer to such a group in this setting as an α-block). Note that here (and in the proof of the subsequent theorem) the ordinary cells are labeled and need not contain an element, with the first element not required to go in a fixed compartment as before.

Alternatively, first consider the n groups of compartments into which we will place the elements. In an ordinary cell, the number of α-blocks is given by β/α, whereas in the special cell, there are γ/α. In total for the $k + 1$ cells, there are $(\beta/\alpha)k + (\gamma/\alpha)$ possible α-blocks. Thus, there are $\binom{(\beta/\alpha)k+(\gamma/\alpha)}{n}$ possibilities for the n groups of compartments. Then select in α^n ways the particular compartment within each group that will contain an element. Finally, after choosing the compartments, we arrange the elements in an arbitrary way, which gives $n!$. □

Theorem 5 ([15])**.** *For n, k non-negative and α, β, γ positive integers such that $\alpha|\beta$ and $\alpha|\gamma$, we have*

$$S(n, k; \alpha, \beta, \gamma) = \frac{n!\alpha^n}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{(\beta/\alpha)j + (\gamma/\alpha)}{n}. \tag{12}$$

Proof. Suppose that there are $k + 1$ labeled cells, where k of the cells have β compartments (ordinary cells) and one has γ compartments (special cell), such that α divides both β and γ. We distribute the elements of $[n]$ into these cells as described in the preceding lemma. Suppose that in a distribution of the elements of $[n]$ into the $k + 1$ cells that $k - j$ stipulated ordinary cells (and possibly some others) do not receive any elements for some $0 \leq j \leq k$.

By the prior lemma, there are $\binom{(\beta/\alpha)j+(\gamma/\alpha)}{n}n!\alpha^n$ such distributions possible. Thus, by the inclusion/exclusion principle, we have that

$$n!\alpha^n\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{(\beta/\alpha)j + (\gamma/\alpha)}{n}$$

gives the total number of distributions in which no ordinary cell is left empty. On the other hand, there are $k!\beta^k S(n,k;\alpha,\beta,\gamma)$ such distributions, where the β^k factor accounts for the fact that the first element in an ordinary cell need not go in a fixed compartment here and $k!$ accounts for the ordinary cells now being labeled. Equating results then gives

$$k!\beta^k S(n,k;\alpha,\beta,\gamma) = n!\alpha^n \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{(\beta/\alpha)j + (\gamma/\alpha)}{n},$$

which implies (12). □

Remark 2. The proofs for the identities in this section where there are the stated restrictions on α, β and γ imply the results for all α, β and γ, real or complex, since both sides of each identity are seen to be polynomials in these variables. A similar remark applies to the identities in the next section.

4. Combinatorial Interpretation of Generalized Bell Polynomials

In accordance with their generalized Stirling numbers, Hsu and Shiue [20] introduced the generalized Bell polynomials as

$$S_n(x;\alpha,\beta,\gamma) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)x^k, \qquad n \geq 0. \tag{13}$$

Members of the sequence $S_n(1;\alpha,\beta,\gamma)$ are referred to as *generalized Bell numbers*. One recovers many well-known number sequences as special cases of $S_n(1;\alpha,\beta,\gamma)$, such as $n! = S_n(1;-1,0,0)$, the Bell numbers $B_n = S_n(1;0,1,0)$, the r-Bell numbers [30] $B_{n,r} = S_n(1;0,1,r)$, the Dowling numbers [4] $D_n = S_n(1;0,m,1)$, etc. (see, e.g., [20]).

There is the following egf formula for $S_n(x;\alpha,\beta,\gamma)$ which readily follows from (3) and (13):

$$\sum_{n=0}^{\infty} S_n(x;\alpha,\beta,\gamma)\frac{t^n}{n!} = (1+\alpha t)^{\frac{\gamma}{\alpha}} \exp\left[\frac{x(1+\alpha t)^{\frac{\beta}{\alpha}} - x}{\beta}\right]. \tag{14}$$

It is clear that in order to interpret $S_n(x;\alpha,\beta,\gamma)$, we only need to augment our model by allowing for the ordinary cells to come in x given colors, where x is a positive integer. Alternatively, one may define the weight $w(\rho)$ of an (α,β,γ)-partition ρ by $w(\rho) = x^i$, where i is the number of the ordinary cells of ρ and x is an indeterminate.

In this direction, we provide here combinatorial proofs of some identities involving $S_n(x;\alpha,\beta,\gamma)$.

Theorem 6. *For n non-negative and α, β, γ, x positive integers with $\alpha|\beta$ and $\alpha|\gamma$, the following identity holds:*

$$S_{n+1}(x;\alpha,\beta,\gamma) = \gamma S_n(x;\alpha,\beta,\gamma-\alpha) + \sum_{j=0}^{n} x\binom{n}{j}(\beta-\alpha|\alpha)_j S_{n-j}(x;\alpha,\beta,\gamma). \quad (15)$$

Proof. This can be shown by computing the egf of both sides of (15) using (14). Alternatively, one can give a combinatorial explanation using the current model based on the position of the smallest element, denoted by 1, within a partition of size $n+1$. There are two cases to consider.

Case 1: The 1 belongs to the special cell. Then the number of possible partitions is $\gamma S_n(x;\alpha,\beta,\gamma-\alpha)$ since α compartments in the special cell would then be closed to the other elements.

Case 2: The 1 belongs to an ordinary cell, say B. We choose a color for B in x ways. Let $j+1$ be the cardinality of B. Then there are $\binom{n}{j}$ choices concerning the j other elements of B, which can be arranged in $(\beta-\alpha|\alpha)_j$ ways within B. The remaining $n-j$ elements in the partition can be arranged in $S_{n-j}(x;\alpha,\beta,\gamma)$ ways. Considering all possible j then accounts for the sum on the right-hand side of (15) and completes the proof. □

Let $S_n(x) = S_n(x;\alpha,\beta,\gamma)$ and $\frac{d}{dx}S_n(x) = S_n'(x)$. The next recursion relates $S_{n+1}(x)$ to the preceding term and its derivative.

Theorem 7 ([23])**.** *For n non-negative and α, β, γ, x positive integers with $\alpha|\beta$ and $\alpha|\gamma$, we have*

$$S_{n+1}(x) = (\gamma - n\alpha + x)S_n(x) + \beta x S_n'(x). \quad (16)$$

Proof. The statement follows from the basic recurrence for $S(n,k;\alpha,\beta,\gamma)$ found in Theorem 1 and the definition of generalized Bell numbers given in (13).

We also provide a combinatorial explanation as follows. First, note that $xS_n'(x)$ gives the number of marked (α,β,γ)-partitions, wherein the ordinary cells are each assigned one of x possible colors and one of these cells is marked.

Within an (α,β,γ)-partition counted by $S_{n+1}(x)$, the $(n+1)$-st element can either occur in the special cell, in an ordinary cell with at least one other element, or in a new ordinary cell by itself. In the first case, we need to choose one of γ compartments, and in the second, one of β compartments, while in the third case no compartment needs to be chosen since the $(n+1)$-st element would be the first element in an ordinary cell (though we need to choose a color for the new cell). This gives

$$\gamma S_n(x) + \beta x S_n'(x) + xS_n(x).$$

However, since each element "closes" α available compartments, we have counted in the previous sum $n\alpha S_n(x)$ cases that are not allowed, and hence must be subtracted. □

Theorem 8 ([13])**.** *For n non-negative and α, β, γ, x positive integers with $\alpha|\beta$ and $\alpha|\gamma$, the following identity holds:*

$$S_{n+1}(x) + n\alpha S_n(x) = \gamma S_n(x) + \sum_{m=0}^{n} xm!\binom{n}{m}\binom{\beta/\alpha}{m}\alpha^m S_{n-m}(x). \qquad (17)$$

Proof. We extend the set of (α, β, γ)-partitions as follows. Let $\mathcal{U}_n$ denote the set of (α, β, γ)-partitions enumerated by $S_n(x)$ and let $\mathcal{V}_n$ be the same as $\mathcal{U}_n$ except that, in addition, the element 1 may lie in the same α-block (possibly in the same compartment) as a member of $[2, n]$. Note however that elements i and j must lie in different α-blocks within a member of $\mathcal{V}_n$ for all $i > j > 1$. Further, the element 1 within a member of $\mathcal{V}_n$ always goes in the favorite compartment when placed in an ordinary cell as before.

Here, in forming members of $\mathcal{U}_n$, we insert the elements of $[n]$ such that whenever an element is inserted into a cell in any one of the compartments labeled $q\alpha + 1, \ldots, (q + 1)\alpha$ for some $q \geq 0$, then all compartments with these labels (which we refer to again as an α-block) are closed. This formulation for members of $\mathcal{U}_n$ is seen to be equivalent to the one described in terms of colored partitions in subsection 2.3. Note however that the element 1 within a member of $\mathcal{V}_n$ does not close any compartments within the cell to which it is added.

Then the right-hand side of (17) enumerates the members of $\mathcal{V}_{n+1}$ by considering the position of the element 1 as follows. If 1 goes in the special cell, then there are $\gamma S_n(x)$ possibilities since 1 can go in any compartment with no restrictions. Otherwise, 1 belongs to an ordinary cell together with m members of $[2, n + 1]$ for some $0 \leq m \leq n$. Then there are $\binom{n}{m}$ ways to choose these members and $m!\binom{\beta/\alpha}{m}\alpha^m$ ways in which to arrange them within an ordinary cell such that no two lie in the same α-block. Then add the element 1 to the favorite compartment of the cell, regardless of whether or not an element already lies in the favorite α-block. The remaining members of $[n + 1]$ are arranged in $S_{n-m}(x)$ ways, with the extra factor of x accounting for the cell containing 1. Considering all possible m then implies that the right side of (17) gives $|\mathcal{V}_{n+1}|$, as claimed.

To complete the proof, we then must show $|\mathcal{V}_{n+1} - \mathcal{U}_{n+1}| = n\alpha S_n(x)$ since $|\mathcal{U}_{n+1}| = S_{n+1}(x)$. Let $\lambda \in \mathcal{U}_n$, represented using elements of $[2, n + 1]$ instead of $[n]$. We then select any $i \in [2, n + 1]$ and place the element 1 in one of the compartments lying within the α-block of λ that contains i. Let λ^* denote the resulting distribution of $[n + 1]$. Clearly, there are $n\alpha S_n(x)$ distinct λ^* each arising uniquely as λ ranges over all members of $\mathcal{U}_n$. Let $\mathcal{U}_n^*$ denote the set of all such distributions λ^*.

We now define a bijection f between $\mathcal{U}_n^*$ and $\mathcal{V}_{n+1} - \mathcal{U}_{n+1}$ as follows, which will complete the proof. Let $\rho \in \mathcal{U}_n^*$. If 1 lies within the special cell of ρ, then let $f(\rho) = \rho$. Otherwise, 1 belongs to some ordinary cell B of ρ and we let $\ell = \min(B - \{1\})$. We consider cases based on the relative positions of 1 and ℓ. If 1 lies in the same compartment as ℓ, then let $f(\rho) = \rho$. If 1 lies in the same α-block as ℓ, but in a different compartment, then let $f(\rho)$ be obtained from ρ by switching the elements 1 and ℓ within their compartments (leaving all other elements undisturbed).

The remaining case is for 1 to lie in an α-block of B containing some element $t > \ell$. In this case, let R and S denote the α-blocks of B containing ℓ and $\{1, t\}$, respectively. Suppose further that ℓ occupies compartment r of R, with 1 and t lying in compartments s_1 and s_2 of S. Note that $s_1 = s_2$ is possible, with r corresponding to the favorite compartment of B since ℓ is minimal in $B - \{1\}$.

To obtain $f(\rho)$ in this case, we switch the elements 1 and ℓ and move the element t so that it occupies compartment $r + s_2 - s_1$ in R, where the compartments in the favorite α-block are assumed to be labeled $1, 2, \ldots, \alpha$ and the addition/subtraction is done modulo α on the set $[\alpha]$. Note that within $f(\rho)$, the element 1 now lies in the favorite α-block in B together with the element t.

To reverse f in the case when 1 lies in an ordinary cell C of $\pi \in \mathcal{V}_{n+1} - \mathcal{U}_{n+1}$, consider cases on whether or not 1 lies in the same α-block as the minimum element of $C - \{1\}$. One may then verify that f is reversible in all cases and hence a bijection between $\mathcal{U}_n^*$ and $\mathcal{V}_{n+1} - \mathcal{U}_{n+1}$, as desired. $\square$

To conclude this section, we consider the generalization of Spivey's Bell formula in terms of $S_n(x)$ (for Spivey's original formula, see [35]). Prior proofs of this formula have been given using various methods; see, e.g., [23, 27, 34, 38]. This identity is also mentioned in [12, 22, 26, 29].

Theorem 9 ([38])**.** *For n, m non-negative and α, β, γ, x positive integers with $\alpha|\beta$ and $\alpha|\gamma$, the following identity holds:*

$$S_{n+m}(x) = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} S(m, j; \alpha, \beta, \gamma)(j\beta - m\alpha|\alpha)_{n-k} S_k(x) x^j. \tag{18}$$

Proof. In order to show (18) using the present model, we slightly reformulate our interpretation for $S(n, k; \alpha, \beta, \gamma)$ as follows. Consider placing n labeled balls in k unlabeled ordinary boxes and a special box such that each ordinary box contains $\frac{\beta}{\alpha}$ labeled cells and the special box $\frac{\gamma}{\alpha}$ cells. All cells in all boxes contain α labeled compartments. Balls are distributed so that every ordinary box receives at least one ball, where the first ball in a box always goes in the first compartment of the first cell. Further, each cell in all boxes can receive at most one ball (to be distributed in one of the compartments contained therein). Note that the special box need not receive any balls in its cells, with no restriction as to the placement of the first ball.

We now proceed with the proof of (18). Assume further $\beta, \gamma \geq m\alpha$. We count (α, β, γ)-partitions of $n+m$ labeled balls into ordinary boxes and a special box as described above, where in addition the ordinary boxes receive one of x colors. We first partition the balls labeled with elements of $[m]$ into exactly $j \geq 1$ ordinary boxes and the special box, which can effected in $x^j S(m, j; \alpha, \beta, \gamma)$ ways.

Suppose now that exactly m' balls in this partition were placed in the special box for some $0 \leq m' \leq m-j$. Now consider arranging the j currently occupied ordinary boxes from left-to-right in ascending order of the smallest labeled balls contained therein. In a left-to-right scan of these boxes, we mark the first m' cells that are encountered which do not contain a ball, where the cells within each box are arranged linearly one after another in ascending order. Next, select $n-k$ balls from those with labels in $[m+1, m+n]$. We insert these balls into the j already occupied ordinary boxes such that in addition to the cells already containing a ball, no ball can be placed in one of the m' marked cells. Then $\beta \geq m\alpha$ implies that this can be implemented in $(j\beta - m\alpha|\alpha)_{n-k}$ ways as now there are $(m-m') + m' = m$ forbidden cells at the onset.

We then form an (α, β, γ)-partition with the k remaining balls such that balls selected for the special box either go in one of the $\frac{\gamma}{\alpha} - m'$ cells not already containing a ball with a label in $[m]$ or in one of the m' marked cells within the first j ordinary boxes. Note that all other balls are to go into new ordinary boxes and thus there are $S_k(x)$ possibilities for the placement of the remaining k balls. Considering all possible $j \geq 1$ and k then yields all partitions of the $n+m$ balls in which at least one ordinary box contains a ball from $[m]$.

Otherwise, all of these balls lie in the special box and there are $S(m, 0; \alpha, \beta, \gamma)$ possibilities for their placement. There are then $S_n(x; \alpha, \beta, \gamma - m\alpha)$ ways in which to arrange the balls with labels from $[m+1, m+n]$. Thus one gets the terms from the right side of (18) for which $j = 0$ since there is the identity

$$S_n(x; \alpha, \beta, \gamma - m\alpha) = \sum_{k=0}^{n} \binom{n}{k} (-m\alpha|\alpha)_{n-k} S_k(x), \tag{19}$$

which can be shown by use of generating functions. For completeness, we provide below a combinatorial proof of (19) using the present model.

This finishes the proof of (18) in the case when $\beta, \gamma \geq m\alpha$. Since both sides of (18) are polynomials in α, β, γ, the proof is complete. □

To prove (19) when α, β, γ are integral, we write the $(-m\alpha|\alpha)_{n-k}$ factor as $(-\alpha)^{n-k} m^{\overline{n-k}}$.

Lemma 2. *For n, m non-negative and α, β, γ, x positive integers with $\alpha|\gamma$, $\alpha|\beta$ and $\gamma \geq m\alpha$, the following identity holds:*

$$S_n(x; \alpha, \beta, \gamma - m\alpha) = \sum_{k=0}^{n} (-\alpha)^{n-k} \binom{n}{k} m^{\overline{n-k}} S_k(x). \tag{20}$$

Proof. Given $0 \leq k \leq n$, let $\mathcal{A}_{n,k}$ denote the set of ordered triples $\lambda = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ defined as follows. Let $\mathbf{a}$ be any subset of $[n]$ of size $n-k$, $\mathbf{b}$ be a distribution of the elements of $\mathbf{a}$ into m urns labeled $1, \ldots, m$ where the order of the elements within an urn matters and $\mathbf{c}$ be an (α, β, γ)-partition of the remaining k elements of $[n]$ not belonging to $\mathbf{a}$ (as enumerated by $S_k(x; \alpha, \beta, \gamma)$ per the first interpretation above). Further, assume that elements of $\mathbf{a}$ each receive one of α colors, with the sign of λ defined as $(-1)^{n-k}$. Let $\mathcal{A} = \cup_{k=0}^{n} \mathcal{A}_{n,k}$. Then the right side of (20) is seen to give the sum of the signs of all members of $\mathcal{A}$.

Assume further that the compartments in the special cell within $\mathbf{c}$ are labeled $1, \ldots, \gamma$, with an α-block comprising a set of consecutive compartments $(j-1)\alpha + 1, \ldots, j\alpha$ for some $j \geq 1$. Also, we denote the colors assigned to members of $\mathbf{a}$ by elements of $[\alpha]$.

We now define an involution on $\mathcal{A}$ as follows. Suppose i_0 is the smallest $i \in [m]$ such that one of the following occurs (possibly both) within $\lambda \in \mathcal{A}$:

(I) urn i receives one or more elements from $[n]$ in the distribution $\mathbf{b}$, or

(II) the α-block in the special cell comprising compartments $[(i-1)\alpha + 1, i\alpha]$ receives some element of $[n]$ in the partition $\mathbf{c}$.

Define S to be the set of elements of $[n]$ belonging either to (i) urn i_0 in $\mathbf{b}$ or (ii) the α-block $[(i_0-1)\alpha + 1, i_0\alpha]$ in $\mathbf{c}$. Note that the latter option applies to at most one element of S, with S always nonempty (by the definition of i_0). Define the *initial* element of S to be the one for which (ii) applies if it exists, or to the leftmost element within box i_0 otherwise.

We then switch options concerning the initial element $s \in S$. That is, if s belongs to the p-th compartment of the α-block $[(i_0-1)\alpha + 1, i_0\alpha]$ in $\mathbf{c}$, then we move s so that it comes at the beginning of the list of elements in urn i_0 in $\mathbf{b}$ and assign s the color $p \in [\alpha]$, and vice versa if s occurs in $\mathbf{b}$.

Since i_0 is invariant, this operation yields a sign-changing involution on $\mathcal{A}$ which is not defined for those λ where i_0 fails to exist, i.e., for $\lambda \in \mathcal{A}_{n,n}$ such that no element of $[n]$ belongs to the first m α-blocks in $\mathbf{c}$. Each such λ then has positive sign and they number $S_n(x; \alpha, \beta, \gamma - m\alpha)$, which completes the proof. $\square$

5. Relations to Other Models

As mentioned earlier, since Stirling numbers occur in many different contexts, it is not surprising that there are several natural generalizations arising in different lines of research. We believe it to be worth demonstrating the connections between some of these generalizations.

5.1. Equivalence with Extended Lah Distributions

In this subsection, we relate our model to one that was considered initially by Mansour et al. [28] in conjunction with the notion of normal ordering.

We recall here the discrete structure defined in Shattuck [34] and termed an *extended Lah distribution.*

Given $0 \leq k \leq n$, let $\mathcal{L}_{n,k}$ denote the set of partitions of $[n]$ into k contents-ordered blocks, i.e., lists. We say that the blocks are in *canonical order* when they are arranged from left to right in ascending order of their respective smallest elements.

We refer to the elements of the set $\mathcal{L}_{n,k}$ as *Lah distributions.* Let $\mathcal{L}_n = \cup_{k=0}^{n} \mathcal{L}_{n,k}$. In order to define extended Lah distributions, we distinguish some elements as follows. Let us call an element $i \in [n]$ within $\rho \in \mathcal{L}_n$ *outstanding* if $i = 1$ or $i \geq 2$ and i is not the smallest element in its block and all smaller elements are to the left of i in the canonical order. An *extended Lah distribution* is a Lah distribution wherein we may mark some subset (possibly empty) of the outstanding elements such that 1 can be marked only when it is the leading element of its block. We denote the set of extended Lah distributions by $\mathcal{L}_n^*$.

Let $\mathcal{L}_{n,k}^*$ denote the set of extended Lah distributions in which there are k nonempty blocks not containing a marked 1. We will refer to a block starting with a marked 1 as a *false block*, while all other blocks are *true.* Thus, members of $\mathcal{L}_{n,k}^*$ contain exactly k true blocks.

In order to incorporate the parameters into the model we need also some statistics. We say that $i \in [n]$ is a *record low* of the Lah distribution ρ if there are no smaller elements to the left of i in its block. Note that the first and the minimum elements of a block are always record lows. Let $\mathrm{nrec}(\rho)$ denote the number of elements that do not correspond to record lows of ρ, i.e., there is at least one smaller element to the left of these elements in their respective blocks.

Suppose now that α, β and γ are fixed positive integers such that $\beta, \gamma \geq n\alpha$ with $\alpha|\beta$ and $\alpha|\gamma$. Let $\mathcal{M}_{n,k}$ denote the set obtained from members of $\mathcal{L}_{n,k}^*$ by assigning one of α colors to each unmarked element of $[n]$ not corresponding to a record low of some block, one of β colors to each record low that is itself not a block minimum and one of γ colors to each marked element.

Assume that the sets of α, β and γ colors used are denoted by elements in $[\alpha]$, $[\beta]$ and $[\gamma]$, where in the latter two sets the elements are arranged clockwise around a circle in the natural order. We may assume further that the elements within a given block draw their color assignments from a particular copy of the set $[\beta]$ (so there are k color sets of size β in all each of which we denote by $[\beta]$ by a slight abuse of notation). Moreover, we apply the same terminology to $\mathcal{M}_{n,k}$ concerning true and false blocks as we did to $\mathcal{L}_{n,k}^*$. We assume that the true blocks of $\rho \in \mathcal{M}_{n,k}$ are arranged from left to right in increasing order of their respective minimum elements and following the single false block of ρ (if it exists).

Define the sign of $\rho \in \mathcal{M}_{n,k}$ by $(-1)^{\mathrm{nrec}(\rho)}$.

By comparison with the recurrence, it is seen that $S(n, k; \alpha, \beta, \gamma)$ gives the sum of the signs of all members of $\mathcal{M}_{n,k}$, i.e.,

$$S(n, k; \alpha, \beta, \gamma) = \sum_{\rho \in \mathcal{M}_{n,k}} (-1)^{\mathrm{nrec}(\rho)}.$$

See [34] for a proof of the recurrence in the case of weighted extended Lah distributions.

We now define a sign-changing involution on $\mathcal{M}_{n,k}$ whose set of survivors will all have positive sign and be in one-to-one correspondence with the set of unfair distributions. To do so, we need to introduce some further definitions as follows. Consider forming members $\rho \in \mathcal{M}_{n,k}$ in n steps, where in the i-th step, some decision is made regarding the placement of i relative to the members of $[i-1]$. In particular, for each $i \in [n]$, we have four possible options:

(a) insert i as an unmarked element directly following some member of $[i-1]$ within a block,

(b) add i at the very beginning of one of the currently occupied true blocks,

(c) insert i as a marked element or

(d) start a new (true) block with i as its smallest element.

Note that for $i = 1$ only options (c) and (d) apply. Also, if (c) holds, then i is to occur at the very end of the last currently occupied block (true or false) if $i \geq 2$, and at the beginning of the first block if $i = 1$ (which would be a false block). Recall that the element i is assigned one of α, β or γ possible colors respectively in options (a), (b) and (c), and no color in (d). Thus, it may be assumed the element i in case (d) is assigned the first color 1 out of the β possible colors.

For each of the cases (a)–(d) above, we consider a vector, which will be referred to as the i-vector and denoted by v_i, encoding certain information concerning the placement of the element i for each $i \in [n]$. Further, we refer to vectors v_i corresponding to cases (a), (b), (c) and (d) as primary, secondary, tertiary and quaternary vectors, respectively (and at times apply the same terminology to the element i itself).

We now define the vectors v_i based on the cases (a)–(d) and according to whether $i = 1$ or $i \geq 2$ as follows.

If $i = 1$, then either (c) or (d) applies.

- If (c), then let the 1-vector v_1 be defined as the $(\alpha+1)$-tuple $(0, p, p+1, \ldots, p+\alpha-1)$, where $p \in [\gamma]$ denotes the number of the color assigned to the element 1 and the addition is done modulo γ on the set $[\gamma]$.

- If (d), then let $v_1 = (1, 1, 2, \ldots, \alpha)$.

Assume now that $i \geq 2$ and that the j-vectors have been defined for all $j \in [i-1]$. We define v_i as follows.

- If (a) applies when inserting the element i, then let v_i be given by the ordered pair (a, b), where $a \in [i-1]$ is the element directly preceding the position in which i is placed and $b \in [\alpha]$ is the color assigned to i.
- If (b) applies when inserting i, then let v_i be the $(\alpha+1)$-tuple given by $(\ell, b_1, \ldots, b_\alpha)$, where i is inserted into the ℓ-th true block from the left in the canonical order for some $\ell \in [k]$. Further, $b_1 \in [\beta]$ is the color assigned to i and $b_2, \ldots, b_\alpha$ are the first $\alpha - 1$ colors encountered, upon proceeding clockwise starting from b_1, that have not appeared as one of the final α components of a secondary or quaternary v_j for some $j \in [i-1]$ whose first component is ℓ.
- If (c) applies, then let $v_i = (0, c_1, \ldots, c_\alpha)$, where $c_1 \in [\gamma]$ is the color assigned to i and $c_2, \ldots, c_\alpha$ are the first $\alpha-1$ colors encountered when moving clockwise from c_1 that have not appeared in a tertiary vector v_j for some $j \in [i-1]$.
- If (d) applies, then let $v_i = (\ell, 1, 2, \ldots, \alpha)$, where i is the smallest element of the ℓ-th true block.

The involution we present now is based on a special value i_0, which we define as follows. Let i_0 be the smallest $i \in [2, n]$ such that

(i) i is primary,

(ii) i is secondary and the first and second components of v_i appear respectively as the first and r-th components of v_j for some $j \in [i-1]$, where j is secondary or quaternary and $2 \leq r \leq \alpha+1$,

(iii) i is tertiary and the first and second components of v_i appear respectively as the first and r-th components of v_j for some $j \in [i-1]$, where j is tertiary and $2 \leq r \leq \alpha+1$.

Note that the j in parts (ii) and (iii) is uniquely determined by the minimality of i_0.

We define an involution between the subset of $\mathcal{M}_{n,k}$ where i_0 is primary, i.e., (i) applies, and the subset where i_0 is not primary, i.e., (ii) or (iii) applies. (Recall that primary means that the element is unmarked and directly follows a smaller element in its block at the time of insertion.)

If (ii) or (iii) applies and j and r are as described above, then let i_0 be primary with ordered pair $(j, r-1)$. (Note that j is the element that is preceding i_0 and $r-1$ is the assigned color.)

Conversely, suppose i_0 is primary with $v_{i_0} = (a, b)$. Then a is not primary and we consider cases based on whether a is tertiary or not.

If a is tertiary (i.e., a is marked), then replace v_{i_0} with the tertiary vector whose second component is the $(b+1)$-st component s of v_a. Note that this uniquely determines the remaining components of v_{i_0}, i.e., simply choose the first $\alpha - 1$ members of $[\gamma]$ encountered when starting from s and going clockwise which have not occurred in a previous tertiary vector v_j for $j \in [i_0 - 1]$.

If a is secondary or quaternary, then let i_0 be secondary with the first and second components of v_{i_0} equal to the first and $(b+1)$-st components of v_a, respectively, where again the remaining $\alpha - 1$ components of v_{i_0} are uniquely determined.

Further, in each of the cases (i)–(iii) above, we keep v_j the same for all $j \in [i_0 - 1]$ and adjust v_d for $d \in [i_0 + 1, n]$ accordingly when d is not primary by only changing the final $\alpha - 1$ components of v_d as needed (note that the possible colors that are left for these components once the status of i_0 has been changed will be different for either tertiary elements $d > i_0$ or secondary elements $d > i_0$ lying in the same block as a).

The operation defined above is sign-changing in all cases since the number of elements that are neither record lows nor marked always changes by one. Further, one may verify that combining all of the operations above yields an involution ϕ on $\mathcal{M}_{n,k}$ since i_0 is invariant.

It is possible to describe directly in terms of the partitions ρ themselves the transformation brought on by the operations defined in terms of the vectors v_i. For example, the case above when $v_{i_0} = (a, b)$ and a is tertiary has the effect of moving the element i_0 so that instead of directly following a, it now occurs at the very end of the last block of the subpartition of ρ obtained by considering only the elements of $[i_0]$, with its color changed from b to t, where t is the $(b+1)$-st component of v_a. The elements of $[i_0+1, n]$ are then inserted sequentially into this new subpartition of $[i_0]$ using the information contained in (the first two components of) their respective vectors.

Note that the set S of survivors of the involution ϕ consists of those distributions $\rho \in \mathcal{M}_{n,k}$ that contain no primary elements and in which the sets obtained by considering the second through the $(\alpha+1)$-st components of all the tertiary v_i are mutually disjoint, as are the comparable sets obtained by considering all secondary or quaternary v_i with the same first component ℓ for each $\ell \in [k]$. Members of S all have positive sign and are seen to be synonymous with the set of unfair distributions described in Section 2.

5.2. Generalized r-Lah Numbers Revisited

Recently, Belbachir et al. [2] presented a combinatorial interpretation for the generalized r-Lah numbers in terms of a weighted partition structure. Let $\Omega_r(n, k)$ be the set of partitions of the elements of $[n+r]$ into $k+r$ contents-ordered blocks (i.e., lists) such that the elements of $[r]$ belong to distinct blocks and weights are assigned to each element as follows:

(a) the first element to be inserted into a block (i.e., the smallest) has weight 1,

(b) an element (other than the smallest) inserted at the beginning of a block is assigned the weight β,

(c) the remaining elements in a block have weight α,

where it is assumed that elements are inserted one-by-one starting with the smallest. The weight $w(d)$ of a distribution $d \in \Omega_r(n,k)$ is defined as the product of the weights of its elements and the total weight of $\Omega_r(n,k)$ is given as the sum of the weights of all the distributions. Let $G_r(n,k;\alpha,\beta)$ denote this total weight. In [2], it was shown that $G_r(n,k;\alpha,\beta)$ coincides with the generalized r-Lah number considered in [33].

We now demonstrate the connection between this model and the current one by establishing the equality

$$G_r(n,k;\alpha,\beta) = S(n,k;-\alpha,\beta,(\alpha+\beta)r). \tag{21}$$

To do so, we define an explicit bijection between the classes of distributions enumerated by both sides of (21) when α and β are positive integers.

Bijective Proof of (21)**.**

Let $\mathcal{S}_r(n,k)$ denote the set of distributions of $[n]$ into k (unlabeled) ordinary cells each with β compartments and r (labeled) special cells each with $\alpha+\beta$ compartments, such that all elements open α new compartments when placed into any compartment P while allowing P to receive additional elements. Each ordinary cell must receive at least one element, with the first element going in a stipulated favorite compartment, whereas no such requirements apply to the special cells. Then we have $|\mathcal{S}_r(n,k)| = S(n,k;-\alpha,\beta,(\alpha+\beta)r)$, since members of $\mathcal{S}_r(n,k)$ are seen to be synonymous with the class of unfair distributions enumerated by $S(n,k;-\alpha,\beta,(\alpha+\beta)r)$.

Note that in the model described above for $G_r(n,k;\alpha,\beta)$, instead of indeterminate weights β and α, we may assume that the elements in (b) and (c) above are assigned colors out of disjoint sets of colors of size β and α, respectively. These sets of colors will be denoted by $[\beta]$ and $[\alpha]$. We note that an element i in (c) must directly follow some member of $[i-1]$ when placed, whereas elements from (a) and (b) correspond to left-right minima in the list of elements for the block (with the element from (a) corresponding to the block minimum). Further, a block containing an element of $[r]$ within a member of $\Omega_r(n,k)$ will be referred to as *special*, with all other blocks (i.e., those comprised exclusively of elements in $I = [r+1, r+n]$) being *non-special*.

We now describe a bijection between $\Omega_r(n,k)$ and $\mathcal{S}_r(n,k)$. Let $\lambda \in \Omega_r(n,k)$. The general idea will be to convert the non-special blocks of λ to ordinary cells in some $\rho = \rho_\lambda \in \mathcal{S}_r(n,k)$ and to convert the special blocks of λ to the special cells of ρ. We first consider performing the former. Let B denote an arbitrary non-special block of λ. Let $m = \min B$ and $a_1 > \cdots > a_s$ denote the left-right minima in the list of elements of B. Note that $s \geq 1$, with $a_s = m$.

We then form using B an ordinary cell, which will be denoted by B'. First, let m be the initial element going into B', which must be placed in the favorite compartment. Then place elements $a_1, \ldots, a_{s-1}$ into the β original compartments of B' such that if a_j for $1 \leq j \leq s-1$ is assigned the color $\beta_j \in [\beta]$, then a_j goes in compartment β_j of B'. (By an *original* compartment in B', we mean one that was present prior to any elements of B being added.)

Now let $b_1 < \cdots < b_t$ denote the elements of B, if any, not corresponding to left-right minima; recall that each b_i is assigned one of α colors. We now insert the elements b_i into B', one at a time, starting with b_1. In order to do so, for each $i \in [t]$, let b_i^* denote the first element in $[b_i - 1]$ that is encountered when one starts from the element b_i in the list of elements of B and proceeds to the left. We now place b_1 into B' so that it lies in the p-th compartment opened up by the element b_1^*, where $p \in [\alpha]$ is the color assigned to b_1. We repeat this for the remaining elements $b_2, \ldots, b_t$. We then subtract r from each of the added elements $a_1, \ldots, a_s, b_1, \ldots, b_t$, all of which belong to I. This completes the construction of B'. The preceding steps are then repeated for all non-special blocks B of λ, which yields the k ordinary cells of ρ.

We now convert the special blocks of λ to special cells of ρ. Let C be a special block of λ and suppose C contains $c \in [r]$. Define the elements a_i and b_j just as we did above for B. Note in this case that $a_s = c$. Let D denote the subset of C consisting of those b_i such that $b_i^* = c$. Observe that members of D are those that would have been placed directly to the right of c when the block C was formed initially by adding elements of I one-by-one.

We construct the special cell C' as follows using C. First, we add to the original compartments of C' the elements $a_1, \ldots, a_{s-1}$ just as we did with B' above. Next, we add the elements of D to C' such that if $d \in D$ was assigned the p-th color, where $p \in [\alpha]$, then d is inserted into the $(\beta + p)$-th original compartment of C'. Finally, we add the remaining elements e among $b_1, \ldots, b_t$ not belonging to D, starting with the smallest such e and working upward. To do so, we insert e into the p-th compartment opened by the element e^*, where $p \in [\alpha]$ is the color assigned to e. Finally, we subtract r from all elements inserted into C'', which completes the construction of C'.

We repeat the procedure for each special block C to obtain the r special cells of

ρ. Putting together the various ordinary and special cells that have arisen above completes the construction of $\rho \in \mathcal{S}_r(n,k)$. Note that if $C = \{c\}$ for some $c \in [r]$, then the resulting special cell C' in ρ contains no elements of $[n]$.

Set $f(\lambda) = \rho$. Then one may verify that the mapping f is reversible and hence yields the desired bijection. □

5.3. Equivalence with the Pit Game Model

Recently, Maltenfort [25] investigated (combinatorial and several algebraic) models for the Hsu-Shiue generalized Stirling numbers in the case when α is a negative integer. We describe now how this relates to our model.

We mention that in the case when α is a negative integer in the unfair distribution model, each ball inserted opens or creates $-\alpha+1$ new compartments (while closing the compartment in which the ball was placed).

The pit game model is based on the basic recursion for $S(n,k;\alpha,\beta,\gamma)$.

In a pit game, distinguishable coins are placed one at a time in a predetermined order such that each coin is placed either into an area known as "the pit" or into one of several labeled urns. The number of urns available changes according to two quantities, the number of coins already placed and the number of those coins that were placed into the pit. For positive integers α, β and γ, an (α,β,γ)-pit game is one in which after i coins have been placed, of which j coins have gone into the pit, there are $\gamma + i\alpha + j\beta$ urns. In other words, at the beginning there are γ urns and when a coin is placed into an urn, the number of urns increases by α and when a coin is placed into the pit, the number of urns increases by $\alpha+\beta$. Maltenfort showed that $S(n,k;-\alpha,\beta,\gamma)$ is the number of pit games involving n coins with k coins in the pit.

We explain briefly the equivalence to the unfair distribution model. Let δ be an unfair distribution of n balls into k nonempty ordinary cells with β compartments and a special cell with γ compartments, such that each ball inserted creates $\alpha+1$ new compartments. What pit game δ^* corresponds to δ? In δ, there is a ball in the favorite compartment of each of the k ordinary cells. These balls correspond to the coins that went into the pit, while the balls that are in any of the other compartments correspond to the coins in the urns.

5.4. Connection to Multiparameter Stirling Numbers

Given a vector

$$\overline{\alpha} = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\},$$

where α_i is real or complex, the generalized factorial is defined as

$$(x;\overline{\alpha})_n = \prod_{i=0}^{n-1}(x-\alpha_i).$$

Using this generalization of the factorial, El-Desouky [16] introduced multiparameter Stirling numbers via

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k;\overline{\alpha})(x;\overline{\alpha})_k \tag{22}$$

and derived the egf formula

$$\sum_{n=0}^{\infty} s(n,k;\overline{\alpha})\frac{t^n}{n!} = \sum_{j=0}^{k} \frac{(1+t)^{\alpha_j}}{\prod\limits_{i=0,i\neq j}^{k}(\alpha_j-\alpha_i)}. \tag{23}$$

Multiparameter Stirling numbers arise in several settings and go by various names in the literature. Cakic et al. [6] further studied these numbers and showed that a special case is closely connected to the Hsu-Shiue generalized Stirling numbers.

Let α^* denote the special case of $\overline{\alpha}$ where $\alpha_i = \frac{i\beta+\gamma}{\alpha}$ for all i. Then the egf in this special case is given by [6]

$$\sum_{n=0}^{\infty} s(n,k;\alpha^*)\frac{t^n}{n!} = \frac{\alpha^k}{k!}\frac{(1+t)^{\frac{\gamma}{\alpha}}}{\beta^k}\left[(1+t)^{\frac{\beta}{\alpha}}-1\right]^k. \tag{24}$$

It also holds [6] that

$$S(n,k;\alpha,\beta,\gamma) = \alpha^{n-k}s(n,k;\alpha^*), \tag{25}$$

which follows from comparing (3) and (24).

We mention here two identities that follow from the preceding results. The combinatorial proofs are straightforward and therefore omitted.

Corollary 1. *If $n \geq 0$ and $k \geq 1$, then*

$$s(n+1,k;\alpha^*) = s(n,k-1;\alpha^*) + \left(\tfrac{k\beta+\gamma}{\alpha}-n\right)s(n,k;\alpha^*). \tag{26}$$

Corollary 2. *If $n, k \geq 0$, then*

$$s(n,k;\alpha^*) = \frac{n!\alpha^k}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{(\beta/\alpha)j+(\gamma/\alpha)}{n}. \tag{27}$$

6. An Extension

In this section, we consider a modification of our model, namely, assuming in each ordinary cell that there is an arbitrary number of favorite compartments. (Note that often this more closely resembles a real world situation.) More precisely, given a positive integer h, let $\alpha'_1, \ldots, \alpha'_h$ denote the set of favorite compartments in a cell. This means that the first element in a particular cell must go in compartment α'_1, the second element in α'_2, and so on, until the h-th element. Then there are no prescribed compartments into which the $(h+1)$-st or subsequent elements must be placed.

Let $\mathbf{S(n,k)} = S(n,k;\alpha,\beta,\gamma,h)$ denote the number of (α,β,γ)-partitions satisfying the above conditions. Note that $\mathbf{S(n,k)}$ reduces to $S(n,k;\alpha,\beta,\gamma)$ when $h=1$.

The $\mathbf{S(n,k)}$ array has the following egf formula for a fixed $k \geq 0$:

$$\sum_{n=0}^{\infty} \mathbf{S(n,k)} \frac{t^n}{n!} = (1+\alpha t)^{\frac{\gamma}{\alpha}} \frac{1}{k!} \left(\frac{(1+\alpha t)^{\frac{\beta}{\alpha}} - 1 + \sum_{i=1}^{h-1} \left((\beta|\alpha)_h - (\beta|\alpha)_i\right) \frac{t^i}{i!}}{(\beta|\alpha)_h} \right)^k, \tag{28}$$

which reduces to (3) when $h=1$. To realize (28), note that there is only one way to arrange i elements in an ordinary cell if $1 \leq i \leq h-1$ and $\frac{(\beta|\alpha)_i}{(\beta|\alpha)_h}$ ways if $i \geq h$.

It does not appear to be the case that $\mathbf{S(n,k)}$ satisfies a two-term recurrence which extends (7) for general h since one lacks knowledge as to the number of elements of $[n]$ lying in an ordinary cell where $n+1$ is to be inserted. However, we do have the following recurrence which generalizes (9).

Theorem 10. *For n, k, α, β, γ, h positive integers such that $\alpha|\beta$ and $\alpha|\gamma$, the following identity holds:*

$$k\mathbf{S(n,k)} = \sum_{j=k-1}^{n-h} \binom{n}{j} \frac{(\beta|\alpha)_{n-j}}{(\beta|\alpha)_h} \mathbf{S(j,k-1)} + \sum_{j=n-h+1}^{n-1} \binom{n}{j} \mathbf{S(j,k-1)}. \tag{29}$$

Proof. For an algebraic proof of (29), one can compute the egf of both sides using (28) for a fixed k and verify that one indeed gets an equality. For a combinatorial proof, consider the number $n-j$ of elements of $[n]$ in the marked ordinary cell within an (α,β,γ)-partition λ enumerated by $\mathbf{S(n,k)}$. If $1 \leq n-j \leq h-1$ (i.e., $n-h+1 \leq j \leq n-1$), then each element must go in a prescribed compartment and thus there are $\binom{n}{n-j}\mathbf{S(j,k-1)}$ possibilities. If $h \leq n-j \leq n-k+1$, then the smallest h elements must go in prescribed compartments, which leaves $(\beta - h\alpha|\alpha)_{n-j-h} = \frac{(\beta|\alpha)_{n-j}}{(\beta|\alpha)_h}$ possibilities for the placement of the remaining elements in the marked cell. Summing over all j then gives the right-hand side of (29). $\square$

Considering whether or not the element 1 lies in a special or in an ordinary cell, and if the latter, the number of additional members of $[n]$ lying in this cell, yields the following alternate recurrence formula.

Theorem 11. *For n, k, α, β, γ, h positive integers such that $\alpha|\beta$ and $\alpha|\gamma$, the* *following identity holds:*

$$\mathbf{S(n,k)} = \gamma S(n-1,k;\alpha,\beta,\gamma-\alpha,h) + \sum_{m=k-1}^{n-h-1} \binom{n-1}{m} \frac{(\beta|\alpha)_{n-m}}{(\beta|\alpha)_h} \mathbf{S(m,k-1)} + \sum_{m=n-h}^{n-1} \binom{n-1}{m} \mathbf{S(m,k-1)}. \quad (30)$$

Let $\mathbf{S_n(x)} = S_n(x;\alpha,\beta,\gamma,h)$ defined by $\mathbf{S_n(x)} = \sum_{k=0}^{n} \mathbf{S(n,k)} x^k$ denote the corresponding extension of the generalized Bell polynomials. By (28) and the definition of $\mathbf{S_n(x)}$, we have the egf formula

$$\sum_{n=0}^{\infty} \mathbf{S_n(x)} \frac{t^n}{n!} = (1+\alpha t)^{\frac{\gamma}{\alpha}} \exp\left(\frac{x(1+\alpha t)^{\frac{\beta}{\alpha}} - x + \sum_{i=1}^{h-1} x\left((\beta|\alpha)_h - (\beta|\alpha)_i\right) \frac{t^i}{i!}}{(\beta|\alpha)_h} \right). \quad (31)$$

Further, multiplying both sides of (30) by x^k, and summing over $0 \le k \le n$, yields

$$\mathbf{S_n(x)} = \gamma S_{n-1}(x;\alpha,\beta,\gamma-\alpha,h) + \sum_{m=0}^{n-h-1} x\binom{n-1}{m} \frac{(\beta|\alpha)_{n-m}}{(\beta|\alpha)_h} \mathbf{S_m(x)} + \sum_{m=n-h}^{n-1} x\binom{n-1}{m} \mathbf{S_m(x)}, \quad (32)$$

which is equivalent to (15) when $h = 1$. Note that (32) may also be shown directly by extending the proof given above for (15).

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7. Appendix

We provide here a fairly complete list of the special cases of the Hsu-Shiue Stirling numbers including some cases that have arisen in the literature after the publication of [20]. We remark that when $\alpha = 0$ or $\beta = 0$ in the identities below for $S(n, k; \alpha, \beta, \gamma)$, it is understood to be the limiting value obtained as the respective parameter approaches zero.

- Binomial coefficients

$$\binom{n}{k} = S(n, k; 0, 0, 1).$$

- Stirling numbers of the first kind [17, Chapter 6]

$$s(n, k) = S(n, k; 1, 0, 0).$$

- Stirling numbers of the second kind [17, Chapter 6]

$$S(n, k) = S(n, k; 0, 1, 0).$$

- Lah numbers [24]

$$L(n,k) = S(n,k;-1,1,0).$$

- Carlitz's degenerate Stirling numbers [7]

$$S_1(n,k|\theta) = S(n,k;-1,-\theta,0) \quad \text{and} \quad S(n,k|\theta) = S(n,k;\theta,1,0).$$

- Carlitz's weighted Stirling numbers [8]

$$R_1(n,k,\lambda) = S(n,k;-1,0,\lambda) \quad \text{and} \quad R(n,k,\lambda) = S(n,k;0,1,\lambda).$$

- Howard's weighted degenerate Stirling numbers [19]

$$S_1(n,k,\lambda|\theta) = S(n,k;-1,-\theta,\lambda-\theta) \quad \text{and} \quad S(n,k,\lambda|\theta) = S(n,k;\theta,1,\lambda).$$

- Gould-Hopper's non-central Lah numbers [18] $S(n,k;0,1,-a+b)$.
- Charalambides-Koutras' non-central C numbers [9] $S(n,k;\frac{1}{s},1,-a+b)$.
- Riordan's non-central Stirling numbers [32] $S(n,k;1,0,b-a)$.
- Tsylova's Stirling numbers [37]

$$A_{\alpha,\beta}(n,k) = S(n,k;\alpha,\beta,0).$$

- Todorov's Stirling numbers [36]

$$a_{n,k}(x) = S(n,k;1,x,0).$$

- Ahuja-Enneking's associated Lah numbers [1]

$$B_r(n,k) = r^n S(n,k;-\frac{1}{r},1,0).$$

- Broder's r-Stirling numbers [5]

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = S(n,k;-1,0,r) \quad \text{and} \quad \begin{Bmatrix} n \\ k \end{Bmatrix}_r = S(n,k;0,1,r).$$

- r-Lah numbers [31]

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = S(n,k;-1,1,2r).$$

- Whitney numbers [4]

$$w_m(n,k) = S(n,k;-m,0,1) \quad \text{and} \quad W_m(n,k) = S(n,k;0,m,1).$$

- r-Whitney numbers [10]

$$w_{m,r}(n,k) = S(n,k;-m,0,r) \quad \text{and} \quad W_{m,r}(n,k) = S(n,k;0,m,r).$$

- r-Whitney-Lah numbers [10]

$$L_{m,r}(n,k) = S(n,k;-m,m,2r).$$

- translated Whitney numbers of both kinds [3]

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(\alpha)} = S(n,k;-\alpha,0,0) \quad \text{and} \quad \begin{Bmatrix} n \\ k \end{Bmatrix}^{(\alpha)} = S(n,k;0,\alpha,0).$$

- translated Whitney-Lah numbers [3]

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{(\alpha)} = S(n,k;-\alpha,\alpha,0).$$

- translated r-Whitney numbers [3]

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(\alpha)} = S(n,k;-\alpha,0,r), \quad \begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(\alpha)} = S(n,k;0,\alpha,r), \quad \text{and}$$
$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)} = S(n,k;-\alpha,\alpha,r).$$

- generalized r-Lah numbers [33]

$$\mathcal{G}_{a,b}(n,k) = S(n,k;-a,b,(a+b)r).$$