#A81 INTEGERS 22 (2022)

# SUMS OF PRODUCTS OF BINOMIAL COEFFICIENTS MOD 2 AND RUN LENGTH TRANSFORMS OF SEQUENCES

**Chai Wah Wu**
*IBM Research AI, IBM T. J. Watson Research Center, Yorktown Heights, New York*
cwwu@us.ibm.com

*Received: 11/24/21, Revised: 5/6/22, Accepted: 8/12/22, Published: 8/24/22*

**Abstract**

We study properties of functions of binomial coefficients mod 2 and derive a set of recurrence relations for sums of products of binomial coefficients mod 2. We show that they result in sequences that are the run length transforms of well known basic sequences. In particular, we obtain formulas for the run length transform of the positive integers, Fibonacci numbers, extended Lucas numbers and Narayana's cows sequence.

## 1. Introduction

When is the binomial coefficient even or odd, i.e., what is $\binom{n}{k} \pmod 2$? It is well known that when Pascal's triangle of binomial coefficients is taken mod 2, the result has a fractal structure in the limit and corresponds to Sierpiński's triangle (also known as Sierpiński's gasket or Sierpiński's sieve) [8, 10, 4, 5].

Lucas' theorem [2, 3] provides a simple way to determine the binomial coefficients modulo a prime. It states that for integers $k$, $n$ and prime $p$, the following relationship holds

$$\binom{n}{k} \equiv \prod_{i=0}^{m} \binom{n_i}{k_i} \pmod p$$

where $n_i$ and $k_i$ are the digits of $n$ and $k$ in base $p$, respectively[1].

When $p = 2$, then $n_i$ and $k_i$ are the bits in the binary expansion of $n$ and $k$, and $\binom{n_i}{k_i}$ is 0 if and only if $n_i < k_i$. This implies that $\binom{n}{k}$ is even if and only if $n_i < k_i$ for some $i$.

The truth table of $n_i < k_i$ is

[1] If the lengths of the base $p$ representations of $n$ and $k$ differ, leading 0's are prepended to the shorter representation.

| $n_i$ | $k_i$ | $n_i < k_i$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 0 |

and is logically equivalent to $k_i \wedge (\neg n_i)$. Let us consider the notation $\wedge$, $\vee$ and $\neg$ to also function as operations on integers by treating them as bitwise operations [1, 6, 11] on the binary representation of numbers[2]. For instance, $11 \wedge 14$ is the bitwise AND of $1011_2$ and $1110_2$, which is equal to $1010_2 = 10$. This implies the following well-known fact [10].

**Theorem 1.** *For integers $n$ and $k$, $\binom{n}{k} \equiv 0 \pmod 2$ if and only if $k \wedge (\neg n) \neq 0$.*

Incidentally, for bits $n_i$ and $k_i$, $n_i < k_i$ is logically equivalent to $\neg(k_i \Rightarrow n_i)$. Consider $\binom{n}{k}\binom{m}{r} \pmod 2$. Clearly this is equivalent to

$$\left(\binom{n}{k} \pmod 2\right)\left(\binom{m}{r} \pmod 2\right).$$

Thus $\binom{n}{k}\binom{m}{r} \equiv 0 \pmod 2$ if and only if $k \wedge (\neg n) \neq 0$ or $r \wedge (\neg m) \neq 0$. This in turn implies the following result.

**Theorem 2.** *The product of two binomial coefficients modulo 2 satisfies*

$$\binom{n}{k}\binom{m}{r} \equiv 0 \pmod 2 \text{ if and only if } (k \wedge (\neg n)) \vee (r \wedge (\neg m)) \neq 0.$$

Analogously for sequences of integers $\{n[j]\}$, $\{k[j]\}$ we have the following result for $\left(\prod_{j=1}^{T}\binom{n[j]}{k[j]}\right) \pmod 2$.

**Theorem 3.** *The product of $T$ binomial coefficients mod 2 satisfies*

$$\prod_{j=1}^{T}\binom{n[j]}{k[j]} \equiv 0 \pmod 2 \text{ if and only if}$$

$$(k[1] \wedge (\neg n[1])) \vee (k[2] \wedge (\neg n[2])) \vee \cdots \vee (k[T] \wedge (\neg n[T])) \neq 0.$$

These equivalences are simply consequences of Lucas' theorem for $p = 2$ but the use of the bitwise notation will be helpful in deriving properties of binomial coefficients mod 2. For instance, the following well-known result can easily be shown.

[2]Again, leading 0's are added to the binary operations $\wedge$ and $\vee$ if the operands differ in bit lengths. Furthermore, negative integers are represented in binary using the 2's complement format [1, 6, 12], in which case leading 1's are prepended.

**Lemma 1.** *The central binomial coefficient* $\binom{2n}{n}$ *is even if and only if* $n > 0$.

*Proof.* First note that $\binom{0}{0} = 1$ is odd. For $n > 0$, let $n = 2^s r$ where $r$ is odd. Then $n \wedge \neg 2n = 2^s(r \wedge \neg 2r)$ since the $s$ least significant bits are 0. As $r$ is odd, $r \wedge \neg 2r \neq 0$, and the conclusion follows from Theorem 1. □

As is common in formulas involving logical operators, $\neg$ has higher precedence than $\wedge$ which in turn has higher precedence than $\vee$.

## 2. Run Length Transform

For a sequence $\{b_i\}$ of bits, let the 1-*runs* $R$ denote the sequence of lengths of consecutive 1's in the sequence. For example, for the bits 011011100111, the consecutive 1's have lengths 2, 3 and 3 and $R = (2, 3, 3)$.

The run length transform of sequences of numbers is defined as follows [7].

**Definition 1.** The *run length transform* of $\{S_n\}_{n\geq 0}$ is given by $\{T_n\}_{n\geq 0}$, where $T_0 = S_0$, and for $n > 0$, $T_n = \Pi_{i\in R} S_i$ with $R$ being the 1-runs of the binary representation of $n$.

In the rest of this paper, as in [7], we assume that $S_0 = 1$. As an example, suppose $n = 463$, which is 111001111 in binary. It has a run of 3 1's and a run of 4 1's, and thus $T_n = S_3 \cdot S_4$. Some fixed points of the run length transform include the sequences $\{1, 0, 0, \ldots\}$ and $\{1, 1, 1, \ldots\}$. In [7], the following result is proved about the run length transform.

**Theorem 4.** *Let* $\{S_n\}_{n\geq 0}$ *be defined by the recurrence* $S_{n+1} = d_0 S_n + d_1 S_{n-1}$ *with initial conditions* $S_0 = 1$, $S_1 = c_1$. *Then the run length transform of* $\{S_n\}$ *is given by* $\{T_n\}_{n\geq 0}$ *satisfying* $T_0 = 1$, $T_{2n} = T_n$, $T_{4n+1} = c_1 T_n$, *and* $T_{4n+3} = d_0 T_{2n+1} + d_1 T_n$.

Note that the sequence $\{S_n\}$ may not uniquely define the values of $d_0$ and $d_1$ in Theorem 4. For instance, for the sequence $\{S_n\} = \{1, 2, 4, 8, \ldots\}$, $d_0$ and $d_1$ can be chosen to be any integers such that $2d_0 + d_1 = 4$. On the other hand, note that the run length transform is injective (one-to-one), since $S_i = T_{2^i - 1}$ for $i \geq 0$ and the sequence $\{S_n\}$ can be derived from the corresponding sequence $\{T_n\}$.

## 3. Recurrence Relations of Products of Binomial Coefficients Modulo 2

**Definition 2.** Consider integers $a_i$, $i = 1, \ldots, 4$, with $0 \leq a_1 + a_2$, and $0 \leq a_3 + a_4$. Define

$$F(n,k) = \binom{a_1 n + a_2 k}{a_3 n + a_4 k}\binom{n}{k} \pmod 2$$

and

$$g(n,k) = ((a_3 n + a_4 k) \wedge \neg(a_1 n + a_2 k)) \vee (k \wedge \neg n).$$

By Theorem 2, $F(n,k) = 1$ if and only if $g(n,k) = 0$. One direct consequence of this is a property we will often use: if $g(m,r) = wg(n,k)$ for some $w \neq 0$, then $F(m,r) = F(n,k)$. The functions $F$ and $g$ depend on the integers $a_i$ whose values are clear from the context. We next show that $F$ satisfies various recurrence relations. In the formulas below, the arithmetical operations $+$ and $\times$ have precedence over the bitwise logical operators $\wedge$, $\vee$ and $\neg$.

**Theorem 5.** *The following relations hold for the function $F$:*

- $F(n,k) = 0$ *if* $k > n$,
- $F(2^r n, 2^r k) = F(n,k)$ *for* $r > 0$,
- $F(2n, 2k+1) = F(4n+1, 4k+2) = F(4n+1, 4k+3) = F(4n+2, 4k+1) = F(4n+2, 4k+3) = F(4n, 4k+1) = F(4n, 4k+2) = F(4n, 4k+3) = 0$,
- *Suppose* $a_3 \in \{0,1\}$. *If* $a_1 = 1$ *or* $a_3 = 0$, *then* $F(4n+1, 4k) = F(4n+3, 4k) = F(2n+1, 2k) = F(n,k)$,
- *If* $a_3 \wedge \neg a_1 \equiv 0 \pmod 4$ *and* $0 \leq a_1, a_3 < 4$, *then* $F(4n+1, 4k) = F(n,k)$,
- *If* $a_3 \wedge \neg a_1 \not\equiv 0 \pmod 4$, *then* $F(4n+1, 4k) = 0$,
- *If* $3a_3 \wedge \neg 3a_1 \not\equiv 0 \pmod 4$, *then* $F(4n+3, 4k) = 0$,
- *If* $a_3 \wedge \neg a_1 \not\equiv 0 \pmod 2$, *then* $F(2n+1, 2k) = 0$.

*Proof.* If $k > n$, then by definition $\binom{n}{k} = 0$ and thus $F(n,k) = 0$.

Note that $g(2n, 2k) = (2(a_3 n + a_4 k) \wedge \neg(2(a_1 n + a_2 k))) \vee (2k \wedge \neg 2n) = 2g(n,k)$ since the least significant bit is 0, i.e., $F(2n, 2k) = F(n,k)$.

Next $F(n,k) = 0$ if $\binom{n}{k} \equiv 0 \pmod 2$, i.e., if $(k \wedge \neg n) > 0$. It is easy to see that $F(2n, 2k+1) = F(4n+1, 4k+2) = F(4n+1, 4k+3) = F(4n+2, 4k+1) = F(4n+2, 4k+3) = 0$ and $F(4n, 4k+i) = 0$ for $1 \leq i \leq 3$.

Since $g(4n+1, 4k) = (4(a_3 n + a_4 k) + a_3 \wedge \neg 4(a_1 n + a_2 k) + a_1) \vee (4k \wedge \neg(4n+1))$ and $4k \wedge \neg(4n+1) \equiv 0 \pmod 4$, the least significant 2 bits of $g(4n+1, 4k)$ are equal to

$a_3 \wedge \neg a_1 \pmod 4$. This means that if $a_1 = 1$ or $a_3 = 0$, then $g(4n+1, 4k) = 4g(n,k)$ and $F(4n+1, 4k) = F(n,k)$. If $a_3 \wedge \neg a_1 \not\equiv 0 \pmod 4$, then $F(4n+1, 4k) = 0$.

Similarly, we write $g(4n+3, 4k) = (4(a_3 n + a_4 k) + 3a_3 \wedge \neg(4(a_1 n + a_2 k) + 3a_1)) \vee (4k \wedge \neg(4n+3))$ which is equal to $4g(n,k)$ if $a_1 = 1$ or $a_3 = 0$, i.e., $F(4n+3, 4k) = F(n,k)$ if $a_1 = 1$ or $a_3 = 0$ and $F(4n+3, 4k) = 0$ if $3a_3 \wedge \neg 3a_1 \not\equiv 0 \pmod 4$.

Finally, $g(2n+1, 2k) = (2(a_3 n + a_4 k) + a_3 \wedge \neg(2(a_1 n + a_2 k) + a_1)) \vee (2k \wedge \neg(2n+1))$. Thus $F(2n+1, 2k) = 0$ if $a_3 \wedge \neg a_1 \not\equiv 0 \pmod 2$. If $a_1 = 1$ or $a_3 = 0$, then $g(2n+1, 2k) = 2g(n,k)$ and $F(2n+1, 2k) = F(n,k)$. $\square$

## 4. Sums of Products of Binomial Coefficients Modulo 2

In this section, we show that for various values of $a_i$'s, the sequence $a(n) = \sum_{k=0}^{n} F(n,k)$ corresponds to the run length transforms of well-known sequences[3].

In particular, we show that the sequences

$$\sum_{k=0}^{n} \left[ \binom{n-k}{2k} \binom{n}{k} \pmod 2 \right],$$

$$\sum_{k=0}^{n} \left[ \binom{n+k}{n-k} \binom{n}{k} \pmod 2 \right],$$

$$\sum_{k=0}^{n} \left[ \binom{n+2k}{2n-k} \binom{n}{k} \pmod 2 \right],$$

and

$$\sum_{k=0}^{n} \left[ \binom{n-k}{6k} \binom{n}{k} \pmod 2 \right]$$

are the run length transform of the Fibonacci numbers, the positive integers, the extended Lucas numbers and Narayana's cows sequence, respectively.

It is clear that $a(n)$ is upper bounded by $a(n) \leq \sum_{k=0}^{n} \left[ \binom{n}{k} \pmod 2 \right]$ with equality when $a_1 = a_4 = 1$, $a_2 = a_3 = 0$ or when $a_1 = a_2 = a_3 = a_4 = 1$. The sequence $\alpha(n) = \sum_{k=0}^{n} \left[ \binom{n}{k} \pmod 2 \right]$ is known as Gould's sequence or Dress' sequence and is the run length transform of the positive powers of 2: $1, 2, 4, 8, 16, 32, \ldots$ (see OEIS [9] sequence A001316).

**Lemma 2.** *The sequence $a(n)$ satisfies the following properties:*

- $a(0) = 1$,

[3]Again, $a(n)$ depends on the integers $a_i$ whose values are deduced from the context.

- $a(2^r n) = a(n)$ *for* $r > 0$,
- *Suppose* $a_3 \in \{0,1\}$. *If* $a_1 = 1$ *or* $a_3 = 0$, *then* $a(4n+1) = a(n)+\sum_{k=0}^{n} F(4n+1, 4k+1)$, $a(4n+3) = a(n)+\sum_{m=1}^{3}\sum_{k=0}^{n} F(4n+3, 4k+m)$, *and* $a(2n+1) = a(n) + \sum_{k=0}^{n} F(2n+1, 2k+1)$,
- *If* $a_3 \wedge \neg a_1 \not\equiv 0 \pmod 4$, *then* $a(4n+1) = \sum_{k=0}^{n} F(4n+1, 4k+1)$,
- *If* $3a_3 \wedge \neg 3a_1 \not\equiv 0 \pmod 4$, *then* $a(4n+3) = \sum_{m=1}^{3}\sum_{k=0}^{n} F(4n+3, 4k+m)$,
- *If* $a_3 \wedge \neg a_1 \not\equiv 0 \pmod 2$, *then* $a(2n+1) = \sum_{k=0}^{n} F(2n+1, 2k+1)$.

*Proof.* First, note that $a(0)$ is trivially equal to 1. Next, $a(2n) = \sum_{k=0}^{2n} F(2n, k) = \sum_{k=0}^{n} F(2n, 2k) + \sum_{k=0}^{n-1} F(2n, 2k+1)$ which is equal to $a(n)$ by Theorem 5.

Suppose $a_3 \in \{0,1\}$ and $a_1 = 1$ or $a_3 = 0$. By Theorem 5, several of the terms $F(n,k)$ vanish or can be rewritten and the following relations hold. Firstly, $a(4n+1) = \sum_{m=0}^{3}\sum_{k=0}^{n} F(4n+1, 4k+m) - F(4n+1, 4n+2) - F(4n+1, 4n+3) = \sum_{m=0}^{3}\sum_{k=0}^{n} F(4n+1, 4k+m) = \sum_{k=0}^{n} F(n,k) + \sum_{k=0}^{n} F(4n+1, 4k+1)$.

Secondly, $a(4n+3) = \sum_{m=0}^{3}\sum_{k=0}^{n} F(4n+3, 4k+m)$ which is equal to

$$\sum_{k=0}^{n} F(n,k) + \sum_{m=1}^{3}\sum_{k=0}^{n} F(4n+3, 4k+m).$$

Next, $a(2n+1) = \sum_{k=0}^{2n+1} F(2n+1, k) = \sum_{k=0}^{n} F(2n+1, 2k) + \sum_{k=0}^{n} F(2n+1, 2k+1) = \sum_{k=0}^{n} F(n,k) + \sum_{k=0}^{n} F(2n+1, 2k+1)$. The other cases follow similarly from Theorem 5. □

In particular, if $a_1 = 1$ or $a_3 = 0$ then $\sum_{k=0}^{n} F(2n+1, 2k+1) = a(2n+1) - a(n)$, an equation that we will often use in what follows.

### 4.1. Run Length Transform of the Fibonacci Sequence

Consider the case $a_1 = 1$, $a_2 = -1$, $a_3 = 0$, $a_4 = 2$.

**Lemma 3.** *For* $a_1 = 1$, $a_2 = -1$, $a_3 = 0$, $a_4 = 2$, *the following relations hold for the function* $F$*:*

- $F(4n+1, 4k+1) = F(4n+3, 4k+3) = 0$,
- $F(4n+3, 4k+1) = F(n,k)$,
- $F(4n+3, 4k+2) = F(2n+1, 2k+1)$.

*Proof.* First, $g(4n+1, 4k+1) = (8k+2 \wedge \neg(4(n-k))) \vee (4k+1 \wedge \neg 4n+1) \neq 0$, i.e $F(4n+1, 4k+1) = 0$.

Next, $g(4n+3, 4k+1) = (8k+2 \wedge \neg(4(n-k)+2)) \vee (4k+1 \wedge \neg 4n+3) = (4(2k \vee n-k)) \vee 4(k \wedge \neg n) = 4g(n,k)$, i.e., $F(4n+3, 4k+1) = F(n,k)$.

Note that $(4k+2 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k \wedge \neg 2n)$ and $(2k+1 \wedge \neg 2n+1) = (2k \wedge \neg 2n)$. Similarly, $(4k+3 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k+1 \wedge \neg 2n+1)$.

Furthermore, $g(4n+3, 4k+2) = (8k+4 \wedge \neg(4(n-k)+1)) \vee (4k+2 \wedge \neg 4n+3) = 2((4k+2 \wedge \neg 2(n-k)) \vee (2k+1 \wedge \neg 2n+1)) = 2g(2n+1, 2k+1)$ where we have used the fact that $(8k+4 \wedge \neg(4(n-k)+1)) = (8k+4 \wedge \neg(4(n-k)))$. This implies that $F(4n+3, 4k+2) = F(2n+1, 2k+1)$.

Finally, $g(4n+3, 4k+3) = (8k+6 \wedge \neg(4(n-k))) \vee (4k+3 \wedge \neg 4n+3)$. Since $8k+6 \wedge \neg 4(n-k) \neq 0$, this implies that $F(4n+3, 4k+3) = 0$. □

**Theorem 6.** *Let* $a(n) = \sum_{k=0}^{n} \left[ \binom{n-k}{2k} \binom{n}{k} \pmod 2 \right]$. *Then* $a(n)$ *satisfies the equations* $a(0) = 1$, $a(2n) = a(n)$, $a(4n+1) = a(n)$ *and* $a(4n+3) = a(2n+1) + a(n)$. *In particular,* $a(n)$ *is the run length transform of the Fibonacci sequence* $1, 1, 2, 3, 5, 8, 13, \ldots$.

*Proof.* By Lemma 2, $a(0) = 1$ and $a(2n) = a(n)$. Next, by Lemma 2 and Lemma 3, we obtain $a(4n+1) = a(n)$. Similarly, $a(4n+3) = a(n) + \sum_{m=1}^{3} \sum_{k=0}^{n} F(4n+3, 4k+m) = a(n) + \sum_{k=0}^{n} F(n,k) + F(2n+1, 2k+1) = a(2n+1) + a(n)$. By Theorem 4, $a(n)$ is the run length transform of the Fibonacci sequence 1,1,2,3,5,8,13,... □

Note that in this case $a(n)$ corresponds to OEIS sequence A246028. Other values of $a_i$ can also generate the same sequence. For instance, it can be shown that the values of

$$\sum_{k=0}^{n} \left[ \binom{2k}{n-k} \binom{n}{k} \pmod 2 \right],$$

$$\sum_{k=0}^{n} \left[ \binom{n+3k}{2k} \binom{n}{k} \pmod 2 \right],$$

and of

$$\sum_{k=0}^{n} \left[ \binom{n+3k}{n+k} \binom{n}{k} \pmod 2 \right]$$

all correspond to the run length transform of the Fibonacci sequence as well.

In what follows, the proofs and derivations of the run length transforms and the intermediate lemmas are similar to the proofs of the above results and are omitted for brevity. Interested readers are referred to [13] for full details.

### 4.2. Run Length Transform of the Truncated Fibonacci Sequence

Next, consider the case $a_1 = a_3 = 0$, $a_2 = 3$, $a_4 = 1$.

**Lemma 4.** *For $a_1 = a_3 = 0$, $a_2 = 3$, $a_4 = 1$, the following relations hold for the function $F$:*

- $F(4n+1, 4k+1) = F(4n+3, 4k+1) = F(n, k)$,
- $F(4n+3, 4k+2) = F(2n+1, 2k+1)$,
- $F(4n+3, 4k+3) = 0$.

**Theorem 7.** *Let $a(n) = \sum_{k=0}^{n} \left[ \binom{3k}{k} \binom{n}{k} \pmod 2 \right]$. Then $a(n)$ satisfies the equations $a(0) = 1$, $a(2n) = a(n)$, $a(4n+1) = 2a(n)$ and $a(4n+3) = a(2n+1)+a(n)$. In particular, $a(n)$ is the run length transform of the truncated Fibonacci sequence $1, 2, 3, 5, 8, 13, \ldots$.*

Note that in this case $a(n)$ corresponds to OEIS sequence A245564. This sequence is also equal to

$$\sum_{k=0}^{n} \left[ \binom{3k2^m}{k2^m} \binom{n}{k} \pmod 2 \right]$$

and

$$\sum_{k=0}^{n} \left[ \binom{3k2^m}{2k2^m} \binom{n}{k} \pmod 2 \right]$$

for all integers $m \geq 0$.

### 4.3. Run Length Transform of $\{1, 1, 2, 4, 8, 16, 32, \ldots\}$

Consider the case $a_1 = 1$, $a_2 = a_3 = 0$, $a_4 = 2$.

**Lemma 5.** *For $a_1 = 1$, $a_2 = a_3 = 0$, $a_4 = 2$, the following relations hold for the function $F$:*

- $F(4n+1, 4k+1) = 0$,
- $F(4n+3, 4k+1) = F(n, k)$,
- $F(4n+3, 4k+2) = F(4n+3, 4k+3) = F(2n+1, 2k+1)$.

**Theorem 8.** *Let $a(n) = \sum_{k=0}^{n} \left[ \binom{n}{2k} \binom{n}{k} \pmod 2 \right]$. Then $a(n)$ satisfies the equations $a(0) = 1$, $a(2n) = a(n)$, $a(4n+1) = a(n)$ and $a(4n+3) = 2a(2n+1)$. In particular, $a(n) = \{1, 1, 1, 2, 1, 1, 2, 4, 1, 1, 1, 2, \ldots\}$ is the run length transform of the sequence $1, 1, 2, 4, 8, 16, 32, \ldots$, i.e., $1$ plus the positive powers of $2$.*

### 4.4. Run Length Transform of $\{1, 2, 2, 2, 2, 2, \dots\}$

Consider the case $a_1 = 1$, $a_2 = a_4 = 2$, $a_3 = 0$.

**Lemma 6.** *For $a_1 = 1$, $a_2 = a_4 = 2$, $a_3 = 0$, the following relations hold for the function $F$:*

- $F(4n+1, 4k+1) = F(n, k)$,
- $F(4n+3, 4k+1) = F(4n+3, 4k+3) = 0$,
- $F(4n+3, 4k+2) = F(2n+1, 2k+1)$.

**Theorem 9.** *Let $a(n) = \sum_{k=0}^{n} \left[ \binom{n+2k}{2k} \binom{n}{k} \pmod 2 \right]$. Then $a(n)$ satisfies the equations $a(0) = 1$, $a(2n) = a(n)$, $a(4n+1) = 2a(n)$ and $a(4n+3) = a(2n+1)$. In particular, $a(n) = \{1, 2, 2, 2, 2, 4, 2, 2, 2, 4, \dots\}$ is the run length transform of the sequence $1, 2, 2, 2, 2, 2, 2, \dots$ (OEIS A040000).*

This sequence is also generated by $\sum_{k=0}^{n} \left[ \binom{n+2k}{n} \binom{n}{k} \pmod 2 \right]$.

### 4.5. Run Length Transform of the Positive Integers

OEIS sequence A106737 is defined as $a(n) = \sum_{k=0}^{n} \left[ \binom{n+k}{n-k} \binom{n}{k} \pmod 2 \right]$. It was noted that the following recursive relationships appear to hold: $a(2n) = a(n)$, $a(4n+1) = 2a(n)$ and $a(4n+3) = 2a(2n+1) - a(n)$. In this section we show that this is indeed the case.

Let $a_1, a_2, a_3 = 1$ and $a_4 = -1$, i.e., $F(n, k) = \binom{n+k}{n-k} \binom{n}{k} \pmod 2$ and $g(n, k) = ((n-k) \wedge \neg(n+k)) \vee (k \wedge \neg n)$.

**Lemma 7.** *For $a_1, a_2, a_3 = 1$ and $a_4 = -1$, the following relations hold for the function $F$:*

- $F(4n+1, 4k+1) = F(n, k)$,
- $F(4n+3, 4k+1) = 0$,
- $F(4n+3, 4k+2) = F(4n+3, 4k+3) = F(2n+1, 2k+1)$.

**Theorem 10.** *For OEIS sequence A106737, $a(0) = 1$, $a(2n) = a(n)$, $a(4n+1) = 2a(n)$ and $a(4n+3) = 2a(2n+1) - a(n)$. Furthermore, $a(n)$ is the run length transform of the positive integers.*

This sequence is also generated by each of the following expressions: $\sum_{k=0}^{n} \left[ \binom{n+k}{2k} \binom{n}{k} \pmod 2 \right]$, $\sum_{k=0}^{n} \left[ \binom{n+2k}{k} \binom{n}{k} \pmod 2 \right]$ and $\sum_{k=0}^{n} \left[ \binom{n+2k}{n+k} \binom{n}{k} \pmod 2 \right]$.

### 4.6. A Fixed Point of the Run Length Transform

The all ones sequence $1, 1, 1, \ldots$ (OEIS sequence A000012) is a fixed point of the run length transform. We next show that it is also expressible as sums of products of binomial coefficients mod 2. To prove this, we look at the case $a_1 = a_4 = 1$, $a_2 = -1$, $a_3 = 0$.

**Lemma 8.** *For $a_1 = a_4 = 1$, $a_2 = -1$, $a_3 = 0$, the following relations hold for the function $F$:*

$$F(4n+1, 4k+1) = F(4n+3, 4k+1) = F(4n+3, 4k+2) = F(4n+3, 4k+3) = 0.$$

**Theorem 11.** *For $n, k \geq 0$, $\binom{n-k}{k}\binom{n}{k}$ is odd if and only if $k = 0$, i.e.,*

$$\sum_{k=0}^{n}\left[\binom{n-k}{k}\binom{n}{k} \pmod 2\right] = 1$$

*for all $n$.*

Theorem 11 can also be interpreted via Sierpiński's triangle generated by Pascal's triangle mod 2 and by looking at it as follows: if starting from the left edge of the triangle and moving $k$ steps to the right reaches a point of Sierpiński's triangle, then continuing moving diagonally $k$ steps must necessarily reach a void of Sierpiński's triangle.

## 5. Third Order Recurrences

**Definition 3.** Let $n$ be an odd positive integer not of the form $2^k - 1$. The *splitting function* $\mu(n) = (a, b, m)$ returns positive integers $a$, $b$, $m$ such that $a2^m + b = n$ with $2^m > 2b$, and $a$ is the smallest such number satisfying this.

A way to describe the numbers $a$ and $b$ in Definition 3 is that they are obtained by splitting the binary expansion of $n$ along the first occurrence of 0. For instance since $413 = 110011101_2$, which can be split into '11' and '011101', which is 3 and 29, thus $\mu(413) = (3, 29, 7)$. Note that $n > 3b$.

Theorem 4 shows that if a sequence satisfies a second order recurrence, then we can easily determine the recurrence relations that the run length transform satisfies. This result can be generalized to $n$-th order recurrences as follows.

**Theorem 12.** *Let $\{S_n\}_{n\geq 0}$ be defined by the $(k+1)$-th order recurrence $S_{n+1} = \sum_{i=0}^{k} d_i S_{n-i}$ with initial conditions $S_i = c_i$ for $i = 0, 1, \ldots, k$. Then the run length transform of $\{S_n\}$ is given by $\{T_n\}_{n\geq 0}$ satisfying*

- $w = 2^{k+1}$,
- $T_0 = c_0$,
- $T_{2n} = T_n$,
- $T_{wn+i} = T_i T_n$, *for* $i = 1, 3, 5, \ldots, 2^k - 1$,
- $T_{wn+2^k+i} = T_{b_i} T_{\frac{wn}{2^{m_i}}+a_i}$ *for* $i = 1, 3, 5, \ldots, 2^k - 3$ *where* $\mu(2^k + i) = (a_i, b_i, m_i)$,
- $T_{wn+w-1} = \sum_{i=0}^{k} d_i T_{2^{k-i}n+2^{k-i}-1}$.

*Proof.* The proof is similar to the proof of Theorem 4 (see [7] for a proof of Theorem 4) and is omitted. □

Even though the right hand side in some of the recurrence relations above is a product of 2 terms of $\{T_n\}$, one of the terms is determined solely by the initial conditions $c_i$'s. More specifically, note that for $i = 1, 3, 5, \ldots, 2^k - 1$, $T_i$ is a product of $S_j$'s where $j \le k$ and thus is a product of some $c_j$'s. Similarly, for $i \le 2^k - 3$, we have $2^{k+1} - 3 > 2^k + i > 3b_i$ and therefore $b_i < 2^k$. Thus $T_{b_i}$ is also the product of some $c_j$'s. In particular, Theorem 12 for the case $k = 1$ corresponds to Theorem 4. For $k = 2$, we have the following result on third order recurrences.

**Corollary 1.** *Let* $\{S_n\}_{n\ge0}$ *be defined by the recurrence* $S_{n+1} = d_0 S_n + d_1 S_{n-1} + d_2 S_{n-2}$ *with initial conditions* $S_0 = c_0$, $S_1 = c_1$, $S_2 = c_2$. *Then the run length transform of* $\{S_n\}$ *is given by* $\{T_n\}_{n\ge0}$ *satisfying*

- $T_0 = c_0$,
- $T_{2n} = T_n$,
- $T_{8n+1} = c_1 T_n$,
- $T_{8n+3} = c_2 T_n$,
- $T_{8n+5} = c_1 T_{2n+1}$, *and*
- $T_{8n+7} = d_0 T_{4n+3} + d_1 T_{2n+1} + d_2 T_n$.

**Theorem 13.** *The following relations hold for the function* $F$ *as defined in Definition 2:*

- $F(8n, 8k + i) = 0$ *for* $i = 1, \ldots, 7$,
- $F(8n + 1, 8k + i) = 0$ *for* $i = 2, \ldots, 7$,
- $F(8n + 3, 8k + i) = 0$ *for* $i = 4, 5, 6, 7$,
- $F(8n + 5, 8k + i) = 0$ *for* $i = 2, 3, 6, 7$,
- *If* $a_3 \wedge \neg a_1 \equiv 0 \pmod 8$ *and* $0 \le a_1, a_3 < 8$, *then* $F(8n + 1, 8k) = F(n, k)$,

- *If* $a_3 \wedge \neg a_1 \not\equiv 0 \pmod 8$, *then* $F(8n+1, 8k) = 0$,
- *If* $3a_3 \wedge \neg 3a_1 \equiv 0 \pmod 8$ *and* $0 \leq 3a_1, 3a_3 < 8$, *then* $F(8n+3, 8k) = F(n, k)$,
- *If* $3a_3 \wedge \neg 3a_1 \not\equiv 0 \pmod 8$, *then* $F(8n+3, 8k) = 0$,
- *If* $5a_3 \wedge \neg 5a_1 \equiv 0 \pmod 8$ *and* $0 \leq 5a_1, 5a_3 < 8$, *then* $F(8n+5, 8k) = F(n, k)$,
- *If* $5a_3 \wedge \neg 5a_1 \not\equiv 0 \pmod 8$, *then* $F(8n+5, 8k) = 0$.

**Lemma 9.** *The sequence* $a(n)$ *satisfies the following properties:*

- *If* $a_3 \wedge \neg a_1 \equiv 0 \pmod 8$ *and* $0 \leq a_1, a_3 < 8$, *then* $a(8n+1) = a(n) + \sum_{k=0}^{n} F(8n+1, 8k+1)$,
- *If* $3a_3 \wedge \neg 3a_1 \equiv 0 \pmod 8$ *and* $0 \leq 3a_1, 3a_3 < 8$, *then* $a(8n+3) = a(n) + \sum_{m=1}^{3} \sum_{k=0}^{n} F(8n+3, 8k+m)$,
- *If* $5a_3 \wedge \neg 5a_1 \equiv 0 \pmod 8$ *and* $0 \leq 5a_1, 5a_3 < 8$, *then* $a(8n+5) = a(n) + \sum_{m\in\{1,4,5\}} \sum_{k=0}^{n} F(8n+5, 8k+m)$,
- *If* $a_3 \wedge \neg a_1 \not\equiv 0 \pmod 8$, *then* $a(8n+1) = \sum_{k=0}^{n} F(8n+1, 8k+1)$,
- *If* $3a_3 \wedge \neg 3a_1 \not\equiv 0 \pmod 8$, *then* $a(8n+3) = \sum_{m=1}^{3} \sum_{k=0}^{n} F(8n+3, 8k+m)$,
- *If* $5a_3 \wedge \neg 5a_1 \not\equiv 0 \pmod 8$, *then* $a(8n+5) = \sum_{m\in\{1,4,5\}} \sum_{k=0}^{n} F(8n+5, 8k+m)$.

### 5.1. Run Length Transform of Narayana's Cows Sequence

Narayana's cows sequence (OEIS A000930) $\{b_n : n \geq 0\}$ is defined as $b_0 = b_1 = b_2 = 1$, $b_n = b_{n-1} + b_{n-3}$. The first few terms are: 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, ... The following results show that

$$a(n) = \sum_{k=0}^{n} \left[ \binom{n-k}{6k} \binom{n}{k} \pmod 2 \right]$$

is the run length transform of Narayana's cows sequence.

**Lemma 10.** *For* $a_1 = 1$, $a_2 = -1$, $a_3 = 0$, $a_4 = 6$, *the following relations hold for the function* $F$*:*

- $F(8n+1, 8k+1) = 0$,
- $F(8n+3, 8k+i) = 0$ *for* $1 \leq i \leq 3$,
- $F(8n+5, 8k+1) = F(8n+5, 8k+5) = 0$,
- $F(8n+5, 8k+4) = F(2n+1, 2k+1)$,
- $F(8n+7, 8k+i) = 0$ *for* $i \in \{3, 5, 6, 7\}$,

- $F(8n+7, 8k) = F(8n+7, 8k+1) = F(n,k)$,
- $F(8n+7, 8k+2) = F(4n+3, 4k+1)$,
- $F(8n+7, 8k+4) = F(4n+3, 4k+2)$,
- $F(4n+3, 4k+3) = 0$.

**Theorem 14.** *Let* $a(n) = \sum_{k=0}^{n} \left[ \binom{n-k}{6k} \binom{n}{k} \pmod 2 \right]$. *Then* $a(n)$ *satisfies the equations* $a(0) = 1$, $a(2n) = a(n)$, $a(8n+1) = a(8n+3) = a(n)$, $a(8n+5) = a(2n+1)$, *and* $a(8n+7) = a(n) + a(4n+3)$. *In particular, the sequence*

$$a(n) = \{1,1,1,1,1,1,1,2,1,1,1,1,1,1,2,3,1,1,1,1,1,1,1,2,1,1,1,1,2,2,\dots\}$$

*is the run length transform of Narayana's cows sequence.*

*Proof.* This is a consequence of Lemma 2, Corollary 1, Lemma 9, and Lemma 10. Note that by Lemma 10, $a(8n+5) = a(n) + \sum_{k=0}^{n} F(2n+1, 2k+1)$ which is equal to $a(2n+1)$ by Lemma 2. Similarly, $a(8n+7) = 2a(n) + \sum_{k=0}^{n} F(4n+3, 4k+1) + F(4n+3, 4k+2)$ which is equal to $a(n) + a(4n+3)$. □

### 5.2. Run Length Transform of $1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \dots$

The following result shows that $a(n) = \sum_{k=0}^{n} \left[ \binom{n+3k}{6k} \binom{n}{k} \pmod 2 \right]$ is equal to the run length transform of the sequence $1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \dots$ (OEIS A008619).

**Theorem 15.** *Let* $a(n) = \sum_{k=0}^{n} \left[ \binom{n+3k}{6k} \binom{n}{k} \pmod 2 \right]$. *Then* $a(n)$ *satisfies the equations* $a(0) = 1$, $a(2n) = a(n)$, $a(8n+1) = a(n)$, $a(8n+3) = 2a(n)$, $a(8n+5) = a(2n+1)$, *and* $a(8n+7) = a(4n+3) + a(2n+1) - a(n)$. *In particular, the sequence*

$$a(n) = \{1,1,1,2,1,1,2,2,1,1,1,2,2,2,2,3,1,1,1,2,1,1,2,2,2,2,2,4,2,2,\dots\}$$

*is the run length transform of the sequence* $1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \dots$.

## 6. Fourth Order Recurrences

We first start with the following Corollary to Theorem 4 for $k = 3$.

**Corollary 2.** *Let* $\{S_n\}_{n \geq 0}$ *be defined by the recurrence* $S_{n+1} = d_0 S_n + d_1 S_{n-1} + d_2 S_{n-2} + d_3 S_{n-3}$ *with initial conditions* $S_0 = c_0$, $S_1 = c_1$, $S_2 = c_2$, *and* $S_3 = c_3$. *Then the run length transform of* $\{S_n\}$ *is given by* $\{T_n\}_{n \geq 0}$ *satisfying*

- $T_0 = c_0$,
- $T_{2n} = T_n$,
- $T_{16n+1} = c_1 T_n$,
- $T_{16n+3} = c_2 T_n$,
- $T_{16n+5} = c_1^2 T_n$,
- $T_{16n+7} = c_3 T_n$,
- $T_{16n+9} = c_1 T_{2n+1}$,
- $T_{16n+11} = c_2 T_{2n+1}$,
- $T_{16n+13} = c_1 T_{4n+3}$,
- $T_{16n+15} = d_0 T_{8n+7} + d_1 T_{4n+3} + d_2 T_{2n+1} + d_3 T_n$.

**Theorem 16.** *The following relations hold for the function $F$ as defined in Definition 2:*

- $F(16n, 16k+i) = 0$ *for* $i = 1, \ldots, 15$,
- $F(16n+1, 16k+i) = 0$ *for* $i = 2, \ldots, 15$,
- $F(16n+3, 16k+i) = 0$ *for* $i = 4, \ldots, 15$,
- $F(16n+5, 16k+i) = 0$ *for* $i = 2, 3, 6, 7, 8, \ldots, 15$,
- $F(16n+7, 16k+i) = 0$ *for* $i = 8, \ldots, 15$,
- $F(16n+9, 16k+i) = 0$ *for* $i = 2, \ldots, 7$ *and* $i = 10, \ldots, 15$,
- $F(16n+11, 16k+i) = 0$ *for* $i = 4, 5, 6, 7, 12, 13, 14, 15$,
- $F(16n+13, 16k+i) = 0$ *for* $i = 2, 3, 6, 7, 10, 11, 14, 15$,
- *If* $ia_3 \wedge \neg ia_1 \equiv 0 \pmod{16}$ *and* $0 \leq ia_1, ia_3 < 16$, *then* $F(16n+i, 16k) = F(n,k)$ *for* $i \in \{1, 3, 5, 7, 9, 11, 13\}$,
- *If* $ia_3 \wedge \neg ia_1 \not\equiv 0 \pmod{16}$, *then* $F(16n+i, 16k) = 0$, *for* $i \in \{1, 3, 5, 7, 9, 11, 13\}$.

### 6.1. Run Length Transform of $1, 1, 2, 1, 3, 4, 7, 11, 18, \ldots$

Consider the sequence 1,1,2,1,3,4,7,11,18,... which is equal to the coefficients in the expansion of $\frac{1-2x^3}{1-x-x^2}$. This sequence (OEIS A329723) is also equal to the sequence formed by prepending the Lucas numbers (OEIS A000032) with the terms 1, 1. The following result shows that $a(n) = \sum_{k=0}^{n} \left[ \binom{n+2k}{2n-k} \binom{n}{k} \pmod 2 \right]$ is equal to the run length transform of this sequence.

**Theorem 17.** *Let* $a(n) = \sum_{k=0}^{n} \left[ \binom{n+2k}{2n-k} \binom{n}{k} \pmod 2 \right]$. *Then* $a(n)$ *satisfies the equations* $a(0) = 1$, $a(2n) = a(n)$, $a(16n+1) = a(n)$, $a(16n+3) = 2a(n)$, $a(16n+5) = a(n)$, *and* $a(16n+7) = a(n)$. *Furthermore,* $a(16n+9) = 2a(2n+1)$, $a(16n+11) = 2a(2n+1)$, *and* $a(16n+13) = a(4n+3)$. *Finally* $a(16n+15) = a(8n+7) + a(4n+3)$. *This implies that the sequence*

$$a(n) = \{1, 1, 1, 2, 1, 1, 2, 1, 1, 1, 1, 2, 2, 2, 1, 3, 1, 1, 1, 2, 1, 1, 2, 1, 2, 2, 2, 4, 1, 1, \dots\}$$

*is the run length transform of the extended Lucas sequence* $1, 1, 2, 1, 3, 4, 7, 11, 18, \dots$.

The sequences considered in the above sections and their run length transforms are summarized in Table 1. In the table, the coefficients $a_i$ describe the run length transform expressed as $\sum_{k=0}^{n} \left[ \binom{a_1 n + a_2 k}{a_3 n + a_4 k} \binom{n}{k} \pmod 2 \right]$.

| **Sequence description** | **OEIS index** | **Sequence terms** | **Coefficients $(a_1, a_2, a_3, a_4)$ of Run Length Transform** | **OEIS index of Run Length Transform** |
|---|---|---|---|---|
| Positive powers of 2 | A000079 | $1, 2, 4, 8, \dots$ | $(1, 0, 0, 1)$ | A001316 |
| Fibonacci sequence | A000045 | $1, 1, 2, 3, 5, 8, \dots$ | $(1, -1, 0, 2)$ | A246028 |
| Truncated Fibonacci sequence | | $1, 2, 3, 5, 8, 13, \dots$ | $(0, 3, 0, 1)$ | A245564 |
| 1 plus the positive powers of 2 | A011782 | $1, 1, 2, 4, 8, 16, \dots$ | $(1, 0, 0, 2)$ | A245195 |
| 1 followed by 2's | A040000 | $1, 2, 2, 2, 2, 2, \dots$ | $(1, 2, 0, 2)$ | A277561 |
| Positive integers | A000027 | $1, 2, 3, 4, 5, 6, \dots$ | $(1, 1, 1, -1)$ | A106737 |
| A sequence of 1's | A000012 | $1, 1, 1, 1, 1, 1, \dots$ | $(1, -1, 0, 1)$ | A000012 |
| Narayana's cows sequence | A000930 | $1, 1, 1, 2, 3, 4, 6, 9, \dots$ | $(1, -1, 0, 6)$ | A329720 |
| Positive integers repeated | A008619 | $1, 1, 2, 2, 3, 3, 4, 4, \dots$ | $(1, 3, 0, 6)$ | A278161 |
| Lucas sequence prepended with 1,1 | A329723 | $1, 1, 2, 1, 3, 4, 7, 11, \dots$ | $(1, 2, 2, -1)$ | A329722 |

Table 1: Table of various sequences and their run length transforms expressed as sums of products of binomial coefficients mod 2.

## References

[1] R. P. Brent and P. Zimmermann, *Modern Computer Arithmetic*, Cambridge Univ. Press, New York, 2010.

[2] N. Fine, Binomial coefficients modulo a prime, *Amer. Math. Monthly* **54** (1947), 589-592.

[3] A. Granville, Arithmetic properties of binomial coefficients I: Binomial coefficients modulo prime powers, in *Canad. Math. Soc. Conf. Proc.*, **20** (1997), 253-275.

[4] J. Leroy, M. Rigo, and M. Stipulanti, Generalized Pascal triangle for binomial coefficients of words, *Adv. Appl. Math.* **80** (2016), 24-47.

[5] P. Mathonet, M. Rigo, M. Stipulanti, and N. Zénaïdi, On digital sequences associated with Pascal's triangle, arXiv.2201.06636 (2022).

[6] D. A. Patterson and J. L. Hennessy, *Computer Organization and Design - The Hardware / Software Interface (Revised 4th Edition)*, The Morgan Kaufmann Series in Computer Architecture and Design, Academic Press, Cambridge, 2012.

[7] N. J. A. Sloane, On the number of ON cells in cellular automata, in S. Butler, J. Cooper, and G. Hurlbert, editors, *Connections in Discrete Mathematics: A Celebration of the Work of Ron Graham*, Cambridge Univ. Press, Cambridge, 2018, 13-38.

[8] I. Stewart, Four encounters with Sierpiński's gasket, *Math. Intelligencer* **17** (1995), 52-64.

[9] The OEIS Foundation Inc., The on-line encyclopedia of integer sequences, 1996-present, URL https://oeis.org/, founded in 1964 by N. J. A. Sloane.

[10] E. W. Weisstein, Sierpiński sieve. From MathWorld–A Wolfram Web Resource, URL http://mathworld.wolfram.com/SierpinskiSieve.html, [Online; accessed 21-April-2022].

[11] Wikipedia contributors, Bitwise operation — Wikipedia, the free encyclopedia, 2022, URL https://en.wikipedia.org/wiki/Bitwise_operation, [Online; accessed 21-April-2022].

[12] Wikipedia contributors, Two's complement — Wikipedia, the free encyclopedia, 2022, URL https://en.wikipedia.org/wiki/Two's_complement, [Online; accessed 10-July-2022].

[13] C. W. Wu, Sums of products of binomial coefficients mod 2 and run length transforms of sequences, arXiv:1610.06166 (2016-2022).