EXTENSION OF AN INEQUALITY FOR FIBONACCI NUMBERS

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Abstract

We extend results given by Popescu and Díaz-Barrero and by Alzer and Luca and determine all real parameters r and s such that the inequality

$$(F_nF_{n+1})^2 \le \sum_{k=1}^{n} F_k^r \sum_{k=1}^{n} F_k^s$$

holds for all $n \ge 1$. Here, F_k denotes the k-th Fibonacci number.

1. Introduction

In this paper, we study an inequality which involves the classical *Fibonacci numbers*, defined recursively by

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-2} + F_{n-1} \quad (n = 2, 3, ...).$$

Explicit representations are given by

$$F_n = \frac{1}{\sqrt{5}}\Big(\varphi^n - (1-\varphi)^n\Big) = \sum_{k=0}^{[(n-1)/2]} \binom{n-k-1}{k},$$

where

$$\varphi = \frac{1}{2}\Big(1+\sqrt{5}\Big) = 1.618...$$

denotes the golden ratio. Inspired by the elegant formula

$$F_nF_{n+1} = \sum_{k=1}^{n} F_k^2,$$

Popescu and Díaz-Barrero [2] used methods from Real Analysis to prove that the inequality

$$(F_nF_{n+1})^2 \leq \sum_{k=1}^{n} F_k^r \sum_{k=1}^{n} F_k^{4-r}$$

is valid for all natural numbers n and integers r. Recently, Alzer and Luca [1] obtained the following extensions of this result.

Theorem A. *Let $r, s \in \mathbb{R}$ with $r + s \geq 4$. Then, for $n \geq 1$,*

$$(F_nF_{n+1})^2 \leq \sum_{k=1}^{n} F_k^r \sum_{k=1}^{n} F_k^s. \tag{1}$$

The sign of equality is valid in (1) if and only if $n = 1, 2$ or $n \geq 3$, $r = s = 2$.

Theorem B. *Let $r, s \in \mathbb{R}$ with $rs \geq 0$. The inequality (1) holds for all $n \geq 1$ if and only if $r + s \geq 4$.*

An application gives that the equation

$$(F_nF_{n+1})^2 = \sum_{k=1}^{n} F_k^r \sum_{k=1}^{n} F_k^s \quad (r, s \in \mathbb{R},\ \ rs \geq 0)$$

holds for all $n \geq 1$ if and only if $r = s = 2$.

The aim of this paper is to extend Theorems A and B and to solve the following problem which is formulated in [1]: determine all real parameters r and s such that (1) is valid for all n.

In the next section, we introduce some helpful notation and in Section 3, we collect five lemmas. Finally, in Section 4, we state and prove our main result.

The numerical calculations have been carried out by using the computer software Maple 13.

2. Preliminaries

We denote by $\lambda = 4.12612...$ the only real number with

$$2^\lambda + 3^\lambda = \frac{221}{2}.$$

The function

$$A(x) = \frac{225}{2 + 2^x + 3^x} - 2$$

is strictly decreasing on $\mathbb{R}$ with $A(\lambda) = 0$. It follows that for each $r \in \mathbb{R}$ with $r < \lambda$ there exists a unique number $\mu(r) \in \mathbb{R}$ such that

$$2^{\mu(r)} + 3^{\mu(r)} = A(r).$$

The function μ is strictly decreasing on $(-\infty, \lambda)$ with

$$\lim_{r\to-\infty} \mu(r) = \lambda, \quad \mu(4) = -2.34572..., \quad \lim_{r\to\ \lambda^-} \mu(r) = -\infty.$$

In what follows, we maintain the notation introduced in this section.

3. Lemmas

First, we present upper and lower bounds for the ratio of Fibonacci numbers.

Lemma 1. *For $n \geq 9$, we have*

$$1.617 \leq \frac{F_{n+1}}{F_n} \leq 1.619. \tag{2}$$

Proof. We use induction to prove (2). Since

$$\frac{F_{10}}{F_9} = \frac{55}{34} = 1.6176...,$$

we conclude that (2) holds for $n = 9$. Next, we assume that (2) is valid. Applying

$$\frac{F_{n+2}}{F_{n+1}} = \frac{F_n + F_{n+1}}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}}$$

gives

$$1.6176... = 1 + \frac{1}{1.619} \leq \frac{F_{n+2}}{F_{n+1}} \leq 1 + \frac{1}{1.617} = 1.6184....$$

This implies that (2) holds with $n + 1$ instead of n. □

The following three lemmas provide lower bounds for power sums of Fibonacci numbers.

Lemma 2. *Let $r \in \mathbb{R}$ with $r > 0$. Then, for $n \geq 9$,*

$$\frac{1 - (p(r))^{n-8}}{1 - p(r)} F_n^r \leq \sum_{k=1}^{n} F_k^r \tag{3}$$

with $p(r) = (1.619)^{-r}$.

Proof. Let $n \geq 9$. Using Lemma 1 gives for $k \in \{9, ..., n\}$,

$$\frac{F_n}{F_k} = \prod_{j=k+1}^{n} \frac{F_j}{F_{j-1}} \leq (1.619)^{n-k}.$$

It follows that

$$\sum_{k=1}^{n} F_k^r \geq \sum_{k=9}^{n} F_k^r \geq F_n^r \sum_{k=9}^{n} (p(r))^{n-k} = F_n^r \frac{1-(p(r))^{n-8}}{1-p(r)}.$$

□

Lemma 3. *For $n \geq 5$, we have*

$$(F_n F_{n+1})^2 \leq 2 \sum_{k=1}^{n} F_k^{\lambda}. \tag{4}$$

Proof. Let

$$S_n = 2\sum_{k=1}^{n} F_k^{\lambda} - (F_n F_{n+1})^2.$$

By direct computation we get

$$S_5 = 156.32..., \quad S_6 = 1588.90..., \quad S_7 = 16816.20..., \quad S_8 = 152504.21.... \tag{5}$$

Let $n \geq 9$. From Lemma 1 we obtain

$$F_n^{\lambda} \geq F_n^2 \cdot F_n^2 \cdot F_n^{0.126} \geq F_n^2 \cdot \left(\frac{F_{n+1}}{1.619}\right)^2 \cdot F_9^{0.126} = a(F_n F_{n+1})^2 \tag{6}$$

with

$$a = \frac{F_9^{0.126}}{(1.619)^2} = 0.594....$$

Applying (3) with $r = \lambda$ and (6) leads to

$$2\sum_{k=1}^{n} F_k^{\lambda} \geq b_n (F_n F_{n+1})^2 \tag{7}$$

with

$$b_n = 2a\frac{1-(p(\lambda))^{n-8}}{1-p(\lambda)} \geq 2a = 1.189.... \tag{8}$$

From (5), (7) and (8) we conclude that (4) holds for $n \geq 5$. □

Lemma 4. *Let $r \in \mathbb{R}$ with $r \geq 4$. Then, for $n \geq 5$,*

$$(F_n F_{n+1})^2 \leq \frac{225}{2+2^r+3^r} \sum_{k=1}^{n} F_k^r. \tag{9}$$

Proof. We denote the expression on the right-hand side of (9) by $G_n(r)$. Then,

$$G_n(r) = 225\Big(1 + \sum_{k=5}^{n} \frac{F_k^r}{h(r)}\Big)$$

with $h(r) = 2 + 2^r + 3^r$. Since $F_k \geq 5$ $(k \geq 5)$, we conclude that

$$\frac{h(r)}{F_k^r} = \frac{2}{F_k^r} + \Big(\frac{2}{F_k}\Big)^r + \Big(\frac{3}{F_k}\Big)^r, \quad k \geq 5,$$

is decreasing on $(0, \infty)$. It follows that G_n is increasing on $(0, \infty)$. Thus, for $r \geq 4$,

$$G_n(r) \geq G_n(4) = \frac{25}{11} \sum_{k=1}^{n} F_k^4. \tag{10}$$

Let $n \geq 11$. We apply Lemma 2 with $r = 4$. This yields

$$G_n(4) \geq \frac{25}{11} \frac{1 - (p(4))^{n-8}}{1 - p(4)} F_n^4 \geq cF_n^4 \tag{11}$$

with

$$c = \frac{25}{11} \frac{1 - (p(4))^3}{1 - p(4)}.$$

From Lemma 1 we obtain

$$F_n^4 = (F_n F_{n+1})^2 \Big(\frac{F_n}{F_{n+1}}\Big)^2 \geq \frac{1}{(1.619)^2} (F_n F_{n+1})^2. \tag{12}$$

Using (11) and (12) gives

$$G_n(4) \geq c^* (F_n F_{n+1})^2 \tag{13}$$

with

$$c^* = \frac{c}{(1.619)^2} = 1.011.... \tag{14}$$

Combining (10), (13) and (14) we conclude that (9) holds for $n \geq 11$. Let

$$H_n = \frac{25}{11} \sum_{k=1}^{n} F_k^4 - (F_n F_{n+1})^2.$$

Then,

$$H_5 = 45.45..., \quad H_6 = 138.54..., \quad H_7 = 1336.90...,$$

$$H_8 = 8072.18..., \quad H_9 = 58095.45..., \quad H_{10} = 390845.48...,$$

so that (10) yields that (9) is also valid for $n = 5, ..., 10$. □

Our fifth lemma gives a necessary condition for which (1) is valid.

Lemma 5. *Let $r, s \in \mathbb{R}$ with $r > 0 > s$. If (1) holds for all $n \geq 1$, then $r \geq 4$.*

Proof. Let $r > 0 > s$. From (1) we obtain for $n \geq 1$,

$$U_n \leq R_n(r,s) \tag{15}$$

with

$$U_n = \varphi^2 \Big(\frac{F_n}{\varphi^n} \frac{F_{n+1}}{\varphi^{n+1}} \Big)^2$$

and

$$R_n(r,s) = X_n(r) Y_n(s) Z_n(r),$$

where

$$X_n(r) = \frac{1}{\varphi^{rn}} \sum_{k=1}^{n} F_k^r, \quad Y_n(s) = \sum_{k=1}^{n} F_k^s, \quad Z_n(r) = \varphi^{(r-4)n}.$$

Since

$$\lim_{n \to \infty} \frac{F_n}{\varphi^n} = \frac{1}{\sqrt{5}},$$

we get

$$\lim_{n \to \infty} U_n = \frac{\varphi^2}{25}. \tag{16}$$

Next, we show that the sequences $(X_n(r))$ and $(Y_n(s))$ are bounded with respect to n. We have for $k \geq 1$,

$$\sqrt{5} F_k = \varphi^k - (-1)^k (\varphi - 1)^k \leq \varphi^k + (\varphi - 1)^k \leq 2\varphi^k$$

and

$$\sqrt{5} F_k \geq \varphi^k - (\varphi - 1)^k \geq \varphi^k - \varphi^k \frac{\varphi - 1}{\varphi} \geq \frac{1}{2} \varphi^k.$$

Let $u = \varphi^r > 1$. Then,

$$\Big(\frac{\sqrt{5}}{2} \Big)^r X_n(r) \leq \frac{1}{\varphi^{rn}} \sum_{k=1}^{n} \varphi^{rk} = \frac{u}{u-1} \frac{u^n - 1}{u^n} \leq \frac{u}{u-1}.$$

Let $v = \varphi^s \in (0,1)$. We obtain

$$(2\sqrt{5})^s Y_n(s) \leq \sum_{k=1}^{n} \varphi^{sk} = \frac{v}{1-v} (1 - v^n) \leq \frac{v}{1-v}.$$

Now, we assume that $r < 4$. Since $0 < \varphi^{r-4} < 1$, we get

$$\lim_{n \to \infty} Z_n(r) = 0.$$

Thus,

$$\lim_{n \to \infty} R_n(r,s) = 0. \tag{17}$$

The limit relations (16) and (17) contradict (15). Therefore, $r \geq 4$. □

4. Main Result

We are now in a position to state and prove our main result.

Theorem. *Let $r, s \in \mathbb{R}$ with $r \geq s$. The inequality*

$$(F_n F_{n+1})^2 \leq \sum_{k=1}^{n} F_k^r \sum_{k=1}^{n} F_k^s \tag{18}$$

holds for all $n \geq 1$ if and only if (I) $rs \geq 0$ *and* $r + s \geq 4$ *or* (II) $rs < 0$ *and either* (i) $4 \leq r < \lambda$ *and* $s \geq \mu(r)$ *or* (ii) $r \geq \lambda$.

Proof. From Theorem A we conclude that if $rs \geq 0$ and $r+s \geq 4$, then (18) is valid for all $n \geq 1$. Let $rs < 0$. If $n = 1$ or $n = 2$, then equality holds in (18). To prove (18) for $n \geq 3$ we consider two cases.

Case 1. $4 \leq r < \lambda$ and $s \geq \mu(r)$.

Let $n = 3$. We obtain

$$(F_3F_4)^2 = 36 = (2 + 2^4) \cdot 2 \leq (2 + 2^r)(2 + 2^s) = \sum_{k=1}^{3} F_k^r \sum_{k=1}^{3} F_k^s.$$

Let $n = 4$. Then,

$$\begin{aligned}(F_4F_5)^2 = 225 &= (2 + 2^r + 3^r)(2 + 2^{\mu(r)} + 3^{\mu(r)}) \\ &\leq (2 + 2^r + 3^r)(2 + 2^s + 3^s) \\ &= \sum_{k=1}^{4} F_k^r \sum_{k=1}^{4} F_k^s.\end{aligned}$$

Let $n \geq 5$. Using Lemma 4 gives

$$(F_nF_{n+1})^2 \leq \frac{225}{2 + 2^r + 3^r} \sum_{k=1}^{n} F_k^r = \sum_{k=1}^{n} F_k^r \sum_{k=1}^{4} F_k^{\mu(r)} \leq \sum_{k=1}^{n} F_k^r \sum_{k=1}^{n} F_k^s.$$

Case 2. $r \geq \lambda$.

Let $n = 3$. Then,

$$(F_3F_4)^2 = 2 \cdot (2 + 2^4) \leq 2 \cdot (2 + 2^r) = 2\sum_{k=1}^{3} F_k^r \leq \sum_{k=1}^{3} F_k^r \sum_{k=1}^{3} F_k^s.$$

Let $n = 4$. We obtain

$$(F_4F_5)^2 = 2 \cdot (2 + 2^\lambda + 3^\lambda) = 2\sum_{k=1}^{4} F_k^\lambda \leq \sum_{k=1}^{4} F_k^r \sum_{k=1}^{4} F_k^s.$$

Let $n \geq 5$. We apply Lemma 3 and get

$$(F_n F_{n+1})^2 \leq 2 \sum_{k=1}^{n} F_k^{\lambda} \leq \sum_{k=1}^{n} F_k^{r} \sum_{k=1}^{n} F_k^{s}.$$

Next, we assume that (18) holds for all $n \geq 1$. Let $r \geq s$.

Case 1. $rs \geq 0$. From Theorem B we conclude that $r + s \geq 4$.

Case 2. $rs < 0$. Applying Lemma 5 gives $r \geq 4$. Let $r < \lambda$. From (18) with $n = 4$ we obtain

$$A(r) = \frac{225}{2 + 2^r + 3^r} - 2 \leq 2^s + 3^s = g(s), \quad \text{say.}$$

Moreover, we have

$$A(r) = g\big(\mu(r)\big).$$

Thus,

$$g\big(\mu(r)\big) \leq g(s).$$

Since g is strictly increasing on $\mathbb{R}$, we get $\mu(r) \leq s$. This completes the proof of the Theorem. □

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References

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