MULTIPLICATIVE FUNCTIONS WHICH ARE ADDITIVE ON SUMS OF TWO NONZERO SQUARES

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Abstract

Let f be a multiplicative function which satisfies

$$f(a^2 + b^2 + c^2 + d^2) = f(a^2 + b^2) + f(c^2 + d^2)$$

for positive integers a, b, c, and d. We show that f is the identity function provided that $f(3)\,f(11) \neq 0$. Otherwise, $f(n) = 0$ for all $n \geq 2$ except for $n = 3, 9$, and 11.

1. Introduction

Let S be a set of arithmetic functions and E be a set of positive integers. If an arithmetic function $f \in S$ is uniquely determined under the condition

$$f(m + n) = f(m) + f(n) \quad \text{for all } m, n \in E,$$

we call E an *additive uniqueness set for* S.

Claudia Spiro [7], who coined the term, proved in 1992 that the set P of all primes is an additive uniqueness set for the set of multiplicative functions f with $f(p_0) \neq 0$ for some $p_0 \in P$. Since her igniting study, a number of mathematicians have been studying various problems related with the additive condition.

P. V. Chung [1] in 1996 classified multiplicative functions satisfying

$$f(m^2 + n^2) = f(m^2) + f(n^2) \quad \text{for all } m, n \in \mathbb{N}.$$

He showed that the set of positive squares is an additive uniqueness set for completely multiplicative functions, but is not one for mere multiplicative functions.

He and B. M. Phong [2] in 1999 proved that the set of triangular numbers is an additive uniqueness set for multiplicative functions. Also, so is the set of tetrahedral numbers.

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In the present article, we consider the set of sums of two nonzero squares. That is, our question is whether a multiplicative function f is the identity function or not, provided f satisfies

$$f(a^2+b^2+c^2+d^2) = f(a^2+b^2) + f(c^2+d^2) \quad \text{for all } a,b,c,d \in \mathbb{N}.$$

Similar researches were performed by K.-H. Indlekofer and B. M. Phong [4] in 2006. They characterized all multiplicative functions f satisfying

$$f(n^2+m^2+k+1) = f(n^2+k) + f(m^2+1)$$

with fixed $k \in \mathbb{N}$ and all $n, m \in \mathbb{N}$.

Recently, in 2016, B. M. Phong [6] studied multiplicative functions f and g satisfying

$$f(m^2+n^2+a+b) = g(m^2+a) + g(n^2+b) \quad \text{for all } m,n \in \mathbb{N},$$

where a and b are some non-negative integers.

But, the condition of our problem can be said to be stronger, so the set of sums of two nonzero squares can be an additive uniqueness set for multiplicative functions f with $f(n) \neq 0$ for some $n \in \mathbb{N} \setminus \{1,3,9,11\}$.

2. Main Theorem

Theorem 1. *If a multiplicative function f satisfies*

$$f(a^2+b^2+c^2+d^2) = f(a^2+b^2) + f(c^2+d^2)$$

for positive integers a, b, c, and d, then f is one of the following:

1. *f is the identity function,*
2. *$f(n) = 0$ for $n \geq 2$,*
3. *$f(3)\,f(9) \neq 0$ and $f(n) = 0$ for other $n \geq 2$,*
4. *$f(9) \neq 0$ and $f(n) = 0$ for other $n \geq 2$,*
5. *$f(11) \neq 0$ and $f(n) = 0$ for other $n \geq 2$.*

Corollary 1. *Let $E = \{1,2,4,5,8,9,10,\dots\}$ be the set of nonzero sums of two squares. If a multiplicative function f satisfies $f(a+b) = f(a) + f(b)$ for $a,b \in E$, then f is the identity function. That is, E is an additive uniqueness set for multiplicative functions.*

We need a condition for an integer to be represented as a sum of 4 nonzero squares to prove the main theorem. In 1911, E. Dubouis [3] classified the general conditions.

Lemma 1 (Dubouis)**.** *Every integer n can be represented as a sum of k nonzero squares except*

$$n = \begin{cases} 1, 3, 5, 9, 11, 17, 29, 41, 2\cdot 4^m, 6\cdot 4^m, 14\cdot 4^m \ (m \geq 0) & \text{if } k = 4, \\ 33 & \text{if } k = 5, \\ 1, 2, \ldots, k-1, k+1, k+2, k+4, k+5, k+7, k+10, k+13 & \text{if } k \geq 5. \end{cases}$$

We now present a proof of Theorem 1.

Proof of Theorem 1. We compute some $f(n)$ for small n's by using the following equalities:

$$\begin{aligned} f(1^2+1^2+1^2+1^2) &= f(4) \\ &= 2\,f(2) \\ f(1^2+1^2+1^2+2^2) &= f(7) \\ &= f(2)+f(5) \\ f(1^2+1^2+2^2+2^2) &= f(2)\,f(5) \\ &= 2\,f(5) \\ &= f(2)+f(8) \\ f(1^2+1^2+1^2+3^2) &= f(3)\,f(4) \\ &= f(2)+f(2)f(5) \\ f(1^2+2^2+2^2+2^2) &= f(13) \\ &= f(5)+f(8) \\ f(1^2+1^2+2^2+3^2) &= f(3)\,f(5) \\ &= f(5)+f(2)f(5) \\ f(2^2+2^2+2^2+3^2) &= f(3)\,f(7) \\ &= f(8)+f(13). \end{aligned}$$

Put $x = f(2)$, $y = f(3)$, and $z = f(5)$. Then, $f(4) = 2x$. From the equalities for $f(10)$ we have that

$$xz = 2z = x + f(8).$$

We divide two cases according to z. First, assume that $z \neq 0$. Then, $x = 2$ and $f(8) = 2z - x$ from the above equality.

Since $f(4) = 2x$, from the equalities for $f(12)$ we obtain that

$$y \cdot 2x = x + xz \text{ or } 2y = 1 + z.$$

Now, from the equality $yz = z(1 + x)$ for $f(15)$ we get $y = 3$ and $z = 5$.

Thus, if $f(5) \neq 0$, then we can easily check that $f(n) = n$ for $n = 1, 2, \ldots, 21$. Note that $f(9)$, $f(11)$, and $f(17)$ can be calculated by using $f(2 \cdot 9)$, $f(2 \cdot 11)$, and $f(2 \cdot 17)$.

Let us consider the second case $z = 0$. Then, $f(8) = f(13) = -x$. From equalities for $f(12)$ and $f(21)$ we obtain that

$$y \cdot 2x = x \text{ and } yx = -2x.$$

Thus, x should vanish and $f(n) = 0$ for $2 \leq n \leq 21$ and $n \neq 3, 9, 11, 17, 19$. But we can find that $f(17) = f(19)$ from $f(19) = f(17) + f(2)$.

In this case, we determine $f(17) = f(19) = f(25) = 0$ from

$$f(25) = f(8) + f(17) = f(5) + f(4)\, f(5).$$

Also, we obtain $f(3)\, f(11) = 0$ from

$$f(33) = f(3)\, f(11) = f(8) + f(25) = f(13) + f(4)\, f(5).$$

Now, we separate the proof into three cases.

(i) $f(n) = n$ for $1 \leq n \leq 21$

Now, we use induction to show that f is the identity function in this case. Assume that $f(n) = n$ for all $n < N$. If $N = a^2 + b^2 + c^2 + d^2$ for some positive integers a, b, c, and d, then $f(a^2 + b^2) = a^2 + b^2$ and $f(c^2 + d^2) = c^2 + d^2$ by induction hypothesis. Thus, $f(N) = N$.

If N cannot be represented as a sum of four nonzero squares, N is 29, 41, $2 \cdot 4^m$, $6 \cdot 4^m$, or $14 \cdot 4^m$ by Lemma 1.

We can obtain $f(29) = 29$ from

$$f(1^2 + 2^2 + 2^2 + 5^2) = f(2)\, f(17) = f(5) + f(29).$$

Similarly, $f(41) = 41$ from

$$f(1^2 + 3^2 + 4^2 + 5^2) = f(3)\, f(17) = f(10) + f(41).$$

If $N = 2 \cdot 4^m$, then $f(2 \cdot 4^m) = 4\, f(2 \cdot 4^{m-1}) = 2 \cdot 4^m$ from

$$\begin{aligned} f(4^{m-1} + 4^{m-1} + 4^m + 4^m) &= f(2 \cdot 4^{m-1})\, f(5) \\ &= f(2 \cdot 4^{m-1}) + f(2 \cdot 4^m) \end{aligned}$$

and the induction hypothesis.

Other cases can be calculated by

$$f(6 \cdot 4^m) = f(3)\, f(2 \cdot 4^m) \text{ and } f(14 \cdot 4^m) = f(7)\, f(2 \cdot 4^m).$$

Hence, $f(n) = n$ for all $n \geq 1$.

(ii) $f(3) = 0$

We have $f(n) = 0$ for $2 \leq n \leq 21$ and $n \neq 9, 11$. We use induction again. Assume that $f(n) = 0$ for $n < N$ except for $n = 9, 11$.

Suppose N is a sum of four nonzero squares. Note that neither 9 nor 11 is a sum of two nonzero squares. So $f(9)$ and $f(11)$ are not used to calculate $f(N)$. Thus, $f(N) = 0$. Specifically,

$$f(99) = f(9)\, f(11) = f(1^2 + 1^2) + f(4^2 + 9^2) = 0$$

and thus, $f(9) = 0$ or $f(11) = 0$.

If N is not a sum of four nonzero squares, then N is 29, 41, $2 \cdot 4^m$, $6 \cdot 4^m$, or $14 \cdot 4^m$. We can obtain $f(N)$ for these N by the same ways as the previous case.

(iii) $f(11) = 0$

In this case, we have that $f(n) = 0$ for $2 \leq n \leq 21$ and $n \neq 3, 9$. By induction, $f(N) = 0$ if N is a sum of four nonzero squares. The exceptional numbers 29, 41, $2 \cdot 4^m$, $6 \cdot 4^m$, and $14 \cdot 4^m$ can be dealt with the same ways as the previous case. But, since 3 and 9 are not sums of two nonzero squares and they are not relatively prime, we cannot determine $f(3)$ and $f(9)$.

Therefore, the proof is completed. □

The corollary immediately follows because $f(2) = f(1 + 1) = f(1) + f(1) = 2$.

References

[1] P. V. Chung, Multiplicative functions satisfying the equation $f(m^2 + n^2) = f(m^2) + f(n^2)$, *Math. Slovaca* **46** (1996), 165–171.

[2] P. V. Chung and B. M. Phong, Additive uniqueness sets for multiplicative functions, *Publ. Math. Debrecen*, **55** (1999), 237–243.

[3] E. Dubouis, Solution of a problem of J. Tannery, *Intermédiaire Math.* **18** (1911), 55–56.

[4] K.-H. Indlekofer and B. M. Phong, Additive uniqueness sets for multiplicative functions, *Ann. Univ. Sci. Budapest. Sect. Comput.* **26** (2006), 65–77.

[5] B. M. Phong, A characterization of the identity function, *Acta Acad. Paed. Agriensis Sect. Math.* **24** (1997), 3–9.

[6] B. M. Phong, Additive uniqueness sets for a pair of multiplicative functions, *Ann. Univ. Sci. Budapest. Sect. Comput.* **45** (2016), 199–221.

[7] C. A. Spiro, Additive uniqueness sets for arithmetic functions, *J. Number Theory* **42** (1992), 232–246.