ON DENSENESS OF CERTAIN DIRECTION AND GENERALIZED DIRECTION SETS

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Abstract

Direction sets, recently introduced by Leonetti and Sanna, are generalizations of ratio sets of subsets of positive integers. In this article, we generalize the notion of direction sets and define *k-generalized direction sets* and *distinct k-generalized direction sets* for subsets of positive integers. We prove a necessary condition for a subset of $\mathcal{S}^{k-1} := \{\underline{x} \in [0,1]^k : ||\underline{x}|| = 1\}$ to be realized as the set of accumulation points of a distinct k-generalized direction set. We provide sufficient conditions for some particular subsets of positive integers so that the corresponding k-generalized direction sets are dense in $\mathcal{S}^{k-1}$. We also consider the denseness properties of certain direction sets and give a partial answer to a question posed by Leonetti and Sanna. Finally we consider a similar question in the framework of an algebraic number field.

1. Introduction and Statements of Results

For a non-empty set $A \subseteq \mathbb{N}$, the *ratio set* of A is defined by $R(A) := \{\frac{a}{b} \in \mathbb{Q} : a, b \in A\}$. One of the most fundamental results in real analysis, viz. $\mathbb{Q}$ is dense in $\mathbb{R}$, when rephrased in terms of ratio sets, reads as the ratio set of $\mathbb{N}$ is dense in $\mathbb{R}_{>0}$. This reformulation of the denseness of $\mathbb{Q}$ in $\mathbb{R}$ has spurred a lot of research in recent times. In particular, the classification of subsets of $\mathbb{N}$ having dense ratio sets in $\mathbb{R}_{>0}$ has been a central question of investigation. In what follows, we say that A is *fractionally dense* in $\mathbb{R}_{>0}$ if $R(A)$ is dense in $\mathbb{R}_{>0}$.

One of the most natural choices for A is the set $\mathbb{P}$ of prime numbers and it is known to be fractionally dense (cf. [16], [19]). Several generalizations of this result have been proven over the years and several interesting subsets of natural numbers have been shown to be fractionally dense (cf. [3] - [7], [11], [14] - [16], [19] - [21], [24] - [27]). In [8], [11] and [23], analogous questions have been dealt with in the set up of algebraic number fields. Very recently, the denseness of ratio sets in the p-adic completion $\mathbb{Q}_p$ have also been considered (cf. [1], [2], [12], [13], [18], [22]).

Very recently, Leonetti and Sanna [17] introduced the notion of *direction sets*, which generalizes the notion of ratio sets as follows. For an integer $k \geq 2$ and $\emptyset \neq A \subseteq \mathbb{N}$, they considered the following sets:

$$\mathcal{S}^{k-1} := \{\underline{x} \in [0,1]^k : ||\underline{x}|| = 1\},\ \mathcal{D}^k(A) := \{\rho(\underline{a}) : \underline{a} \in A^k\}$$

and

$$\mathcal{D}^{\underline{k}}(A) := \{\rho(\underline{a}) : \underline{a} \in A^{\underline{k}}\},$$

where $\rho : \mathbb{R}^k \setminus \{\underline{0}\} \to \mathcal{S}^{k-1}$ is the map defined by $\rho(\underline{x}) = \frac{\underline{x}}{||\underline{x}||}$ and $A^{\underline{k}} = \{\underline{a} \in A^k : a_i \neq a_j \text{ for all } i \neq j\}$. The sets $\mathcal{D}^k(A)$ and $\mathcal{D}^{\underline{k}}(A)$ are called the k-direction sets of A. We note that, for $k = 2$, we can identify $\mathcal{S}^1$ with $[0, +\infty]$ via a bijective map and thus the question of denseness in $\mathbb{R}_{>0}$ can be translated into that in $\mathcal{S}^1$. Therefore, direction sets are indeed generalizations of ratio sets. Leonetti and Sanna [17, Theorem 1.2] proved a necessary and sufficient criterion that determines whether a set $X \subseteq \mathcal{S}^{k-1}$ can be realized as the set of accumulation points of $\mathcal{D}^{\underline{k}}(A)$ for some $A \subseteq \mathbb{N}$. Moreover, they proved a sufficient condition (cf. [17, Theorem 1.5]) that asserts whether $\mathcal{D}^k(A)$ is dense in $\mathcal{S}^{k-1}$.

In this article, we further generalize the notion of direction sets and introduce generalized k-direction sets as follows.

Definition 1. Let $k \geq 2$ be an integer and let $U_1, \ldots, U_k$ be non-empty subsets of $\mathbb{N}$. We define the *k-generalized direction set* for the k-tuple $(U_1, \ldots, U_k)$ to be $\mathcal{D}^k(U_1, \ldots, U_k) := \{\rho(u_1, \ldots, u_k) : u_j \in U_j \text{ for } j = 1, \ldots, k\}$. Also, we define the *distinct k-generalized direction set* to be $\mathcal{D}^{\underline{k}}(U_1, \ldots, U_k) := \{\rho(u_1, \ldots, u_k) : u_j \in U_j \text{ for } j = 1, \ldots, k \text{ and } u_i \neq u_j \text{ for all } i \neq j\}$.

Our first theorem is an analogue of Theorem 1.2 of [17] for distinct k-generalized direction sets. For any set $X \subseteq \mathcal{S}^{k-1}$, we denote by X' the set of accumulation points of X. Also, we denote by S_k the symmetric group on k elements $\{1, \ldots, k\}$. For a permutation $\pi \in S_k$, we define $\pi(x_1, \ldots, x_k) := (x_{\pi(1)}, \ldots, x_{\pi(k)})$ for all $\underline{x} = (x_1, \ldots, x_k)$ in $\mathcal{S}^{k-1}$. Also, for any subset I of $\{1, \ldots, k\}$, we define $\rho_I(\underline{x}) := \rho(\underline{y})$ where $\underline{y} = (y_1, \ldots, y_k)$ is defined as $y_i := x_i$ if $i \in I$ and $y_i := 0$ if $i \notin I$. We say that I *meets* $\underline{x}$ if $x_i \neq 0$ for some $i \in I$.

Our theorem provides a necessary condition for a set $X \subseteq \mathcal{S}^{k-1}$ to be realized as the set of accumulation points of $\mathcal{D}^{\underline{k}}(U_1, \ldots, U_k)$. This indeed extends the necessary conditions of Theorem 1.2 of [17] when all the U_i's are equal. It would be interesting to investigate whether the necessary conditions of the following theorem are also sufficient. We state our first theorem as follows.

Theorem 1. *Let $k \geq 2$ be an integer. For subsets $U_1, \ldots, U_k$ of $\mathbb{N}$, let $X = \mathcal{D}^{\underline{k}}(U_1, \ldots, U_k)'$. Then, we have:*

(i) X is a closed subset of $\mathcal{S}^{k-1}$.

(ii) If $U_{i_1} = \cdots = U_{i_m}$ for some $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k\}$, then for $\pi \in S_k$ with $\pi(j) = j$ for all $j \notin \{i_1, \ldots, i_m\}$, we have $\pi(\underline{x}) \in X$ for every $\underline{x} \in X$.

(iii) If $|U_i| \geq k$ for each $i \in \{1, \ldots, k\}$, then for every $I \subseteq \{1, \ldots, k\}$ that meets $\underline{x}$, we have $\rho_I(\underline{x}) \in X$.

We recall that for a non-empty set $A \subseteq \mathbb{N}$, the natural density of A is defined as $d(A) := \lim_{X \to \infty} \frac{\#\{n \in A : n \leq X\}}{X}$, provided the limit exists. The next theorem provides a sufficient condition for $\mathcal{D}^k(U_1, \ldots, U_k)$ to be dense in $\mathcal{S}^{k-1}$.

Theorem 2. *Let $k \geq 2$ be an integer and let $U_1, \ldots, U_k \subseteq \mathbb{N}$ be such that $d(U_i)$ exists and equals $\delta_i > 0$ for all $i = 1, \ldots, k$. Assume that $\bigcap_{i=1}^{k} U_i$ is an infinite set. Then $\mathcal{D}^k(U_1, \ldots, U_k)$ is dense in $\mathcal{S}^{k-1}$.*

The next theorem extends Theorem 1.5 of [17], which asserts that if for a set $A \subseteq \mathbb{N}$, there exists an increasing sequence $\{a_n\}_{n=1}^{\infty} \subseteq A$ with $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1$, then $\mathcal{D}^k(A)$ is dense in $\mathcal{S}^{k-1}$. We generalize this for $\mathcal{D}^k(U_1, \ldots, U_k)$ as follows.

Theorem 3. *Let $k \geq 2$ be an integer and let $U_1, U_2, \ldots, U_k$ be non-empty subsets of $\mathbb{N}$. If there exist increasing sequences $u_i^{(n)} \subseteq U_i$ for all $i \in \{1, \ldots, k\}$ such that $\lim_{n \to \infty} \frac{u_i^{(n-1)}}{u_i^{(n)}} = 1$, then $\mathcal{D}^k(U_1, \ldots, U_k)$ is dense in $\mathcal{S}^{k-1}$.*

Remark 1. For an integer $k \geq 2$ and for each $i \in \{1, \ldots, k\}$, let a_i and m_i be integers with $\gcd(a_i, m_i) = 1$. Let $\mathbb{P}_{m_i} := \{p \in \mathbb{P} : p \equiv a_i \pmod{m_i}\}$. For $U_i = \mathbb{P}_{m_i}$, using Dirichlet's theorem for primes in arithmetic progressions, we see that the hypotheses of Theorem 3 are satisfied. Therefore, $\mathcal{D}^k(\mathbb{P}_{m_1}, \ldots, \mathbb{P}_{m_k})$ is dense in $\mathcal{S}^{k-1}$.

Theorem 4. *Let $k \geq 2$ be an integer. For each $i \in \{1, \ldots, k\}$, let $f_i(X_1, \ldots, X_m) \in \mathbb{Z}[X_1, \ldots, X_m]$ be polynomials of total degree d_i such that the sum of the coefficients of degree d_i terms is positive. Let $U_i := \{f_i(n_1, \ldots, n_m) | (n_1, \ldots, n_m) \in \mathbb{N}^m\} \cap \mathbb{N}$. Then $\mathcal{D}^k(U_1, \ldots, U_k)$ is dense in $\mathcal{S}^{k-1}$.*

In [5], it is proven that there is a 3-partition of $\mathbb{N} = A \cup B \cup C$, such that none of $R(A), R(B)$ and $R(C)$ is dense in $\mathbb{R}_{>0}$. That is, none of $\mathcal{D}^2(A), \mathcal{D}^2(B)$ and $\mathcal{D}^2(C)$ is dense in $\mathcal{S}^1$. In [17], Leonetti and Sanna asked for a possible generalization of this result for $k \geq 3$ [17, Question 1.9]. We give a partial answer to their question in the next theorem.

Theorem 5. *Let $k \geq 3$ be an integer. Then there exists a 3-partition $\mathbb{N} = A \cup B \cup C$ of $\mathbb{N}$ such that none of $\mathcal{D}^k(A), \mathcal{D}^k(B)$ or $\mathcal{D}^k(C)$ is dense in $\mathcal{S}^{k-1}$.*

Remark 2. In view of Theorem 5, it remains to be seen whether for a 2-partition $\mathbb{N} = A \cup B$, either $\mathcal{D}^k(A)$ or $\mathcal{D}^k(B)$ is dense in $\mathcal{S}^{k-1}$ or not. We note that Theorem 3 cannot be directly applied to resolve this issue. In other words, we exhibit a 2-partition of $\mathbb{N}$, neither part of which contains a sequence with the ratio of consecutive terms converging to 1. This can be seen by considering $A = \bigcup_{k=0}^{\infty} [3^k, 2 \cdot 3^k) \cap \mathbb{N}$ and $B = \bigcup_{k=0}^{\infty} [2 \cdot 3^k, 3^{k+1}) \cap \mathbb{N}$. For, if $\{a_n\}_{n=1}^{\infty} \subseteq A$ is an infinite sequence, then there are infinitely many indices i for which $a_i \in [3^k, 2 \cdot 3^k)$ and $a_{i+1} \in [3^\ell, 2 \cdot 3^\ell)$ for $k < \ell$. Then it follows that $\frac{a_i}{a_{i+1}} < \frac{2 \cdot 3^k}{3^\ell} \leq \frac{2}{3}$. Therefore, the elements of the sequence $\{\frac{a_n}{a_{n+1}}\}_{n=1}^{\infty}$ cannot get arbitrarily close to 1. A similar argument works for B as well. This itself is an interesting question to decide whether $\mathcal{D}^k(A)$ or $\mathcal{D}^k(B)$ is dense in $\mathcal{S}^{k-1}$.

One of the interesting questions in the literature of fractionally dense sets is to look for sets $A \subseteq \mathbb{N}$ such that the ratio set $R(A)$ is dense in $\mathbb{R}_{>0}$ but A contains no 3-term arithmetic progressions. One such set is $A = \{2^m : m \geq 2\} \cup \{3^n : n \geq 2\}$, which is known to be fractionally dense in $\mathbb{R}_{>0}$ but A contains no 3-term arithmetic progressions (cf. [3, Proposition 6]).

Theorem 6. *There exists a set $A \subseteq \mathbb{N}$ such that A contains no 3-term arithmetic progressions and $\mathcal{D}^k(A)$ is dense in $\mathcal{S}^{k-1}$.*

Remark 3. We shall see in the proof of Theorem 6 that we can obtain infinitely many sets $A \subseteq \mathbb{N}$ having no arithmetic progression of length 3 such that $\mathcal{D}^k(A)$ is dense in $\mathcal{S}^{k-1}$.

Next, we discuss the denseness of some particular type of sets whose properties have been recently considered in [10]. For an arithmetic function $f : \mathbb{N} \to \mathbb{N}$ and a positive real number X, let $f_X := \#\{n \leq X : n = kf(k) \text{ for some } k \in \mathbb{N}\}$. Keeping this notation, we state the results of [10] as follows.

Theorem 7. *[10] (i) Let* $\omega(n) = \sum_{\substack{p|n \\ p \in \mathbb{P}}} 1$ *be the prime divisor function. Then*

$$\omega_X = \frac{X}{\log\log X} + o\left(\frac{X}{\log\log X}\right).$$

(ii) Let $\phi(n) = \#\{1 \le k \le n : \gcd(k,n) = 1\}$ *be the Euler's totient function. Then*

$$\phi_X = cX^{\frac{1}{2}} + o(X^{\frac{1}{2}}),$$

where $c = \prod_p \left(1 + \frac{1}{p(p-1+\sqrt{p^2-p})}\right) \sim 1.365\ldots$.

Now, we state our result as follows.

Theorem 8. *Let* $A = \{n\omega(n) : n \in \mathbb{N}\}$ *and* $B = \{n\phi(n) : n \in \mathbb{N}\}$*. Then for any integer* $k \ge 2$*, we have that both* $\mathcal{D}^k(A)$ *and* $\mathcal{D}^k(B)$ *are dense in* $\mathcal{S}^{k-1}$*.*

2. Proof of Theorems

In this section, we prove our theorems. We first prove Theorem 1.

Proof of Theorem 1. Since X is the set of accumulation points of a subset of $\mathcal{S}^{k-1}$, we immediately conclude that X is closed and (i) is satisfied.

To prove (ii), let $U_{i_1} = \cdots = U_{i_m}$ for some $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k\}$ and let $\underline{x} = (x_1, x_2, \ldots, x_k) \in X = \mathcal{D}^{\underline{k}}(U_1, \ldots, U_k)'$. Then there exists a sequence $\rho(\underline{a}^{(n)}) \in \mathcal{D}^{\underline{k}}(U_1, \ldots, U_k)$ converging to $\underline{x}$ such that $\rho(\underline{a}^{(n)}) \ne \underline{x}$ for infinitely many n, where $\underline{a}^{(n)} \in \prod_{i=1}^{k} U_i$. For $\pi \in S_k$ with $\pi(j) = j$ for all $j \notin \{i_1, \ldots i_m\}$, we consider $\underline{b}^{(n)} := \pi(\underline{a}^{(n)}) \in \mathcal{D}^{\underline{k}}(U_1, \ldots, U_k)$. Then $\rho(\underline{b}^{(n)})$ converges to $\pi(\underline{x})$. Consequently, we have $\pi(\underline{x}) \in X$ for every $\underline{x} \in X$ and thus (ii) is satisfied.

Now, assume that I is a non-empty subset of $\{1, \ldots, k\}$ that meets $\underline{x}$. We can consider a sub-sequence of $\underline{a}^{(n)}$ such that each $a_i^{(n)}$ is non-decreasing for each $i \in \{1, \ldots, k\}$. If $j \in \{1, \ldots, k\} \setminus I$, then we can choose distinct $c_j \in U_j$ such that for sufficiently large positive integer n_0, a sequence $\underline{d}^{(n)} \in U_1 \times \cdots \times U_k$ with distinct coordinates can be defined for all $n \ge n_0$ with $d_i^{(n)} := a_i^{(n)}$ for $i \in I$ and $d_i^{(n)} := c_i$ for $i \notin I$. This choice is possible because of the assumption $|U_i| \ge k$ for each i. It then follows that $\rho(\underline{d}^{(n)})$ converges to $\rho_I(\underline{x})$. Thus (iii) holds. This completes the proof of Theorem 1. □

Proof of Theorem 2. Let $\underline{x} \in (x_1, \ldots, x_k) \in \mathcal{S}^{k-1}$ and let $I_i = (a_i, b_i)$ be open intervals such that $x_i \in I_i$ for each $i \in \{1, \ldots, k\}$. Then $\prod_{i=1}^{k}(a_i, b_i) \cap \mathcal{S}^{k-1}$ is a basic open set in $\mathcal{S}^{k-1}$ containing $\underline{x}$. For a real number $X > 1$, let $U_i(X) := \#\{u_i \in U_i | u_i \leq X\}$. By the hypothesis, we have that $\lim_{X\to\infty} \frac{U_i(X)}{X} = \delta_i > 0$. This implies that $U_i(X) = \delta_i X + o(X)$. Therefore,

$$\lim_{X\to\infty} \frac{U_i(a_iX)}{U_i(b_iX)} = \lim_{X\to\infty} \frac{\delta_i a_i X + o(a_iX)}{\delta_i b_i X + o(b_iX)} = \frac{a_i}{b_i} < 1.$$

Thus for every sufficiently large real number X, there exists $u_i \in U_i$ such that $a_iX < u_i \leq b_iX$. That is, $a_i < \frac{u_i}{X} \leq b_i$. Since $\bigcap_{i=1}^{k} U_i$ is an infinite set, we can choose a large enough element $u \in \bigcap_{i=1}^{k} U_i$ such that $a_iu < u_i \leq b_iu$ for all $i = 1, \ldots, k$. This, in turn, implies that $\frac{u_i}{u} \in (a_i, b_i)$. Using the fact that $\rho(\underline{\alpha}) = \frac{\underline{\alpha}}{\|\underline{\alpha}\|}$ is continuous function, we see that $\rho(u_1, \ldots, u_k) \in \prod_{i=1}^{k} I_i \cap \mathcal{S}^{k-1}$. In other words, $\mathcal{D}^k(U_1, \ldots, U_k)$ is dense in $\mathcal{S}^{k-1}$. $\square$

We next prove Theorem 3 which extends Theorem 1.5 of [17].

Proof of Theorem 3. Let $\underline{x} = (x_1, \ldots, x_k) \in \mathcal{S}^{k-1}$ with $x_i > 0$ for all $i \in \{1, \ldots, k\}$. We pick an integer m such that $m > \frac{u_i^{(1)}}{\min\{x_1,\ldots,x_k\}}$ for all $i \in \{1, \ldots, k\}$. By the choice of m, we have $\frac{u_i^{(1)}}{x_i} < m$. Since $u_i^{(n)}$ is an increasing sequence of positive integers, for each i, there exists a smallest integer m_i such that $\frac{u_i^{(m_i)}}{x_i} > m$. Therefore, for all $i \in \{1, \ldots, k\}$, there exist m_i such that $\frac{u_i^{(m_i-1)}}{x_i} \leq m < \frac{u_i^{(m_i)}}{x_i}$. That is, $x_i < \frac{u_i^{(m_i)}}{m} \leq \frac{u_i^{(m_i)}}{u_i^{(m_i-1)}} x_i$. Since $m_i \to \infty$ as $m \to \infty$, it follows that $\lim_{m\to\infty} \dfrac{u_i^{(m_i)}}{m} = x_i$. Consequently, $\frac{1}{m}\underline{u} = (\frac{1}{m}u_1^{(m_1)}, \ldots, \frac{1}{m}u_k^{(m_k)})$ converges to $\underline{x}$. Since ρ is a continuous map, we conclude that $\rho(\underline{u}) = \rho(\frac{1}{m}\underline{u})$ converges to $\underline{x}$. Consequently, $\mathcal{D}^k(U_1, \ldots, U_k)$ is dense in $\mathcal{S}^{k-1}$. $\square$

Proof of Theorem 4. For a fixed integer $i \in \{1, \ldots, k\}$, we consider the polynomial $g_i(X)$ obtained by replacing all the variables of g_i by the variable X. We get, $g_i(X) = a_{d_i}X^{d_i} + a_{d_i-1}X^{d_i-1} + \cdots + a_0 \in \mathbb{Z}[X]$. Since $a_{d_i} > 0$, we conclude that for a sufficiently large positive real number X, we have $g_i(X) > 0$. Let $B_i := \{g_i(n) | n \in \mathbb{N}\} \cap \mathbb{N}$. We have $\frac{g_i(X-1)}{g_i(X)} = \frac{a_{d_i}(X-1)^{d_i} + \cdots + a_0}{a_{d_i}X^{d_i} + \cdots + a_o}$ which tends to 1 as X tends to ∞. Also, since $g_i(X)$ is a polynomial in one variable, the sequence $\{g_i(n)\}_{n=1}^{\infty}$ is eventually increasing. Therefore, by using Theorem 3, we obtain that $\mathcal{D}^k(B_i)$ is dense in $\mathcal{S}^{k-1}$. Since $B_i \subseteq U_i$, we conclude that $\mathcal{D}^k(U_1, \ldots, U_k)$ is dense in $\mathcal{S}^{k-1}$. $\square$

We now prove Theorem 5 which gives a partial answer to [17, Question 1.9].

Proof of Theorem 5. We consider the following three sets as in [5] (see also [3]):

$$A := \bigcup_{k=0}^{\infty} [5^k, 2 \cdot 5^k) \cap \mathbb{N},$$
$$B := \bigcup_{k=0}^{\infty} [2 \cdot 5^k, 3 \cdot 5^k) \cap \mathbb{N},$$
$$C := \bigcup_{k=0}^{\infty} [3 \cdot 5^k, 5 \cdot 5^k) \cap \mathbb{N}.$$

It is clear that A, B and C indeed give a partition of $\mathbb{N}$. If $\mathcal{D}^k(A)$, $\mathcal{D}^k(B)$ or $\mathcal{D}^k(C)$ is dense in $\mathcal{S}^{k-1}$, then by Theorem 1.4 of [17], which states that if $\mathcal{D}^k(A)$ is dense in $\mathcal{S}^{k-1}$ for some $A \subseteq \mathbb{N}$, then $\mathcal{D}^{k-1}(A)$ is dense in $\mathcal{S}^{k-2}$, we see inductively that $\mathcal{D}^2(A)$ (or $\mathcal{D}^2(B)$ or $\mathcal{D}^2(C)$) is dense in $\mathcal{S}^1$, which is false (cf. [3, Proposition 3]). Therefore, we get a 3-partition of $\mathbb{N}$ such that none of $\mathcal{D}^k(A)$, $\mathcal{D}^k(B)$ or $\mathcal{D}^k(C)$ is dense in $\mathcal{S}^{k-1}$. This completes the proof of Theorem 5. □

Proof of Theorem 6. In [9], it was proven that the equation $x^n + y^n = 2z^n$ has no non-trivial solution in $\mathbb{Z}$ if $n \geq 3$. In other words, the set $A := \{m^r : r, m \in \mathbb{Z}, r \geq 3\}$ does not contain any 3-term arithmetic progressions. Since for a fixed value of $r \geq 3$, we have $\frac{m^r}{(m+1)^r} \to 1$ as $m \to \infty$, by Theorem 1.5 of [17], we conclude that $\mathcal{D}^k(A)$ is dense in $\mathcal{S}^{k-1}$. □

Proof of Theorem 8. Let $\underline{x} = (x_1, \ldots, x_k) \in \mathcal{S}^{k-1}$ and let $\prod_{i=1}^{k}(a_i, b_i)$ be a basic neighborhood of $\underline{x}$. Then by Theorem 7, we see that

$$\lim_{X\to\infty} \frac{\omega_{a_i X}}{\omega_{b_i X}} = \lim_{X\to\infty} \frac{a_i X}{\log\log a_i X} \cdot \frac{\log\log b_i X}{b_i X} = \frac{a_i}{b_i} < 1 \text{ for all } i \text{ with } 1 \leq i \leq k.$$

Therefore, for sufficiently large X, there exists $\alpha_i \in A$ such that $a_i X < \alpha_i < b_i X$ for all i. That is, $\left(\frac{\alpha_1}{X}, \ldots, \frac{\alpha_k}{X}\right) \in \prod_{i=1}^{k}(a_i, b_i)$. Hence $\rho(\alpha_1, \ldots, \alpha_k) = \rho\left(\frac{\alpha_1}{X}, \ldots, \frac{\alpha_k}{X}\right) \in \prod_{i=1}^{k}(a_i, b_i)$. Consequently, $\mathcal{D}^k(A)$ is dense in $\mathcal{S}^{k-1}$.

Similarly, for $\mathcal{D}^k(B)$, we note that

$$\lim_{X\to\infty} \frac{\phi_{a_i X}}{\phi_{b_i X}} = \frac{\sqrt{a_i}}{\sqrt{b_i}} < 1 \text{ for all } i \text{ with } 1 \leq i \leq k$$

and thereafter it follows a similar line of argument. □

3. Concluding Remarks: Case of Algebraic Number Fields

The ratio sets have been studied in the context of algebraic number fields in [8], [11] and [23]. It is interesting to extend the notion of direction sets in the set up of number fields and formulate analogous questions for the same.

Let $K \subsetneq \mathbb{R}$ be a number field of degree $d \geq 2$ and let $\mathcal{O}_K$ be its ring of integers. Let $\mathcal{O}_K^0 := \{\alpha \in \mathcal{O}_K : \mathrm{Tr}_{K/\mathbb{Q}}(\alpha) = 0\}$ be the set of elements in $\mathcal{O}_K$ with trace 0. Since $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank d and Tr is an additive group homomorphism from $\mathcal{O}_K$ to $\mathbb{Z}$, we see that $\mathcal{O}_K \cong \mathcal{O}_K^0 \oplus \mathbb{Z}$. In particular, $\mathcal{O}_K^0$ is a free $\mathbb{Z}$-module of rank $d-1$. Therefore, $\mathcal{O}_K^0$ itself is dense in $\mathbb{R}$ whenever $d \geq 3$. Also, for $d = 2$, we see that the ratio set of $\mathcal{O}_K^0$ is $\mathbb{Q}$. Consequently, the direction set of $\mathcal{O}_K^0$ is dense in $\mathcal{S}^{k-1}$ for any integer $k \geq 2$.

We note that $\mathcal{O}_K^0 \cap \mathbb{N} = \emptyset$. In view of this, we ask the following question.

Question 1. Let $d \geq 2$ and $k \geq 2$ be integers and let K be a number field of degree d. Characterize the sets $\mathcal{A} \subseteq \mathcal{O}_K$ such that $\mathcal{A} \cap \mathbb{N}$ is finite and $\mathcal{D}^{k-1}(\mathcal{A})$ is dense in $\mathcal{S}^{k-1}$.

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