LOOP DECOMPOSITIONS OF RANDOM WALKS AND NONTRIVIAL IDENTITIES OF BERNOULLI AND EULER POLYNOMIALS

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Abstract

Let a and b, with $a < b$, be two level sites of a general random walk. If we partition $[a, b]$ into n arbitrary subintervals with endpoints $a < a_1 < a_2 < \cdots < a_{n-1} < b$, then the hitting time from a to b can also be decomposed by the hitting times between adjacent pairs of sites. As the walk moves back and forth between these endpoints, loops are created. Hence, we can express the generating function of the hitting time as a loop decomposition for an arbitrary number of consecutive sites. By applying this decomposition to a 1-dimensional reflected Brownian motion with equally distributed sites, we derive identities of Bernoulli and Euler polynomials in terms of their higher-order generalizations. Similar results from a 3-dimensional Bessel process are also obtained.

1. Introduction

Random walks of various types have been comprehensively studied and widely applied in physics, engineering, and mathematics. Initially, the connection between random walk models and special polynomials was not obvious. Most of these polynomials, especially Bernoulli and Euler polynomials, have appeared more in the

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context of number theory and combinatorics than probability and statistics. Recent work by the first author and Vignat [4], however, reveals that a strong connection between random walks and Bernoulli and Euler polynomials does exist. More specifically, by considering the 1-dimensional reflected Brownian motion and 3-dimensional Bessel process, one can prove certain non-trivial identities involving Bernoulli and Euler polynomials of order p. Both Bernoulli and Euler polynomials of higher-order, denoted by $B_n^{(p)}(x)$ and $E_n^{(p)}(x)$, respectively, are defined via their exponential generating functions

$$\left(\frac{t}{e^t-1}\right)^p e^{xt} = \sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{t^n}{n!} \quad \text{and} \quad \left(\frac{2}{e^t+1}\right)^p e^{xt} = \sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{t^n}{n!}. \tag{1}$$

In particular, $B_n(x) = B_n^{(1)}(x)$ and $E_n(x) = E_n^{(1)}(x)$ are the ordinary Bernoulli and Euler polynomials; Bernoulli numbers $B_n = B_n(1)$ and Euler numbers $E_n = 2^n E_n(1/2)$ are special values of the corresponding polynomials. See, for example, [6, Chapter 24] for details.

The above discovery arose from earlier work [3, Equation (3.8)], in which the first author, Moll, and Vignat expressed the Euler polynomials as a linear combination of higher-order Euler polynomials: for any positive integer N,

$$E_n(x) = \frac{1}{N^n}\sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)}\left(\frac{\ell-N}{2}+Nx\right).$$

It is surprising that the positive coefficients $p_\ell^{(N)}$ also appear as transition probabilities in the context of a random walk over a finite number of sites [3, Note 4.8]. This has inspired several other results regarding the connection between certain random walks and $E_n^{(p)}(x)$ as well as $B_n^{(p)}(x)$.

Consider an arbitrary random walk on $\mathbb{R}$ starting from a and ending at b, with $a < b$. The hitting time from a to b can be decomposed by partitioning $[a, b]$ into n arbitrary sub-intervals with endpoints $a < a_1 < a_2 < \cdots < a_{n-1} < b$, and then by considering the hitting times between pairs of adjacent endpoints. Results in [4] are obtained by considering this decomposition for $n = 2, 3$, i.e., walks with one or two loops. This paper generalizes these results for any finite number of loops:

- In Theorem 1, we derive an explicit formula for the moment generating function of the hitting times between a and b via the decomposition for an arbitrary number of consecutive sites in between. We provide two proofs of this result. One uses combinatorics and the other uses induction (the rationale for this is given at the beginning of Section 3.2).
- We apply Theorem 1 to derive new identities involving $B_n^{(p)}(x)$ and $E_n^{(p)}(x)$.

This paper is organized as follows. In Section 2, we introduce basic notation for random walks including the generating function of the hitting times. Also, we describe the models of 1-loop and 2-loop cases as examples. In Section 3, we generalize the model to include n-loops with proofs. In Section 4, we recall the three umbral symbols - Bernoulli, Euler and uniform - along with their important properties and identities. These formulas are crucial in deriving the identities explored in Section 5. In the last section, we consider the 1-dimensional reflected Brownian motion model. Then, as an analogue, the loop decomposition and identities in the 3-dimensional Bessel process model are derived.

2. Preliminaries: Loops

Although one can find a similar summary in [4], we include some background here in an effort to make this paper self-contained. We note a slight change in the notation: see Definition 1 below.

2.1. Notation for Paths and Loops

We begin with some notation for the moment generating functions of random walks among loops. Here we do not restrict to any specific random walk model but consider a general setup. For a more detailed introduction to hitting times of a random walk, one can consult, for example, [5, 7].

Definition 1. Consider sites $a < b$ and a third site c, different from a and b.

- Let $\phi_{a\to b}$ denote the *moment generating function of the hitting time of site b starting from site a*; also let $\phi_{b\to a}$ be the *counterpart from b to a*. Moreover, let
$$L_{a,b} := \phi_{a\to b}\phi_{b\to a}$$
be the moment generating function of the hitting time that (i) starts from a; (ii) hits b first; (iii) and finally returns to a.

  Note the symmetry $L_{b,a} = \phi_{b\to a}\phi_{a\to b} = L_{a,b}$. Thus, this is the *loop* between sites a and b.

- Also, let $\phi_{a\to b|\not c}$ denote the moment generating function of the hitting time of site b starting from site a *before hitting site* c, and similarly for $\phi_{b\to a|\not c}$. It follows that $\phi_{a\to b|\not c} = \phi_{a\to b}$ if $c > b$ and $\phi_{a\to b|\not c} = 0$ if $a < c < b$.

- If a is the mth site, denoted by a_m, and b is the nth site, a_n, we shall use the simplified expression $\phi_{m\to n}$ instead of $\phi_{a_m\to a_n}$. Moreover,
$$\phi_{m\to n|\not k} = \phi_{a_m\to a_n|\not a_k}.$$

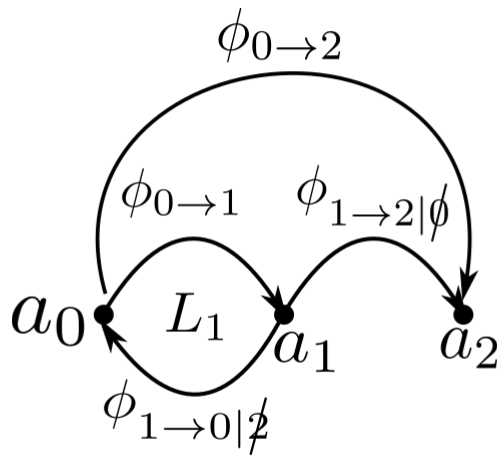

Figure 1: 1-loop case

- Similarly, we use $t_{a\to b}$ for the hitting time from site a to site b. All the notation above for ϕ naturally extends to t. For instance, $t_{m\to n|\not{k}}$ is the hitting time from the mth site to the nth site before hitting the kth site.
- Finally, it is both important and convenient for us to denote the moment generating function of the hitting time of the loop between the (consecutive) $(n-1)$th site and the nth site by

$$L_n = \phi_{(n-1)\to n|\not{(n-2)}} \cdot \phi_{n\to(n-1)|\not{(n+1)}}.$$

Example 1. We first recall the 1-loop and 2-loop cases, already studied in [4]. The 1-loop case can be viewed in Figure 1, in which we assume that the initial site is a_0. Namely, there is no other site to the left of a_0. The hitting time decomposition can be expressed as

$$t_{0\to 2} = t_{0\to 1} + \underbrace{(t_{1\to 0|\not{2}} + t_{0\to 1}) + \cdots + (t_{1\to 0|\not{2}} + t_{0\to 1})}_{k \text{ copies}} + t_{1\to 2|\not{0}}.$$

Here, $k = 0, 1, 2, \ldots$, and there can be k copies of L_1 in the moment generating functions. When considering the moment generating functions of both sides, independence turns the summation into products. Therefore, we have

$$\phi_{0\to 2} = \phi_{0\to 1}\phi_{1\to 2|\not{0}} \sum_{k=0}^{\infty} \left(\phi_{1\to 0|\not{2}}\phi_{0\to 1}\right)^k = \frac{\phi_{0\to 1}\phi_{1\to 2|\not{0}}}{1 - L_1}. \tag{2}$$

(See also [4, Equation (2.5)].) In addition, the 2-loop case,

$$\phi_{0\to 3} = \frac{\phi_{0\to 1}\phi_{1\to 2|\not{0}}\phi_{2\to 3|\not{1}}}{1 - (L_1 + L_2)}, \tag{3}$$

is obtained through combinatorial enumeration [4, Equation (2.6)].

We give the loop decomposition for n-loops in the following section.

3. General n-Loop Decomposition

In this section we give the expression of the general n-loop formula (see Figure 2, the black paths) as the generalization of Equation (2) and Equation (3). Again we assume the walk begins at the site a_0 and only moves to consecutive sites, say $a_1 < a_2 < \cdots < a_{m+1}$, to its right. In other words, the walk considers a_0 as its starting point, and there are no sites to the left of a_0.

Theorem 1.

$$\phi_{0\to(n+1)} = \phi_{0\to1}\prod_{j=1}^{n}\phi_{j\to(j+1)|\cancel{j-1}}\left(\sum_{k\geq0}\sum_{**}\prod_{t=1}^{k}L_{i_t}\right)$$

$$= \phi_{0\to1}\prod_{j=1}^{n}\phi_{j\to(j+1)|\cancel{j-1}}\sum_{k\geq0}\left((L_1+\cdots+L_n)+\sum_{*'}(-1)^{l+1}(L_{j_1}L_{j_2}\cdots L_{j_l})\right)^k$$

$$= \phi_{0\to1}\prod_{j=1}^{n}\phi_{j\to(j+1)|\cancel{j-1}}\frac{1}{1-(L_1+L_2+\cdots+L_n)+\sum_{*'}(-1)^l(L_{j_1}L_{j_2}\cdots L_{j_l})},$$

where

$$** = \{(i_1,i_2,\cdots,i_k) : 1\leq i_t\leq n, \textit{and } \ i_t-1\leq i_{t+1}\};$$

and

$$*' = \{n\geq j_1>\cdots>j_l\geq1,\ l\geq2,\ j_m-j_{m+1}\geq2\}.$$

Remark 1. The terms in the $*'$ sum do not contain consecutive loops. In each term, the loops are listed in *descending* order. The combinatorial interpretation of this will be provided later in the proof. Before the proof of Theorem 1, we present several examples for small numbers of loops.

Example 2. The formulas for $n = 2,3,4,5$, given by Theorem 1, are as follows:

$$\phi_{0\to3} = \frac{\phi_{0\to1}\phi_{1\to2|\not 0}\phi_{2\to3|\not 1}}{1-(L_1+L_2)},$$

$$\phi_{0\to4} = \frac{\phi_{0\to1}\phi_{1\to2|\not 0}\phi_{2\to3|\not 1}\phi_{3\to4|\not 2}}{1-(L_1+L_2+L_3-L_3L_1)},$$

$$\phi_{0\to5} = \frac{\phi_{0\to1}\phi_{1\to2|\not 0}\phi_{2\to3|\not 1}\phi_{3\to4|\not 2}\phi_{4\to5|\not 3}}{1-(L_1+L_2+L_3+L_4-L_4L_2-L_4L_1-L_3L_1)},$$

$$\phi_{0\to 6} = \frac{\phi_{0\to 1}\phi_{1\to 2|\not 0}\phi_{2\to 3|\not 1}\phi_{3\to 4|\not 2}\phi_{4\to 5|\not 3}\phi_{5\to 6|\not 4}}{1-(L_1+\cdots+L_5-L_5L_3-L_5L_2-L_5L_1-L_4L_2-L_4L_1-L_3L_1+L_5L_3L_1)}.$$

If there are at most two loops, L_1 and L_2, then $*'$ is empty. So we can recover both Equation (2) and Equation (3).

3.1. Combinatorial Proof

We now give a combinatorial proof of Theorem 1 by the inclusion-exclusion principle.

Proof of Theorem 1. First, we provide a possible combinatorial interpretation of the terms appearing in Theorem 1.

Let $n \geq 1$. An arbitrary decomposition of $t_{0\to(n+1)}$ can be written as

$$t_{0\to(n+1)} = \sum_{j=0}^{2k+n+1} t_{i_j\to i_{j+1}|\not{i_j^*}}, \tag{4}$$

where

(i) $i_0 = 0$, $i_1 = 1$, $i_{2k+n+1} = n+1$;

(ii) if $0 \leq j < 2k+n$, then $0 \leq i_j, i_{j+1} \leq n$, and $|i_j - i_{j+1}| = 1$;

(iii) if $i_j = 0$, then $t_{i_j\to i_{j+1}|\not{i_j^*}} = t_{i_j\to i_{j+1}} = t_{0\to 1}$;

(iv) if $i_j \neq 0$, then $|i_j^* - i_j| = 1$, and $|i_j^* - i_{j+1}| = 2$.

Notice that as we move through the sum in Equation (4) in the direction of increasing j, every time we encounter a stopping time which corresponds to a step to the left, namely from a site i_j to site $i_{j+1} = i_j - 1$, a loop is formed. We denote the hitting time of this loop by l_{i_j}, i.e., $l_{i_j} = t_{i_j\to i_j-1|\not{i_j+1}} + t_{i_j-1\to i_j|\not{i_j-2}}$. These loops can only occur between consecutive sites, and the order in which they appear is important: different walks with the same collection of loops will have different orderings. The ordering of the loops has an additional constraint, given by the assumption that the walk only moves between consecutive sites. That is, if

$i_{j+1} < i_j$, then $i_{j+1} = i_j - 1$. No such constraint is needed if $i_{j+1} > i_j$. Hence, every decomposition of $t_{0\to(n+1)}$ in Equation (4) can be uniquely written as

$$t_{0\to(n+1)} = t_{0\to1} + \sum_{j=1}^{n} t_{j\to(j+1)|\not{j-1}} + \sum_{t=1}^{k} l_{i_t}, \tag{5}$$

where the order in which we sum the l_{i_t}'s is important, $1 \le i_t \le n$, and $i_t - 1 \le i_{t+1}$. This correspondence can be reversed, and Equation (5) can be used to express the moment generating function of $t_{0\to(n+1)}$ in terms of the moment generating functions of loops. That is,

$$\phi_{0\to n+1} = \phi_{0\to1} \prod_{j=1}^{n} \phi_{j\to(j+1)|\not{j-1}} \left(\sum_{k\ge0} \sum_{**} \prod_{t=1}^{k} L_{i_t} \right), \tag{6}$$

where

$$** = \{(i_1, i_2, \cdots, i_k) : \text{ the } i_t\text{'s satisfy } (b)\}.$$

Let k be arbitrary but fixed. We apply the principle of inclusion-exclusion to recover $\sum_{**} \prod_{t=1}^{k} L_{i_t}$ from $(L_1 + L_2 + \cdots + L_n)^k$.

Note that if $n = 1, 2$, $1 \le i_t \le n$, then $i_t - 1 \le i_{t+1}$ necessarily holds, and from Equation (6) we immediately obtain

$$\phi_{0\to2} = \phi_{0\to1}\phi_{1\to2|\not{0}} \sum_{k=0}^{\infty} L_1^k = \frac{\phi_{0\to1}\phi_{1\to2|\not{0}}}{1 - L_1},$$

and

$$\phi_{0\to3} = \phi_{0\to1}\phi_{1\to2|\not{0}}\phi_{2\to3|\not{1}} \sum_{k=0}^{\infty} (L_1 + L_2)^k = \frac{\phi_{0\to1}\phi_{1\to2|\not{0}}\phi_{2\to3|\not{1}}}{1 - (L_1 + L_2)},$$

which coincide with Equation (2) and Equation (3), respectively (see also Example 2).

Let $n \ge 3$. Before applying the principle of inclusion-exclusion, it is convenient to introduce additional notation. Since property (b) can be expressed in terms of the order in which multiplication of the L_j's is performed, we order the factors of

$$(L_1{+}L_2{+}\cdots{+}L_n)^k = (L_1{+}L_2{+}\cdots{+}L_n)\,(L_1{+}L_2{+}\cdots{+}L_n)\,\cdots\cdots\,(L_1{+}L_2{+}\cdots{+}L_n)$$

by labeling them in increasing order from left to right. That is, we write

$$(L_1 + L_2 + \cdots + L_n)^k = \prod_{t=1}^{k} (L_1 + L_2 + \cdots + L_n)_{(t)},$$

and use these additional subscripts to label the forbidden products of pairs of moment generating functions of loops in the expansion of $(L_1 + L_2 + \cdots + L_n)^k$. We write $(L_iL_j)_{(s)}$ when $i-j>1$, where L_i comes from the factor $(L_1+L_2+\cdots+L_n)_{(s)}$, and L_j comes from $(L_1+L_2+\cdots+L_n)_{(s+1)}$. Moreover, when we write $(L_iL_j)_{(s)}\,(L_rL_u)_{(v)}$, we assume that $v\geq s+1$, and $v=s+1$ if and only if $L_j=L_r$. In this latter case,

$$(L_iL_j)_{(s)}\,(L_jL_u)_{(s+1)} \quad \text{reduces to} \quad (L_iL_jL_u)_{(s)}. \tag{7}$$

The simplification in (7) naturally extends to products of several forbidden pairs of moment generating functions of loops, thus producing forbidden tuples $(L_{i_1}L_{i_2}\cdots L_{i_t})$, where $i_j\geq i_{j+1}+2$, $1\leq j\leq t-1$.

By the principle of inclusion-exclusion we have that $\sum_{k\geq 0}\sum_{**}\prod_{t=1}^{k} L_{i_t}$ can be written as

$$\sum_{k\geq 0}\left[(L_1+\cdots+L_n)^k+\sum_{1\leq l\leq k-1}(-1)^l\sum_{(l)}\Big(\prod_{j=1}^{l}(L_{i_j}L_{t_j})_{s_j}(L_1+\cdots+L_n)^{n-\#_l}\Big)\right], \tag{8}$$

where $\#_l$ denotes the number of distinct L_r in $\prod_{j=1}^{l}(L_{i_j}L_{t_j})_{s_j}$, and $\sum_{(l)}$ runs over all possible $(i_j,t_j)_{s_j}$, such that $1\leq j\leq l$, $s_j<s_{j+1}$, and $i_j>t_j+1$.

Next we rewrite the quantity in (8) in a very simple form. First, we apply the reduction in (7) and its extensions, whenever possible. Then we drop the subscripts (now they do not provide any additional information), but we keep parentheses around each forbidden quantity. Finally, we combine the terms. We view each forbidden tuple, i.e., $(L_{i_1}L_{i_2}),(L_{i_3}L_{i_4}L_{i_5}),\ldots$, as distinct variables; for every k, we collect all monomials of degree k in these variables. We claim that this procedure reduces (8) to

$$\sum_{k\geq 0}\left((L_1+L_2+\cdots+L_n)+\sum_{*'}(-1)^{l+1}(L_{j_1}L_{j_2}\cdots L_{j_l})\right)^k, \tag{9}$$

where $*'=\{n\geq j_1>\cdots>j_l\geq 1,\ l\geq 2,\ j_m-j_{m+1}\geq 2\}$.

We prove the equivalence of these quantities by showing that, for every $k\geq 0$, there is a one-to-one correspondence between the terms of degree k in (8) and (9). Let $k=m$ be arbitrary but fixed. An arbitrary term in the expansion of

$$\left((L_1+L_2+\cdots+L_n)+\sum_{*'}(-1)^{l+1}(L_{j_1}L_{j_2}\cdots L_{j_l})\right)^m$$

will have the form

$$(-1)^s\binom{m}{t_1,\cdots,t_{r+1}}\left(\prod_{i=1}^{s_1}L_{j_i^{(1)}}\right)^{t_1}\left(\prod_{i=1}^{s_2}L_{j_i^{(2)}}\right)^{t_2}\cdots\left(\prod_{i=1}^{s_r}L_{j_i^{(r)}}\right)^{t_r}(L_1+\cdots+L_n)^{t_{r+1}}$$

for arbitrary positive integers $t_1, t_2, \cdots, t_{r+1}$, with $t_1+t_2+\cdots+t_{r+1} = m$, arbitrary forbidden tuples $(j_1^{(1)}, \cdots, j_{s_1}^{(1)}), \cdots, (j_1^{(r)}, \cdots, j_{s_r}^{(r)})$, and $s = (s_1+1)t_1 + \cdots + (s_r + 1)t_r$. In (8), the terms

$$(-1)^s \left(\prod_{i=1}^{s_1} L_{j_i^{(1)}} \right)^{t_1} \left(\prod_{i=1}^{s_2} L_{j_i^{(2)}} \right)^{t_2} \cdots \left(\prod_{i=1}^{s_r} L_{j_i^{(r)}} \right)^{t_r} (L_1 + \cdots + L_n)^{t_{r+1}} \tag{10}$$

are introduced when the principle of inclusion-exclusion is applied to remove the forbidden pairs of moment generating functions of loops from

$$(L_1 + \cdots + L_n)^{t_1 s_1 + t_2 s_2 + \cdots + t_r s_r + t_{r+1}}.$$

The number of these terms can be obtained by viewing each pair of parentheses $(\cdots)$ in (10) as a distinct object. The superscript $(\cdots)^{t_i}$ implies $(\cdots)$ is repeated t_i times, and we count the number of ways they can be arranged on a line where order is important. It follows that this number is

$$\frac{(t_1 + t_2 + \cdots + t_r + t_{r+1})!}{t_1! t_2! \cdots t_r!\, t_{r+1}!} = \binom{m}{t_1, \cdots, t_{r+1}}.$$

This shows that all the terms of degree m in (9) can be found in (8). Moreover, one can observe that these are the only terms of degree m in (8), thus proving the equivalence between (8) and (9). Therefore,

$$\begin{aligned}
\phi_{0\to(n+1)} &= \phi_{0\to 1} \prod_{j=1}^{n} \phi_{j\to(j+1)|\cancel{j\to 1}} \left(\sum_{k\geq 0} \sum_{**} \prod_{t=1}^{k} L_{i_t} \right) \\
&= \phi_{0\to 1} \prod_{j=1}^{n} \phi_{j\to(j+1)|\cancel{j\to 1}} \sum_{k\geq 0} \left((L_1 + \cdots + L_n) + \sum_{*'} (-1)^{l+1} (L_{j_1} L_{j_2} \cdots L_{j_l}) \right)^k \\
&= \phi_{0\to 1} \prod_{j=1}^{n} \phi_{j\to(j+1)|\cancel{j\to 1}} \frac{1}{1 - (L_1 + L_2 + \cdots + L_n) + \sum_{*'} (-1)^l (L_{j_1} L_{j_2} \cdots L_{j_l})}.
\end{aligned}$$

The proof is now complete. □

3.2. Induction

In this subsection, we provide an alternative proof of Theorem 1 by using induction on n. We also believe this approach is necessary given that in earlier work [4] the authors encountered difficulties when extending the result from the 1-loop to 2-loop case and were ultimately unable to derive the general n-loop formula. For simplicity, some tedious, detailed calculations are omitted.

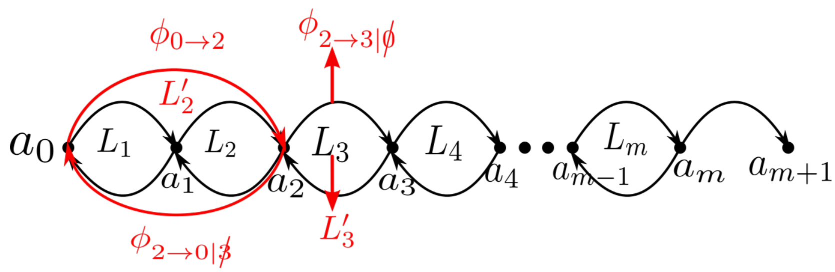

Figure 2: m-loop

In an effort to simplify notation, we use

$$\sum_{(k,l,n)} = \sum_{*} (-1)^{l+1} L_{j_1} \cdots L_{j_l}, \tag{11}$$

where $* = \{k = j_1 < \cdots < j_l \leq n, 1 \leq l \leq n-k+1, j_{m+1} - j_m \geq 2\}$, in which we have reordered the loops in ascending order.

Remark 2. Notice that we have reversed the order of the subscripts in the newly defined product notation. Indeed, $\sum\limits_{*}$ and $\sum\limits_{*'}$ are mathematically equivalent. Although we believe $\sum\limits_{*'}$ does a better job in conveying the combinatorial idea behind the loop identity, it is for the simplicity of expression that we choose to use the reverse order in the following proof.

We observe that

$$\sum_{(k,l,n)} = L_k - L_k \sum_{j=k+2}^{n} \sum_{(j,l,n)} = L_k \left(1 - \sum_{j=k+2}^{n} \sum_{(j,l,n)}\right). \tag{12}$$

Hence, we can express Theorem 1 as

$$\phi_{0\to n+1} = \phi_{0\to 1} \prod_{j=1}^{n} \phi_{j\to(j+1)|\not j\text{-}\not 1} \frac{1}{1 - \sum\limits_{k=1}^{n} \sum\limits_{(k,l,n)}}. \tag{13}$$

Proof of Theorem 1, by induction. For the case $n = 1$, Equation (13) reduces to

$$\phi_{0\to 2} = \phi_{0\to 1} \phi_{1\to 2|\not\phi} \frac{1}{1 - L_1}, \tag{14}$$

the same as Equation (2). Suppose Equation (13) holds for $n = m-1$. Then, we need to show

$$\phi_{0\to(m+1)} = \phi_{0\to 1} \prod_{j=1}^{m} \phi_{j\to(j+1)|\not j\text{-}\not 1} \frac{1}{1 - \sum\limits_{k=1}^{m} \sum\limits_{(k,l,m)}}. \tag{15}$$

We combine the first two loops together, which reduces to $m-1$ new loops labeled as $L'_2, L'_3, L'_4, \ldots$; see Figure 2. Note that $\phi_{0\to 2}$ is given by Equation (14); and similarly

$$\phi_{2\to 0|\not 3} = \frac{\phi_{1\to 0|\not 2}\phi_{2\to 1|\not 3}}{1-\phi_{1\to 2|\not 0}\phi_{2\to 1|\not 3}}.$$

Hence,

$$L'_2 = \phi_{0\to 2}\phi_{2\to 0|\not 3} = \frac{L_1 L_2}{(1-L_1)(1-L_2)}.$$

In addition,

$$L'_3 = \phi_{2\to 3|\not 0}\phi_{3\to 2|\not 4} = \phi_{2\to 3|\not 1}\sum_{k=0}^{\infty}(\phi_{2\to 1|\not 3}\phi_{1\to 2|\not 0})^k\phi_{3\to 2|\not 4} = \frac{L_3}{1-L_2},$$

and $L'_k = L_k$, for all $4 \le k \le m$. Define

$$\sum_{(k,l,n)}{}' = \sum_{*}(-1)^{l+1}L'_{j_1}\cdots L'_{j_l},$$

where $* = \{k = j_1 < \cdots < j_l \le n, 1 \le l \le n-k+1, j_{m+1}-j_m \ge 2\}$, which is the same as Equation (11), with all L's replaced by L'. Now apply Equation (13) for sites $0, 2, 3, \ldots, m+1$, (i.e., with $m-1$ loops,) to obtain

$$\begin{aligned}
\phi_{0\to(m+1)} &= \phi_{0\to 2}\phi_{2\to 3|\not 0}\prod_{j=3}^{m}\phi_{j\to(j+1)|\cancel{j-1}}\frac{1}{1-\sum\limits_{k=2}^{m}\sum\limits_{(k,l,m)}{}'} \\
&= \phi_{0\to 1}\phi_{1\to 2|\not 0}\frac{1}{1-L_1}\phi_{2\to 3|\not 1}\frac{1}{1-L_2}\prod_{j=3}^{m}\phi_{j\to(j+1)|\cancel{j-1}}\frac{1}{1-\sum\limits_{k=2}^{m}\sum\limits_{(k,l,m)}{}'} \\
&= \phi_{0\to 1}\prod_{j=1}^{m}\phi_{j\to(j+1)|\cancel{j-1}}\frac{1}{(1-L_1)(1-L_2)\left(1-\sum\limits_{k=2}^{m}\sum\limits_{(k,l,m)}{}'\right)}.
\end{aligned}$$

Therefore, Equation (15) is equivalent to

$$(1-L_1)(1-L_2)\left(1-\sum_{k=2}^{m}\sum_{(k,l,m)}{}'\right) = 1-\sum_{k=1}^{m}\sum_{(k,l,m)}. \tag{16}$$

Applying Equation (12) and simplifying will complete the proof. □

4. Preliminaries: Umbral Random Symbols

Again, one can find a similar summary of this section in [4]. Let $\mathcal{B}$, $\mathcal{E}$, and $\mathcal{U}$ be the *Bernoulli*, *Euler*, and *uniform (umbral) symbols*, respectively. They are defined as follows.

4.1. Bernoulli $\mathcal{B}$

The Bernoulli symbol $\mathcal{B}$ satisfies the evaluation rule

$$(x+\mathcal{B})^n = B_n(x). \tag{17}$$

In fact, $\mathcal{B}$ can be viewed as a random variable [2, Theorem 2.3], i.e., $\mathcal{B} = iL_B - 1/2$, for $i^2 = -1$, and L_B is the random variable on $\mathbb{R}$, with density $p_B(t) = \pi \operatorname{sech}^2(\pi t)/2$. Hence, the evaluation rule given by Equation (17) is equivalent to the expectation operator. Moreover, for any suitable function f, i.e., one for which integrals are absolutely convergent,

$$f(x+\mathcal{B}) = \mathbb{E}\left[f\left(x+iL_B-\frac{1}{2}\right)\right] = \frac{\pi}{2}\int_{\mathbb{R}} f\left(x+it-\frac{1}{2}\right)\operatorname{sech}^2(\pi t)dt.$$

In particular, $f(x) = x^n$ yields Equation (17). In addition, we have

$$B_n^{(p)}(x) = \left(x+\mathcal{B}^{(p)}\right)^n = (x+\mathcal{B}_1+\cdots+\mathcal{B}_p)^n$$

for a set of p *independent* umbral symbols (or random variables) $(\mathcal{B}_i)_{i=1}^p$, satisfying:

- if $i \neq j$, so that $\mathcal{B}_i$ and $\mathcal{B}_j$ are independent, then we evaluate
$$\mathcal{B}_i^n \mathcal{B}_j^m = B_n B_m;$$
- and if $i = j$, then
$$\mathcal{B}_i^n \mathcal{B}_j^m = \mathcal{B}_i^{n+m} = B_{n+m}.$$

Now, using Equation (1), we deduce that

$$e^{\mathcal{B}t} = \frac{t}{e^t-1}, \quad e^{t(2\mathcal{B}+1)} = \frac{t}{\sinh t}, \quad \text{and} \quad e^{t(2\mathcal{B}^{(p)}+p)} = \left(\frac{t}{\sinh t}\right)^p. \tag{18}$$

4.2. Euler $\mathcal{E}$

The Euler symbol $\mathcal{E}$ can be similarly defined via the random variable interpretation that $\mathcal{E} = iL_E - 1/2$, where L_E's density is given by $p_E(t) = \operatorname{sech}(\pi t)$. Then, $(\mathcal{E} + x)^n = E_n(x)$. In particular, for a sum of independent symbols $\mathcal{E}^{(p)} = \mathcal{E}_1 + \cdots + \mathcal{E}_p$,

$$E_n^{(p)}(x) = \left(x+\mathcal{E}^{(p)}\right)^n.$$

Therefore, Equation (1) yields

$$e^{t\mathcal{E}} = \frac{2}{e^t + 1}, \quad e^{t(2\mathcal{E}+1)} = \operatorname{sech} t, \quad \text{and} \quad e^{t(2\mathcal{E}^{(p)}+p)} = \operatorname{sech}^p t. \tag{19}$$

Moreover, from the generating functions Equations (18), and (19), we have

$$e^{2\mathcal{B}t} = \frac{2t}{e^{2t} - 1} = \frac{t}{e^t - 1} \cdot \frac{2}{e^t + 1} = e^{t(\mathcal{B}+\mathcal{E})},$$

i.e., $2\mathcal{B} = \mathcal{B} + \mathcal{E}$. Hence, for any suitable function f, $f(x + 2\mathcal{B}) = f(x + \mathcal{B} + \mathcal{E})$.

4.3. Uniform $\mathcal{U}$

The uniform symbol $\mathcal{U}$ is the uniform random variable on $[0, 1]$, i.e., $\mathcal{U} \sim U[0, 1]$, and the evaluation is

$$\mathcal{U}^n = \int_0^1 t^n dt = \frac{1}{n+1}. \tag{20}$$

We then have

$$e^{t\mathcal{U}} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} = \frac{e^t - 1}{t}, \quad e^{t(2\mathcal{U}-1)} = \frac{\sinh t}{t}, \quad \text{and} \quad e^{t(2\mathcal{U}^{(p)}-p)} = \left(\frac{\sinh t}{t}\right)^p, \tag{21}$$

as the sum of independent symbols $\mathcal{U}^{(p)} = \mathcal{U}_1 + \cdots + \mathcal{U}_p$. An important link between $\mathcal{B}$ and $\mathcal{U}$ is the cancellation rule. Note that

$$e^{t(\mathcal{U}+\mathcal{B})} = e^{t\mathcal{U}} e^{t\mathcal{B}} = \frac{e^t - 1}{t} \cdot \frac{t}{e^t - 1} = 1.$$

So for a suitable function f, $f(x + \mathcal{B} + \mathcal{U}) = f(x)$.

In what follows, we use independent copies of the three symbols. In order to distinguish them, we denote independent uniform symbols by $\mathcal{U}, \mathcal{U}', \ldots$ and $\mathcal{U}^{(p)}, \mathcal{U}'^{(p)}, \ldots$. We follow a similar convention for the other two symbols.

5. Identities of Bernoulli and Euler Polynomials

Now, with the loop decomposition in Equation (13) and evaluation of symbols given by Equations (18), (19), and (21), we can derive certain identities. Note that, for arbitrary sites, even in the case of two loops, it does not seem possible to further simplify the expressions completely in terms of Bernoulli and Euler polynomials. Instead, we can express them in terms of the three symbols; see for example, [4, Theorems 3.4 and 4.2]. We consider the case where the sites are *equally distributed*, namely $a_j = j$, for $j = 0, 1, 2, \ldots, n$, throughout this section.

5.1. 1-Dim Reflected Brownian Motion on $\mathbb{R}_+$

In this case, for three consecutive sites $a < b < c$, the generating functions of the corresponding hitting times can be found in [1, p. 198 and p. 355] with variable w:

$$\begin{aligned}
\phi_{a\to b} &= \frac{\cosh(aw)}{\cosh(bw)},\\
\phi_{b\to a|\not{c}} &= \frac{\sinh\left((c-b)w\right)}{\sinh\left((c-a)w\right)},\\
\phi_{b\to c|\not{a}} &= \frac{\sinh\left((b-a)w\right)}{\sinh\left((c-a)w\right)}.
\end{aligned}$$

In this case, we begin with $a_0 = 0$ as the initial site and then apply the formulas above for $n \geq 1$. We obtain

$$\phi_{0\to n} = \frac{1}{\cosh(nw)} \quad \text{and} \quad \phi_{n\to n+1|\cancel{n-1}} = \phi_{n\to n-1|\cancel{n+1}} = \frac{1}{2\cosh(w)}. \tag{22}$$

Before we state and prove the general formula, we first compute an example of the 3-loop case, which is not included in [4].

Example 3. As stated in Example 2,

$$\phi_{0\to 4} = \frac{\phi_{0\to 1}\phi_{1\to 2|\not{0}}\phi_{2\to 3|\not{1}}\phi_{3\to 4|\not{2}}}{1-(L_1+L_2+L_3-L_1L_3)}.$$

Now we apply Equation (22) to obtain

$$\begin{aligned}
\frac{1}{\cosh(4w)} = \frac{1}{8\cosh^4 w}\sum_{k=0}^{\infty}\Bigg(&\frac{\sinh w}{\sinh(2w)\cosh w} + \frac{2\sinh^2 w}{\sinh^2(2w)}\\
&-\frac{\sinh w}{\sinh(2w)\cosh w}\cdot\frac{\sinh^2 w}{\sinh^2(2w)}\Bigg)^k.
\end{aligned}$$

The left-hand side is simply $\operatorname{sech}(4w) = \exp\{4w(2\mathcal{E}+1)\}$.

For the right-hand side, we first simplify

$$\begin{aligned}
\frac{\sinh w}{\sinh(2w)\cosh w} + \frac{2\sinh^2 w}{\sinh^2(2w)} &= \frac{1}{\cosh^2 w},\\
\frac{\sinh w}{\sinh(2w)\cosh w}\cdot\frac{\sinh^2 w}{\sinh^2(2w)} &= \frac{1}{8\cosh^4 w}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{1}{8\cosh^4 w}\sum_{k=0}^{\infty}\left(\frac{1}{\cosh^2 w}-\frac{1}{8\cosh^4 w}\right)^k \\
&= \frac{1}{8\cosh^4 w}\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\left(\frac{1}{\cosh^2 w}\right)^{k-\ell}\left(\frac{1}{8\cosh^4 w}\right)^{\ell} \\
&= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\frac{1}{8^{\ell+1}}\cdot\left(\frac{1}{\cosh w}\right)^{2k+2\ell+4} \\
&= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\frac{1}{8^{\ell+1}}\exp\left\{w(2\mathcal{E}^{(2k+2\ell+4)}+2k+2\ell+4)\right\}.
\end{aligned}$$

Namely,

$$\exp\{8\mathcal{E}w+4w\} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\frac{1}{8^{\ell+1}}\exp\left\{w(2\mathcal{E}^{(2k+2\ell+4)}+2k+2\ell+4)\right\},$$

i.e.,

$$\exp\{8\mathcal{E}w\} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\frac{1}{8^{\ell+1}}\exp\left\{w(2\mathcal{E}^{(2k+2\ell+4)}+2k+2\ell)\right\}.$$

Multiplying both sides by $\exp\{xw\}$ and comparing the coefficients of w^n, we see that the left-hand side yields

$$\exp\{(8\mathcal{E}+x)w\} \Rightarrow (8\mathcal{E}+x)^n = 8^n E_n\left(\frac{x}{8}\right);$$

and the right-hand side gives

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\frac{1}{8^{\ell+1}}(2\mathcal{E}^{(2k+2\ell+4)}+2k+2\ell+x)^n \\
&= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\frac{2^n}{8^{\ell+1}}E_n^{(2k+2\ell+4)}\left(\frac{x}{2}+k+\ell\right).
\end{aligned}$$

Therefore, we have

$$E_n(x) = \frac{1}{4^n}\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\frac{2^n}{8^{\ell+1}}E_n^{(2k+2\ell)}(4x+k+\ell).$$

Remark 3. From Equation (22), we see that

$$\phi_{0\to1} = 2\phi_{n\to n+1|\cancel{n-1}} = 2\phi_{n\to n-1|\cancel{n+1}},$$

so that the L_1 and L_j's, for $j = 2, 3, \ldots$, are different. The following combinatorial enumeration is the key to expressing the general formulas.

Definition 2. Consider n sites $a_1 < a_2 < \cdots < a_n$.

1. If the sub-indices of $a_{j_1} a_{j_2} \cdots a_{j_m}$ satisfy $1 \leq j_1 \leq \cdots \leq j_m \leq n, j_k - j_{k-1} \geq 2$, we call it a nonadjacent product of order n and length m with initial state j_1.
2. We define $N(\ell, n)$ as the number of all nonadjacent products of order n and length ℓ (without specific initial state), i.e., terms with $m = \ell$ in item 1.
3. Finally, we let $n(k, \ell, m)$ be the number of different nonadjacent products of order n and length ℓ with initial state a_k. Namely, it counts terms with $j_1 = k$ and $m = \ell$ in item 1.

Finally, by convention, both $N(\ell, n)$ and $n(a, \ell, m)$ can be zero if no such product exists.

Lemma 1. *Let ℓ be an integer such that $3 \leq \ell \leq n$, then*

$$N(\ell, n) = N(\ell, n-1) + N(\ell - 1, n-2). \tag{23}$$

Proof. All the nonadjacent products of order n and length ℓ can be divided into two parts. The first part consists of the existing nonadjacent products before adding a_n, which are all nonadjacent products of order n and length ℓ. The second part consists of the new nonadjacent products after adding a_n, which are all nonadjacent products of order $n-2$ and length $\ell - 1$. □

Lemma 2. *If $2 \leq \ell \leq n$, then*

$$n(1, \ell, n) = \sum_{k=3}^{n} n(k, \ell - 1, n) = N(\ell - 1, n - 2).$$

Proof. A nonadjacent product of order n, length ℓ, and initial state a_1, has the form $a_1 a_{j_2} a_{j_3} \cdots a_{j_\ell}$ with $a_{j_2} \geq 3$. So the number of such products equals the number of all products $a_{j_2} a_{j_3} \cdots a_{j_\ell}$ with $a_{j_2} \geq 3$. By eliminating a_1 and a_2, this also equals to the number of all nonadjacent products of order $n-2$ and length $\ell - 1$. □

Note that $\binom{n}{k} = 0$ if $k > n$ or $k < 0$. Thus, we can identify $N(\ell, n)$ as the binomial coefficients.

Theorem 2. *If $1 \leq \ell \leq n$, then*

$$N(\ell, n) = \binom{n - \ell + 1}{\ell}.$$

Proof. It suffices to show that $\binom{n-l+1}{l}$ satisfies the same initial conditions and recurrence relation as $N(\ell, n)$, since

$$\binom{n - \ell + 1}{\ell} = \binom{n - \ell}{\ell} + \binom{n - \ell}{\ell - 1}$$

coincides with Equation (23). For the initial conditions, notice that when $\ell = 1$,

$$N(1,n) = n = \binom{n}{1},$$

the number of loops $L_1, \ldots, L_n$. $\square$

Recall the *multinomial coefficients*: for $k_1, \ldots, k_m \in \mathbb{N}$, $k_1 + \cdots + k_m \leq n$,

$$\binom{n}{k_1, \ldots, k_m} = \frac{n!}{k_1! \cdots k_m!(n - k_1 - \cdots - k_m)!}.$$

In particular, for $m = 1$, we have the binomial coefficients $\binom{n}{k}$. Also, recall the *ceiling function*:

$$\lceil x \rceil := \min_n \{n \in \mathbb{N} : n \geq x\}.$$

Theorem 3. *Let $M = \lceil m \rceil - 1$, and let M' be the largest odd number less than or equal to M. Then*

$$\begin{aligned} E_n\left(\frac{x}{m+1}\right) = \frac{1}{(m+1)^n} \sum_{k=0}^{\infty} \frac{(m+1)^k}{2^{2k+m}} & \sum_{n_1,\ldots,n_M=0}^{k} \binom{k}{n_1, \ldots, n_M} \\ & \times (-1)^{n_1+n_3+\cdots+n_{M'}} \frac{4^{n_1+\cdots+n_M}}{(m+1)^{n_1+\cdots+n_M}} \\ & \times \left(\frac{\binom{m-2}{1}}{2^3} + \frac{\binom{m-2}{2}}{2^4}\right)^{n_1} \cdots \left(\frac{\binom{m-M-1}{M}}{2^{2M+1}} + \frac{\binom{m-M-1}{M+1}}{2^{2M+2}}\right)^{n_M} \\ & \times E_n^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)}\left(k + n_1 + 2n_2 + \cdots + Mn_M + x\right). \end{aligned} \tag{24}$$

Proof. Let $n = m$ in Equation (13) and apply Equation (22) to obtain

$$\begin{aligned} \operatorname{sech}((m+1)w) = \operatorname{sech} w \frac{\sinh^m w}{\sinh^m(2w)} \sum_{k=0}^{\infty} \Bigg(& n(1,1,m) \frac{\operatorname{sech}^2 w}{2} \\ & + N(1,m-1) \frac{\operatorname{sech}^2 w}{2^2} - n(1,2,m) \frac{\operatorname{sech}^4 w}{2^3} \\ & - N(2,m-1) \frac{\operatorname{sech}^4 w}{2^4} + \cdots + (-1)^M n(1, M+1, m) \\ & \times \frac{\operatorname{sech}^{(2M+2)} w}{2^{2M+1}} + (-1)^M N(M+1, m-1) \frac{\operatorname{sech}^{(2M+2)} w}{2^{2M+2}} \Bigg)^k \end{aligned}$$

$$
\begin{aligned}
&= \frac{\operatorname{sech}^{(m+1)} w}{2^m} \sum_{k=0}^{\infty} \left(\frac{\operatorname{sech}^2 w}{2} + \frac{\binom{m-1}{1} \operatorname{sech}^2 w}{2^2} \right. \\
&\quad - \frac{\binom{m-2}{1} \operatorname{sech}^4 w}{2^3} - \frac{\binom{m-2}{2} \operatorname{sech}^4 w}{2^4} + \cdots + (-1)^M \\
&\quad \left. \times \frac{\binom{m-M-1}{M} \operatorname{sech}^{(2M+2)} w}{2^{2M+1}} + (-1)^M \frac{\binom{m-M-1}{M+1} \operatorname{sech}^{(2M+2)} w}{2^{2M+2}} \right)^k .
\end{aligned}
$$

Applying Equation (19), we get

$$
\begin{aligned}
e^{(m+1)w(2\mathcal{E}+1)} &= \sum_{k=0}^{\infty} \frac{(m+1)^k}{2^{2k+m}} \sum_{n_1,\ldots,n_M=0}^{k} \binom{k}{n_1,\ldots,n_M} \\
&\quad \times \frac{(-1)^{n_1+n_3+\cdots+n_{M'}} 4^{n_1+\cdots+n_M}}{(m+1)^{n_1+\cdots+n_M}} \\
&\quad \times \left(\frac{\binom{m-2}{1}}{2^3} + \frac{\binom{m-2}{2}}{2^4} \right)^{n_1} \cdots \left(\frac{\binom{m-M-1}{M}}{2^{2M+1}} + \frac{\binom{m-M-1}{M+1}}{2^{2M+2}} \right)^{n_M} \\
&\quad \times e^{w(2\mathcal{E}^{(2k+2n_1+4n_2+\cdots+2Mn_M+m+1)}+2k+2n_1+4n_2+\cdots+2Mn_M+m+1)}.
\end{aligned}
$$

Multiplying by e^{wx} produces

$$
\begin{aligned}
e^{w(2m\mathcal{E}+2\mathcal{E}+x)} &= \sum_{k=0}^{\infty} \frac{(m+1)^k}{2^{2k+m}} \sum_{n_1,\ldots,n_M=0}^{k} \binom{k}{n_1,\ldots,n_M} \\
&\quad \times \frac{(-1)^{n_1+n_3+\cdots+n_{M'}} 4^{n_1+\cdots+n_M}}{(m+1)^{n_1+\cdots+n_M}} \\
&\quad \times \left(\frac{\binom{m-2}{1}}{2^3} + \frac{\binom{m-2}{2}}{2^4} \right)^{n_1} \cdots \left(\frac{\binom{m-M-1}{M}}{2^{2M+1}} + \frac{\binom{m-M-1}{M+1}}{2^{2M+2}} \right)^{n_M} \\
&\quad \times e^{w(2\mathcal{E}^{(2k+2n_1+4n_2+\cdots+2Mn_M+m+1)}+2k+2n_1+4n_2+\cdots+2Mn_M+x)}.
\end{aligned}
$$

Now, we identify the coefficients of w^n on both sides.

The left-hand side is simply

$$
(2m\mathcal{E}+2\mathcal{E}+x)^n = (2m+2)^n \left(\mathcal{E} + \frac{x}{2m+2}\right)^n = (2m+2)^n E_n\left(\frac{x}{2m+2}\right).
$$

For the right-hand side, we only need to focus on the term

$$
\begin{aligned}
&(2\mathcal{E}^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)} + 2k + 2n_1 + 4n_2 + \cdots + 2Mn_M + x)^n \\
&= 2^n E_n^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)} \left(k + n_1 + 2n_2 + \cdots + Mn_M + \frac{x}{2}\right).
\end{aligned}
$$

Therefore, simplification with $x \mapsto 2x$ gives the desired identity. □

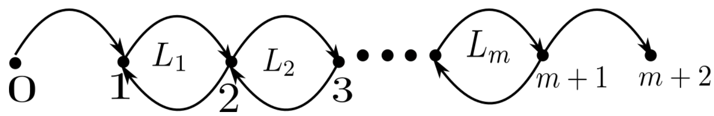

Figure 3: 3-dim Bessel Process

Example 4. The formulas derived from the 4-loop and 5-loop cases are as follows:

$$E_n\left(\frac{x}{5}\right)=\frac{1}{5^n}\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\frac{5^k(-1)^\ell}{2^{2k+2\ell+4}}\binom{k}{\ell}E_n^{(2\ell+2k+5)}(x+\ell+k),$$

$$E_n\left(\frac{x}{6}\right)=\frac{1}{6^n}\sum_{k=0}^{\infty}\sum_{n_1,n_2=0}^{k}\frac{(-1)^{n_1}3^{k-n_1-n_2}}{2^{k+3n_1+4n_2+5}}\binom{k}{n_1,n_2}$$
$$\times E_n^{(2k+2n_1+4n_2+6)}(x+k+n_1+2n_2).$$

5.2. 3-Dim Bessel Process on $\mathbb{R}^3$

We can also find the generating functions for three consecutive sites $a<b<c$ [1, pp. 463–464] with variable w:

$$\phi_{a\to b}=\frac{b\sinh(aw)}{a\sinh(bw)},$$
$$\phi_{b\to a|\not c}=\frac{a\sinh((c-b)w)}{b\sinh((c-a)w)},$$
$$\phi_{b\to c|\not a}=\frac{c\sinh((b-a)w)}{b\sinh((c-a)w)}.$$

As stated in [4, Remark 4.1], $\phi_{a\to 0|\not b}=0$ for $0<a<b$. In this case, the first loop occurs between sites 1 and 2 (instead of 0 and 1). Hence, after hitting level 1, it never returns to 0 and we can treat 1 as the new starting level. Then, Equation (13) applies after shifting all the indices by 1.

Theorem 4. *For the 3-dimensional Bessel process on sites $0,1,\ldots,m+2$, we have*

$$\phi_{0\to(m+2)}=\phi_{0\to1}\prod_{j=1}^{m+1}\phi_{j\to(j+1)|\not j-1}\frac{1}{1-\sum\limits_{k=1}^{m}\sum\limits_{(k,l,n)}}. \tag{25}$$

Let $M=\lceil m\rceil-1$, and let M' be the largest odd number less than or equal to M.

Then Equation (25) *yields*

$$\begin{aligned}
&B_{n+1}\left(\frac{2+x}{m+2}\right)-B_{n+1}\left(\frac{x}{m+2}\right)\\
&=\frac{n+1}{(m+2)^n}\sum_{k=0}^{\infty}\frac{m^k}{2^{2k+m}}\sum_{n_1,\dots,n_M=0}^{k}\binom{k}{n_1,\dots,n_M}(-1)^{n_1+n_3+\cdots+n_{M'}}\\
&\quad\times\frac{\binom{m-1}{2}^{n_1}\binom{m-2}{3}^{n_2}\cdots\binom{m-M}{M}^{n_M}}{4^{n_1+2n_2+\cdots+Mn_M}m^{n_1+\cdots+n_M}}\\
&\quad\times E_n^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)}(k+n_1+2n_2+\cdots+Mn_M+x).
\end{aligned}$$

Proof. The proof of Equation (25) is trivial, by shifting all the indices as $a_m \mapsto a_{m+1}$. Note that L_1 is between site a_1 and a_2; see Figure 3. For the identity above, the steps are similar to those used in Equation (24), so we do not include all the calculations. By Equation (25) and the standard formula $\sinh(2w) = 2\sinh w\cosh w$, we have

$$\begin{aligned}
\frac{(m+2)w}{\sinh((m+2)w)}=\frac{(m+2)w}{\sinh(2w)}\sum_{k=0}^{\infty}\Bigg(&\frac{\binom{m}{1}\operatorname{sech}^2 w}{4}-\frac{\binom{m-1}{2}\operatorname{sech}^4 w}{4^2}\\
&+\cdots+(-1)^M\frac{\binom{m-M}{M+1}\operatorname{sech}^{2M+2} w}{4^{M+1}}\Bigg)^k.
\end{aligned}$$

From Equation (18) and Equation (19), we deduce that

$$\begin{aligned}
e^{(m+2)w(2\mathcal{B}+1)}=e^{2w(2\mathcal{B}'+1)}&\sum_{k=0}^{\infty}\frac{(m+2)m^k}{2^{2k+m+1}}\sum_{n_1,\dots,n_M=0}^{k}\binom{k}{n_1,\dots,n_M}\\
&\times(-1)^{n_1+n_3+\cdots+n_{M'}}\frac{\binom{m-1}{2}^{n_1}\binom{m-2}{3}^{n_2}\cdots\binom{m-M}{M}^{n_M}}{4^{n_1+2n_2+\cdots+Mn_M}m^{n_1+\cdots+n_M}}\\
&\times e^{w\left(2\mathcal{E}^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)}+2k+2n_1+4n_2+\cdots+2Mn_M+m\right)}.
\end{aligned}$$

In order to cancel $e^{4w\mathcal{B}'}$ on the right-hand side, we multiply by $e^{4w\mathcal{U}+wx}$ to obtain

$$\begin{aligned}
e^{w((2m+4)\mathcal{B}+4\mathcal{U}+x)}=&\sum_{k=0}^{\infty}\frac{(m+2)m^k}{2^{2k+m+1}}\sum_{n_1,\dots,n_M=0}^{k}\binom{k}{n_1,\dots,n_M}\times\\
&(-1)^{n_1+n_3+\cdots+n_{M'}}\frac{\binom{m-1}{2}^{n_1}\binom{m-2}{3}^{n_2}\cdots\binom{m-M}{M}^{n_M}}{4^{n_1+2n_2+\cdots+Mn_M}m^{n_1+\cdots+n_M}}\times\\
&e^{w(2\mathcal{E}^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)}+2k+2n_1+4n_2+\cdots+2Mn_M+x)}.
\end{aligned}$$

When identifying the coefficient of w^n, the right-hand side directly gives the Euler polynomials of higher-orders. By Equation (20), the left-hand side is

$$((2m+4)\mathcal{B}+4\mathcal{U}+x)^n = \int_0^1 ((2m+4)\mathcal{B}+4u+x)^n du$$
$$= \frac{((2m+4)\mathcal{B}+4u+x)^{n+1}}{4(n+1)}\Bigg|_{u=0}^{u=1}$$
$$= \frac{(2m+4)^{n+1}}{4(n+1)}\left(B_{n+1}\left(\frac{x+4}{2m+4}\right) - B_{n+1}\left(\frac{x}{2m+4}\right)\right).$$

Simplification and substitution $x \mapsto 2x$ completes the proof. □

Example 5. The identities from 3 and 4 loops are given by

$$B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right)$$
$$= \frac{n+1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell\frac{1}{12^\ell}E_n^{(2k+2\ell+3)}(k+\ell+x), \tag{26}$$

and

$$B_{n+1}\left(\frac{x+2}{6}\right) - B_{n+1}\left(\frac{x}{6}\right)$$
$$= \frac{n+1}{6^n}\sum_{k=0}^{\infty}\frac{4^k}{2^{2k+4}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell\frac{3^\ell}{4^{2\ell}}E_n^{(2k+2\ell+4)}(k+\ell+x).$$

Remark 4. One can alternatively verify the identities above by direct calculation of the generating functions of Bernoulli and Euler polynomials. For example, Equation (26) is equivalent to

$$\sum_{n=0}^{\infty}\left(B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right)\right)\frac{t^{n+1}}{(n+1)!}$$
$$= t\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell\frac{1}{12^\ell}E_n^{(2k+2\ell+3)}(k+\ell+x)\frac{\left(\frac{t}{5}\right)^n}{n!},$$

where the left-hand side is

$$\sum_{n=0}^{\infty}\left(B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right)\right)\frac{t^{n+1}}{(n+1)!}$$
$$= \sum_{n+1=0}^{\infty}\left(B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right)\right)\frac{t^{n+1}}{(n+1)!} - B_0\left(\frac{x+2}{5}\right) + B_0\left(\frac{x}{5}\right)$$
$$= \left(\frac{t}{e^t-1}\right)e^{\frac{x+2}{5}\cdot t} - \left(\frac{t}{e^t-1}\right)e^{\frac{x}{5}\cdot t} = \frac{t(e^{\frac{2t}{5}}-1)}{e^t-1}e^{\frac{tx}{5}},$$

and the right-hand side is

$$
\begin{aligned}
&t\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}\frac{1}{12^{\ell}}E_n^{(2k+2\ell+3)}(k+\ell+x)\frac{\left(\frac{t}{5}\right)^n}{n!}\\
&=t\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}\frac{1}{12^{\ell}}\sum_{n=0}^{\infty}E_n^{(2k+2\ell+3)}(k+\ell+x)\frac{\left(\frac{t}{5}\right)^n}{n!}\\
&=t\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}\frac{1}{12^{\ell}}\left(\frac{2}{e^{\frac{t}{5}}+1}\right)^{2k+2\ell+3}e^{(k+\ell+x)\frac{t}{5}}\\
&=te^{\frac{tx}{5}}\left(\frac{1}{e^{\frac{t}{5}}+1}\right)^3\sum_{k=0}^{\infty}3^k e^{\frac{kt}{5}}\left(\frac{1}{e^{\frac{t}{5}}+1}\right)^{2k}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}\frac{1}{3^{\ell}}\left(\frac{1}{e^{\frac{t}{5}}+1}\right)^{2\ell}e^{\frac{\ell t}{5}}\\
&=te^{\frac{tx}{5}}\left(\frac{1}{e^{\frac{t}{5}}+1}\right)^3\sum_{k=0}^{\infty}3^k e^{\frac{kt}{5}}\left(\frac{1}{e^{\frac{t}{5}}+1}\right)^{2k}\left(1-\frac{1}{3}\left(\frac{1}{e^{\frac{t}{5}}+1}\right)^2 e^{\frac{t}{5}}\right)^k\\
&=te^{\frac{tx}{5}}\frac{e^{\frac{t}{5}}+1}{e^{\frac{4t}{5}}+e^{\frac{3t}{5}}+e^{\frac{2t}{5}}+e^{\frac{t}{5}}+1}=\frac{t(e^{\frac{2t}{5}}-1)}{e^t-1}e^{\frac{tx}{5}}.
\end{aligned}
$$

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