HYPERGRAPH RAMSEY NUMBERS INVOLVING TREES, STARS, AND COMPLETE HYPERGRAPHS

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Abstract

We investigate multicolor hypergraph Ramsey numbers for various combinations of r-uniform trees, stars, and complete hypergraphs, providing upper and lower bounds. For trees and stars, several existence theorems are proved and numerous bounds are given for the corresponding Ramsey numbers. In a few cases of Ramsey numbers involving stars, exact evaluations are determined. We conclude by proving a simplification of the work needed to prove the conjecture that every r-uniform tree is n-good.

1. Introduction

The Ramsey theory of graphs has seen many generalizations since its introduction in the mid-1900s. Ramsey numbers involving trees, stars, and complete graphs have been thoroughly studied (see [8], [9], [11], [13], [18], [24], [25], and [26]). Over the past few decades, generalizations involving uniform hypergraphs have gained the attention of many researchers (e.g., see [4], [5], [6], [16], [17], [19], [20], [21], [22], [23], [27], [28], and Section 7 of Radziszowski's dynamic survey [29]) and the present paper focuses on further developing this theory for trees, stars, and complete hypergraphs. Before we describe our results, we must review the relevant definitions and background required for our investigation.

If S is any set, denote by S^r the set of all r-element subsets of S. We write $|S|$ for the cardinality of a set S. An *r-uniform hypergraph* $H = (V(H), E(H))$ consists of a nonempty set $V(H)$ of vertices, along with a set of hyperedges $E(H) \subseteq V(H)^r$. In the case where $|V(H)| = n$ and $E(H) = V(H)^r$, the resulting hypergraph is called the *complete r-uniform hypergraph of order n*, denoted by $K_n^{(r)}$. A *Berge path of length* $m-1$ is a hypergraph containing distinct vertices $x_1, x_2, \ldots, x_m$ and distinct hyperedges $e_1, e_2, \ldots, e_{m-1}$ such that $x_i, x_{i+1} \in e_i$ for $1 \leq i \leq m-1$. Such a path, along with a hyperedge e_m such that $x_1, x_m \in e_m$ forms a *Berge cycle of length* m. An r-uniform hypergraph is called *connected* if for every pair of distinct vertices a and b, there exists a Berge path that contains both a and b. An r-uniform hypergraph that is not connected is called *disconnected*.

A connected r-uniform hypergraph that does not contain any Berge cycles is called an *r-uniform tree*. Numerous equivalent definitions for an r-uniform tree are given in [3] and [6] . In particular, we note that every r-uniform tree is *minimally connected* (the removal of any hyperedge while retaining all vertices results in a disconnected hypergraph), but not every minimally connected r-uniform hypergraph is an r-uniform tree. It is also well-known that for $r \geq 3$, a connected r-uniform hypergraph may not contain a spanning tree. Given an r-uniform tree T, a hyperedge that contains $r-1$ vertices of degree 1 is called a *leaf.* From the first definition given in Theorem 2.1 in [6], it is easily shown that every r-uniform tree contains at least one leaf and that a tree of order m has size $\frac{m-1}{r-1}$. Denote by $\mathcal{T}_m^{(r)}$ the unique r-uniform tree of order m that contains a single vertex of degree $\frac{m-1}{r-1}$ and all other vertices of degree 1 (every hyperedge is a leaf).

If $1 \leq c \leq r-1$, then the *star* $S_{c,n}^{(r)}$ is defined to be the r-uniform hypergraph with vertex set given by the disjoint union $C \dot{\cup} U$ such that $|C| = c$ and $|U| = n-c$, and the hyperedge set consists of all hyperedges that include all c vertices in C and $r-c$ vertices from U. The set C is called the *center* of the star and $S_{c,n}^{(r)}$ has size $\binom{n-c}{r-c}$. When $r=2$, the star $S_{1,n}^{(2)}$ can be identified with the bipartite graph $K_{1,n-1}$, which is a tree. Whenever $n > r \geq 3$, the hypergraph $S_{c,n}^{(r)}$ is not an r-uniform tree and is not minimally connected unless it has size greater than 1 and $c = r-1$.

If $H_1, H_2, \ldots, H_t$ are r-uniform hypergraphs, then the *t-color Ramsey number* $R(H_1, H_2, \ldots, H_t; r)$ is the least natural number p such that every t-coloring of the hyperedges of $K_p^{(r)}$ contains a subhypergraph isomorphic to H_i in color i for some $1 \leq i \leq p$. A general lower bound for $R(H_1, H_2, \ldots, H_t; r)$ can be obtained by considering the *weak chromatic number* $\chi_w(H)$ of a hypergraph H. For any such H, $\chi_w(H)$ is defined to be the minimum number of colors needed to color the vertices of H such that no hyperedge is monochromatic. Such a coloring is called a *weak proper vertex coloring.* Among all weak proper vertex colorings of H using $\chi_w(H)$ colors, denote by $s(H)$ the minimum cardinality of a color class. The number $s(H)$ is called the *chromatic surplus* of H.

Generalizing the lower bound originally proved by Chvátal and Harary in [10], Burr [7] proved that

$$R(G_1, G_2; 2) \geq (|V(G_1)| - 1)(\chi(G_2) - 1) + s(G_2),$$

where G_1 is a connected graph and χ is the usual chromatic number (which agrees with the weak chromatic number when $r = 2$). This result was extended to r-uniform hypergraphs in Theorem 3.1 of [6], and in Theorem 5 of [2], it was shown that

$$R(H_1, H_2, \dots, H_t; r) \geq (R(H_1, H_2, \dots, H_{t-1}; r) - 1)(\chi_w(H_t) - 1) + s(H_t). \quad (1)$$

It was actually shown in [2] that this inequality holds for Gallai-Ramsey numbers, but the same proof works for standard Ramsey numbers as well. When equality holds in (1), we say that the multiset $\mathcal{H} = \{H_1, H_2, \dots, H_{t-1}\}$ is H_t-*good*. If $H_t = K_n^{(r)}$, we say that $\mathcal{H}$ is n-*good* and if $t = 2$, we say that H_1 is H_2-*good*. It is known that when $r = 2$, all trees are n-good [9] and it is conjectured that this is also the case when $r \geq 3$ (see [6]).

A useful tool for studying hyperedge colorings of hypergraphs is the link of a set of vertices. For an r-uniform hypergraph H of order $m \geq r$, and a set $S \subset V(H)$ with cardinality $1 \leq t \leq r-1$, define the *link* L_S to be the $(r-t)$-uniform hypergraph with vertex set $V(H) - S$ and hyperedge set

$$\{x_1 x_2 \cdots x_{r-t} \mid S \cup \{x_1, x_2, \dots, x_{r-t}\} \in E(H)\}.$$

In particular, there is a natural extension of a hyperedge coloring of H to a hyperedge coloring of L_S for any S. In the special case where $S = \{x\}$, we write L_x in place of L_S.

This paper proceeds as follows. In Section 2, we begin our investigation by proving a few results involving the existence of stars and trees in r-uniform hypergraphs. We then turn our attention to providing several general multicolor Ramsey number inequalities in Section 3, and shift our focus to the cases of stars and trees in Section 4. Here, we prove several exact evaluations, including the t-color Ramsey number

$$R(S^{(r)}_{r-1,r+1}, S^{(r)}_{r-1,r+1}, \dots, S^{(r)}_{r-1,r+1}, S^{(r)}_{r-1,n}; r) = n + 1$$

and the 2-color Ramsey number $R(S^{(4)}_{3,6}, S^{(4)}_{3,7}; 4) = 9$. In Section 5, we direct our attention to the conjecture that all r-uniform trees are n-good (see [6]). While we are unable to prove the conjecture in its entirety, we prove a simplification of the work needed to prove it and conclude with a potential method for completing its proof.

2. Some Existence Results for Trees and Stars

If H is an r-uniform hypergraph, $1 \leq \ell \leq r - 1$, and $S \in V(H)^\ell$, then define the ℓ-*degree of* S to be

$$deg_\ell(S) := |\{e \in E(H) \mid S \subset e\}|.$$

The *minimum ℓ-degree of H* is given by

$$\delta_\ell(H) := \min\{deg_\ell(S) \mid S \subset V(H)^\ell\}$$

and the *maximum ℓ-degree of H* is given by

$$\Delta_\ell(H) := \max\{deg_\ell(S) \mid S \subset V(H)^\ell\}.$$

Note that if $T_m^{(r)}$ is an r-uniform tree of order m, then $m = k(r-1)+1$, where k is the size of $T_m^{(r)}$. It is well known that if a graph has a minimum vertex degree of k, then it is possible to find a subgraph isomorphic to any tree T of size k (e.g., see Lemma 1 of [11]). The following lemma generalizes this result to r-uniform trees and includes Theorem 2.2 of [6] as a special case. Note that the statement of Theorem 2.2 in [6] should have been given as a strict inequality.

Lemma 1. *Let $T_m^{(r)}$ be an r-uniform tree of order $m \geq r$ and size k and let H be any r-uniform hypergraph of order $p \geq m$ that satisfies*

$$\delta_\ell(H) > \binom{p-\ell}{r-\ell} - \binom{p-\ell-(k-1)(r-1)}{r-\ell}.$$

Then H contains a subhypergraph isomorphic to $T_m^{(r)}$.

Proof. We proceed by induction on $k \geq 1$. When $k = 1$, we have

$$\delta_\ell(H) > \binom{p-\ell}{r-\ell} - \binom{p-\ell}{r-\ell} = 0.$$

Thus, a hyperedge exists forming a $T_r^{(r)}$. Now assume that the lemma is true for all trees of size $k-1$, let $T_m^{(r)}$ be a tree of size k, and assume that H is an r-uniform hypergraph with order $p \geq m$ that satisfies

$$\delta_\ell(H) > \binom{p-\ell}{r-\ell} - \binom{p-\ell-(k-1)(r-1)}{r-\ell}.$$

Let T' be an r-uniform tree formed by removing a single leaf from $T_m^{(r)}$ and denote by x the vertex in T' that was incident with the removed leaf. By the inductive hypothesis, H contains a subgraph isomorphic to T'. By our assumption that $p \geq m$, there exists at least $r-1$ vertices in H that are not in T'. Label $r-1$ of these vertices $y_1, y_2, \ldots, y_{r-1}$. Now consider the set $S = \{x, y_1, y_2, \ldots, y_{r-1}\}$. The maximum number of hyperedges that contain S and some other vertices from T' in H is

$$\binom{p-\ell}{r-\ell} - \binom{p-\ell-(k-1)(r-1)}{r-\ell},$$

so the assumed inequality implies that some other hyperedge that contains S, and no other vertices from T', must exist. Such a hyperedge can be added to T' to form a copy of $T_m^{(r)}$. □

In particular, we find that Lemma 1 is not dependent on the order of H when $\ell = r-1$.

Corollary 1. *Let $T_m^{(r)}$ be an r-uniform tree of order $m \geq r$ and size k and let H be any r-uniform hypergraph that satisfies*

$$\delta_{r-1}(H) > (k-1)(r-1).$$

Then H contains a subhypergraph isomorphic to $T_m^{(r)}$.

If H is an r-uniform hypergraph, then the *Turán number* $ex(n, H)$ is defined to be the maximum number of hyperedges in an H-free r-uniform hypergraph on n vertices. While Turán numbers are clearly useful in directly arguing the existence of a certain subhypergraph, we can combine their use with that of the link of a set of vertices to make arguments such as the one demonstrated in the next lemma.

Lemma 2. *Let $p \geq n \geq r \geq 3$ and H be an r-uniform hypergraph of order p. If*

$$\Delta_\ell(H) \geq ex(p-\ell, K_{n-\ell}^{(r-\ell)}) + 1,$$

then H contains a subhypergraph isomorphic to $S_{\ell,n}^{(r)}$.

Proof. Assuming that $\Delta_\ell(H) \geq ex(p-\ell, K_{n-\ell}^{(r-\ell)}) + 1$, $V(H)$ contains some ℓ-set S having degree at least $ex(p-\ell, K_{n-\ell}^{(r-\ell)}) + 1$. Consider the link L_S, which is an $(r-\ell)$-uniform hypergraph of order $p-\ell$. Since this hypergraph contains at least $ex(p-\ell, K_{n-\ell}^{(r-\ell)})+1$ hyperedges, it necessarily contains a subhypergraph isomorphic to $K_{n-\ell}^{(r-\ell)}$, which corresponds with a copy of $S_{\ell,n}^{(r)}$ in H, having center S. □

The number $ex(n, H)$ is called a Turán number because of the foundational work of Turán [30], where he proved that

$$ex(n, K_m) = \left\lfloor \left(1 - \frac{1}{m-1}\right) \frac{n^2}{2} \right\rfloor.$$

Combining this result with Lemma 2 gives the following corollary in the case where $\ell = r-2$.

Corollary 2. *Let H be an r-uniform hypergraph of order p for $p \geq n \geq r \geq 3$. If*

$$\Delta_{(r-2)}(H) \geq \left\lfloor \left(1 - \frac{1}{n-r+1}\right) \frac{(p-r+2)^2}{2} \right\rfloor + 1,$$

then H contains a subhypergraph isomorphic to $S_{r-2,n}^{(r)}$.

3. General Bounds on Multicolor Ramsey Numbers

In this section, we prove a couple of general bounds for multicolor hypergraph Ramsey numbers, generalizing known results from [19] and [24].

Theorem 1. *Let $H_1^{(r)}, H_2^{(r)}, \ldots, H_t^{(r)}$ be r-uniform hypergraphs. Let $T_m^{(r)}$ be an r-uniform tree. Then,*

$$R(T_m^{(r)}, H_1^{(r)}, H_2^{(r)}, \ldots, H_t^{(r)}; r) \leq \frac{(m-1)(R(H_1^{(r)}, H_2^{(r)}, \ldots, H_t^{(r)}; r) - 1)}{r-1} + 1.$$

Proof. The Ramsey number $R(T_m^{(r)}, H_1^{(r)}, H_2^{(r)}, \ldots, H_t^{(r)}; r)$ is defined as the least natural number p such that every t-coloring of the hyperedges of $K_p^{(r)}$ results in a subhypergraph isomorphic to a red $T_m^{(r)}$ or a H_i in color i for some $1 \leq i \leq t$. Then the Ramsey number $R(H_1^{(r)}, H_2^{(r)}, \ldots, H_t^{(r)}; r)$ is the least natural number ℓ such that every t-coloring of the hyperedges of $K_\ell^{(r)}$ results in a subhypergraph isomorphic to H_i in color i for some $1 \leq i \leq t$. Now, consider the complete r-uniform hypergraph on $R(T_m^{(r)}, K^{(r)}_{R(H_1^{(r)}, H_2^{(r)}, \ldots, H_t^{(r)}; r)}; r)$ vertices. This hypergraph will necessarily contain a subhypergraph isomorphic to H_i in color i for some $1 \leq i \leq t$. Therefore,

$$R(T_m^{(r)}, H_1^{(r)}, H_2^{(r)}, \ldots, H_t^{(r)}; r) \leq R(T_m^{(r)}, K^{(r)}_{R(H_1^{(r)}, H_2^{(r)}, \ldots, H_t^{(r)}; r)}; r).$$

Now,

$$R(T_m^{(r)}, K^{(r)}_{R(H_1^{(r)}, H_2^{(r)}, \ldots, H_t^{(r)}; r)}; r) \leq \frac{(m-1)(R(H_1^{(r)}, H_2^{(r)}, \ldots, H_t^{(r)}; r) - 1)}{r-1} + 1$$

by Theorem 4.2 of [19], and our proof is completed. □

The following theorem was motivated by the upper bound proved in Theorem 2.1 of Omidi and Raeisi's paper [24].

Theorem 2. *Let $n = R(K_{n_1}^{(r)}, K_{n_2}^{(r)}, \ldots, K_{n_t}^{(r)}; r)$. If $H_1, H_2, \ldots, H_s$ are any r-uniform hypergraphs, then*

$$R(H_1, H_2, \ldots, H_s, K_{n_1}^{(r)}, K_{n_2}^{(r)}, \ldots, K_{n_t}^{(r)}; r) \leq R(H_1, H_2, \ldots, H_s, K_n^{(r)}; r).$$

Proof. Let $p = R(H_1, H_2, \ldots, H_s, K_n^{(r)}; r)$ and consider an $(s+t)$-coloring of $K_p^{(r)}$. Grouping together the colors $s+1, s+2, \ldots, s+t$, there exists a copy of H_i in color i, for some $1 \leq i \leq s$, or there is a $K_n^{(r)}$ spanned by hyperedges only using colors $s+1, s+2, \ldots, s+t$. In the latter case, since $n = R(K_{n_1}^{(r)}, K_{n_2}^{(r)}, \ldots, K_{n_t}^{(r)}; r)$, it follows that there is a copy of $K_{n_j}^{(r)}$ in color j for some $s+1 \leq j \leq s+t$. □

4. Trees and Stars

We now turn our attention to multicolor hypergraph Ramsey numbers involving trees and stars. Recall that we denote by $\mathcal{T}_m^{(r)}$ the r-uniform tree of order m having a single vertex of degree $\frac{m-1}{r-1}$ and all other vertices of degree 1. Motivated by the proof of Theorem 6 of [18], we obtain the following theorem.

Theorem 3. *Let $r \geq 3$. For each $1 \leq i \leq p$, write $m_i = \ell_i(r-1)+1$ and set*

$$M = 1 + \sum_{i=1}^{t} (\ell_i - 1).$$

Then for all $n \geq r$,

$$R(\mathcal{T}_{m_1}^{(r)}, \mathcal{T}_{m_2}^{(r)}, \ldots, \mathcal{T}_{m_p}^{(r)}, K_n^{(r)}; r) \leq R(\mathcal{T}_{M(r-1)+1}^{(r)}, K_n^{(r)}; r).$$

Proof. Let $k = R(\mathcal{T}_{M(r-1)+1}^{(r)}, K_n^{(r)}; r)$ and consider a $(t+1)$-coloring of the hyperedges of $K_k^{(r)}$. Grouping the first t colors together, there exists a $\mathcal{T}_{M(r-1)+1}^{(r)}$ spanned by hyperedges using only the first t colors or there exists a copy of $K_n^{(r)}$ in the $(t+1)^{st}$ color. In the former case, the M hyperedges of $\mathcal{T}_{M(r-1)+1}^{(r)}$ are partitioned among the first t colors. By the general pigeonhole principle, there must be some color i such that at least ℓ_i of the hyperedges are in color i, forming a copy of $\mathcal{T}_{\ell_i(r-1)+1}^{(r)}$. □

If we consider the case $\ell_1 = \ell_2 = 2$ and assume that $(r-1)|(n-1)$, then Theorem 3, along with Equation (1), and Corollary 4 of [12], implies that

$$\begin{aligned} 2(r-1)\left(\left\lceil \frac{n}{r-1} \right\rceil - 1\right) + s(K_n^{(r)}) &\leq R(\mathcal{T}_{2r-1}^{(r)}, \mathcal{T}_{2r-1}^{(r)}, K_n^{(r)}; r) \\ &\leq 3(r-1)\left(\left\lceil \frac{n}{r-1} \right\rceil - 1\right) + s(K_n^{(r)}). \end{aligned}$$

Observe that $S_{r-1,n}^{(r)}$ has size $n-(r-1)$, with each hyperedge containing a vertex of degree 1.

Theorem 4. *If $n_i \geq r \geq 3$ for $1 \leq i \leq t$, then*

$$R(S_{r-1,n_1}^{(r)}, S_{r-1,n_2}^{(r)}, \ldots, S_{r-1,n_t}^{(r)}; r) \leq n_1 + n_2 + \cdots + n_t - (t-1)r.$$

Proof. Let

$$N = \sum_{i=1}^{t} n_i$$

and consider a t-coloring of $K_{N-(t-1)r}^{(r)}$. If $S \in (V(K_{N-(t-1)r}^{(r)}))^{r-1}$, then S is contained in exactly $N - tr + 1$ hyperedges. If a monochromatic copy of $S_{r-1,n_i}^{(r)}$ is avoided in color i, for all $1 \le i \le t$, then an $(r-1)$-set of vertices can be contained in at most $N - tr$ hyperedges, giving a contradiction. Hence, there must exist a copy of $S^{(r)}_{r-1,n_i}$ in color i, for some $1 \le i \le t$. □

We will use the following well-known theorem proved by Baranyai [1] concerning the 1-factorization of complete r-uniform hypergraphs. Recall that a 1-factor of a hypergraph is a spanning collection of disjoint hyperedges (i.e., a complete matching). We say that an r-uniform hypergraph H is 1-factorable if its hyperedge set can be partitioned into 1-factors.

Theorem 5 ([1])**.** *Let $n \ge r \ge 2$. If $r|n$, then $K^{(r)}_n$ is 1-factorable.*

Theorem 6. *Let $n \ge r \ge 3$ and suppose that $r|n$. If $n_i \ge r$ for all $1 \le i \le t$ satisfies*

$$\binom{n_1-1}{r-1} + \binom{n_2-1}{r-1} + \cdots + \binom{n_t-1}{r-1} - t \ge \binom{n-1}{r-1},$$

then

$$R(S^{(r)}_{c_1,n_1}, S^{(r)}_{c_2,n_2}, \ldots, S^{(r)}_{c_t,n_t}; r) \ge n+1.$$

Proof. From Theorem 5, it follows that $K^{(r)}_n$ is 1-factorable. Since $K^{(r)}_n$ has size $\binom{n}{r}$ and each 1-factor contains exactly $\frac{n}{r}$ hyperedges, it follows that there are

$$\frac{\binom{n}{r}}{n/r} = \binom{n-1}{r-1}$$

1-factors in such a factorization. Color at most $\binom{n_i-1}{r-1} - 1$ 1-factors in color i for each $1 \le i \le t$ to obtain a p-coloring of the hyperedges of $K^{(r)}_n$. This is possible because of the inequality given in the statement of the theorem. If $A_i \subset V(K^{(r)}_n)^{c_i}$, where $1 \le c_i \le r-1$, then A_i is contained in exactly $\binom{n-c_i}{r-c_i}$ hyperedges, no two of which are contained in the same 1-factor. So, each such A_i is incident with at most $\binom{n_i-1}{r-1} - 1$ hyperedges in color i, preventing a copy of $S^{(r)}_{c_i,n_i}$ in color i for all $1 \le i \le t$. □

Combining Theorems 4 and 6, we obtain the following corollary.

Corollary 3. *Let $r \ge 3$, $t \ge 2$, and assume that $r|n$. Then the t-color Ramsey number satisfies*

$$n+1 \le R(S^{(r)}_{r-1,r+1}, S^{(r)}_{r-1,r+1}, \ldots, S^{(r)}_{r-1,r+1}, S^{(r)}_{r-1,n}; r) \le n+t-1.$$

In particular, it follows that $R(S^{(r)}_{r-1,r+1}, S^{(r)}_{r-1,n}; r) = n+1$.

Proof. Theorem 4 implies that $n+t-1$ is an upper bound for the t-color Ramsey number

$$R(S^{(r)}_{r-1,r+1}, S^{(r)}_{r-1,r+1}, \ldots, S^{(r)}_{r-1,r+1}, S^{(r)}_{r-1,n}; r).$$

To see that $n+1$ is a lower bound, we must check that the hypotheses of Theorem 6 are satisfied. Observe that

$$\binom{r}{r-1}+\binom{r}{r-1}+\cdots+\binom{r}{r-1}+\binom{n-1}{r-1}-t=(t-1)r-t+\binom{n-1}{r-1}\geq\binom{n-1}{r-1}$$

whenever $r \geq 3$. □

Observing that for $n \geq r$, a weak proper vertex coloring of $S^{(r)}_{t,n}$ can be obtained by coloring a single vertex in the center one color and all remaining vertices another color, it follows that $\chi_w(S^{(r)}_{t,n}) = 2$ and $s(S^{(r)}_{t,n}) = 1$. Applying Theorem 3.1 of [6], we obtain the trivial bound

$$R(S^{(r)}_{t_1,m}, S^{(r)}_{t_2,n}; r) \geq \max\{m, n\}.$$

This bound, along with Theorems 4 and 6 provide the bounds given in Table 1 for Ramsey numbers of the form $R(S^{(r)}_{r-1,m}, S^{(r)}_{r-1,n}; r)$, when $r = 3$.

$m \setminus n$	4	5	6	7	8	9	10	11	12
4	[4,5]	[5,6]	7	[7, 8]	[8, 9]	10	[10,11]	[11,12]	13
5	[5,6]	7	[7,8]	[7,9]	[8,10]	[10,11]	[10,12]	[11,13]	[13,14]
6	7	[7,8]	[7,9]	[7,10]	[10,11]	[10,12]	[10,13]	[11,14]	[13,15]
7	[7,8]	[7,9]	[7,10]	[10,11]	[10,12]	[10,13]	[10,14]	[13,15]	[13,16]
8	[8,9]	[8,10]	[10,11]	[10,12]	[10,13]	[10,14]	[13,15]	[13,16]	[13,17]
9	10	[10,11]	[10,12]	[10,13]	[10,14]	[13,15]	[13,16]	[13,17]	[13,18]
10	[10,11]	[10,12]	[10,13]	[10,14]	[13,15]	[13,16]	[13,17]	[13,18]	[13,19]
11	[11,12]	[11,13]	[11,14]	[13,15]	[13,16]	[13,17]	[13,18]	[13,19]	[16,20]
12	13	[13,14]	[13,15]	[13,16]	[13,17]	[13,18]	[13,19]	[16,20]	[16,21]

Table 1: Exact values/ranges for $R(S^{(3)}_{2,m}, S^{(3)}_{2,n}; 3)$ when $4 \leq m \leq 12$ and $4 \leq n \leq 12$.

Theorem 7. *Let $T^{(r)}_m$ be an r-uniform tree with m vertices. For $r \geq 2$,*

$$R(T^{(r)}_m, S^{(r)}_{t,n}) \leq m + n - r.$$

Proof. We proceed by induction on the number k of hyperedges in $T^{(r)}_m$. If $k = 1$, then $m = r$ and $T^{(r)}_r$ consists of a single hyperedge. It follows that

$$R(T^{(r)}_r, S^{(r)}_{t,n}) = n.$$

Now assume that the theorem is true for all trees of size $k-1$. Let $T^{(r)}_m$ be a tree of order m and size k ($m = r + (k-1)(r-1)$). Denote by T' the tree of size $k-1$ formed by removing a leaf from $T^{(r)}_m$ and let x be the vertex in T' that was incident with the removed leaf in $T^{(r)}_m$. Consider a red/blue coloring of the hyperedges in $K^{(r)}_{m+n-r}$. By the inductive hypothesis, there exists a red T' or a blue $S^{(r)}_{t,n}$. In the latter case, we are done. In the former case, note that there are

$$m + n - r - (m - (r-1)) = n - 1$$

vertices in the $K^{(r)}_{m+n-r}$ that are not included in the red T'. Denote this set of vertices by $U = \{u_1, u_2, \ldots, u_{n-1}\}$. Now consider the hyperedges that include $x, u_1, u_2, \ldots, u_{t-1}$ and $r-t$ vertices from $\{u_t, u_{t+1}, \ldots, u_{n-1}\}$. If any such hyperedge is red, then it can be used to form a red copy of $T^{(r)}_m$. Otherwise, all such hyperedges are blue and form a blue $S^{(r)}_{t,n}$ with center $\{x, u_1, u_2, \ldots, u_{t-1}\}$. □

Theorem 8. *Let $r \geq 3$, $p \geq 0$, $q \geq 0$, and $p+q \geq 2$. For each $1 \leq i \leq p$, write $m_i = \ell_i(r-1)+1$. Then*

$$R(\mathcal{T}^{(r)}_{m_1}, \mathcal{T}^{(r)}_{m_2}, \ldots, \mathcal{T}^{(r)}_{m_p}, S^{(r)}_{1,n_1}, S^{(r)}_{1,n_2}, \ldots, S^{(r)}_{1,n_q}; r)$$

$$\leq R\left(\ell_1 K^{(r-1)}_{r-1}, \ell_2 K^{(r-1)}_{r-1}, \ldots, \ell_p K^{(r-1)}_{r-1}, K^{(r-1)}_{n_1}, K^{(r-1)}_{n_2}, \ldots, K^{(r-1)}_{n_q}; r-1\right)+1.$$

Proof. Let

$$k = R\left(\ell_1 K^{(r-1)}_{r-1}, \ell_2 K^{(r-1)}_{r-1}, \ldots, \ell_p K^{(r-1)}_{r-1}, K^{(r-1)}_{n_1}, K^{(r-1)}_{n_2}, \ldots, K^{(r-1)}_{n_q}; r-1\right) + 1.$$

Begin by selecting a vertex x from $K^{(r)}_k$. Then the link L_x has order $k-1$ and contains a subhypergraph isomorphic to $\ell_j K^{(r-1)}_{r-1}$ in color j for some $1 \leq j \leq p$ or a subhypergraph isomorphic to $K^{(r-1)}_{n_k}$ in color $k+p$ for some $1 \leq k \leq q$. If L_x contains a subhypergraph isomorphic to $\ell_j K^{(r-1)}_{r-1}$ in color j, then each hyperedge contained in $\ell_j K^{(r-1)}_{r-1}$ corresponds with an r-uniform hyperedge that includes x in $K^{(r)}_k$ in color j. It follows that $K^{(r)}_k$ contains a subhypergraph isomorphic to a $\mathcal{T}^{(r)}_{m_p}$ in color j. If L_x contains a subhypergraph isomorphic to $K^{(r-1)}_{n_k}$ in color $k+p$, then each hyperedge contained in $K^{(r-1)}_{n_k}$ corresponds with an r-uniform hyperedge that includes x in $K^{(r)}_k$ in color $k+p$. It follows that $K^{(r)}_k$ contains a subhypergraph isomorphic to $S^{(r)}_{1,n_k}$ of color $k+p$. □

Theorem 9. *Let $r \geq 3$, $q \geq 2$, and $1 \leq t \leq r-1$. Then*

$$R(S^{(r)}_{t,n_1}, S^{(r)}_{t,n_2}, \ldots, S^{(r)}_{t,n_q}; r) \leq R(K^{(r-t)}_{n_1-t}, K^{(r-t)}_{n_2-t}, \ldots, K^{(r-t)}_{n_q-t}; r-t) + t.$$

Proof. Let $k = R(K^{(r-t)}_{n_1-t}, K^{(r-t)}_{n_2-t}, \ldots, K^{(r-t)}_{n_q-t}; r-t) + t$. Begin by selecting a set of t vertices from $K^{(r)}_k$ and denote these vertices by $x_1, x_2, \ldots, x_t$. Then the link $L_{\{x_1,x_2,\ldots,x_t\}}$ has order $k-t$ and contains a subhypergraph isomorphic to $K^{(r-t)}_{n_i-t}$ in color i, for some $1 \leq i \leq q$. Each hyperedge contained in $K^{(r-t)}_{n_i-t}$ corresponds with an r-uniform hyperedge that includes $x_1, x_2, \ldots, x_t$ in $K^{(r)}_k$ in color i. Then, $K^{(r)}_k$ contains a subhypergraph isomorphic to $S^{(r)}_{t,n_i}$ in color i. □

5. Trees Versus Complete Hypergraphs

In [6], it was conjectured that all r-uniform trees are n-good. This statement is equivalent to the inequality

$$R(T_m^{(r)}, K_n^{(r)}; r) \leq (m-1)\left(\left\lceil \frac{n}{r-1} \right\rceil - 1\right) + s(K_n^{(r)}).$$

In Corollary 4.3 of [6], it was shown that this conjecture is true when $(r-1)|(n-1)$. So, the case where $n = q(r-1)+1$ is proven, but most of the remaining cases in which

$$n = q(r-1)+k, \quad \text{where } k = 0 \text{ or } 2 \leq k < r-1,$$

still remain. The following lemma will assist in reducing the remaining cases.

Lemma 3. *For all $n \geq r$, and any r-uniform tree $T_m^{(r)}$ of order m,*

$$R(T_m^{(r)}, K_n^{(r)}; r) < R(T_m^{(r)}, K_{n+1}^{(r)}; r).$$

Proof. Let $p = R(T_m^{(r)}, K_n^{(r)}; r)$. Then there exists a red/blue coloring of the hyperedges of $K_{p-1}^{(r)}$ that lacks a red $T_m^{(r)}$ and a blue $K_n^{(r)}$. Fix such a coloring and add in a single vertex x, connected to all existing vertices via blue hyperedges. Since no red hyperedges have been added, the resulting $K_p^{(r)}$ lacks a red $T_m^{(r)}$. Also, no blue $K_{n+1}^{(r)}$ has been created since removing vertex x would then result in a blue $K_n^{(r)}$, which was assumed not to exist. It follows that $R(T_m^{(r)}, K_{n+1}^{(r)}; r) > p$. □

Theorem 10. *Let $T_m^{(r)}$ be an r-uniform tree of order m. If $T_m^{(r)}$ is n-good for some $(r-1)|n$, then $T_m^{(r)}$ is n'-good for all $n' \in \{n-1, n-2, \ldots, n-(r-2)\}$.*

Proof. Let $n = q(r-1)$, where $q \geq 2$ and note that the lower bound given in Theorem 3.1 of [6] implies that

$$R(T_m^{(r)}, K_n^{(r)}; r) \geq (m-1)\left(\frac{n}{r-1} - 1\right) + r - 1 \tag{2}$$

$$R(T_m^{(r)}, K_{n-1}^{(r)}; r) \geq (m-1)\left(\frac{n}{r-1} - 1\right) + r - 2$$

$$R(T_m^{(r)}, K_{n-2}^{(r)}; r) \geq (m-1)\left(\frac{n}{r-1} - 1\right) + r - 3$$

$$\vdots$$

$$R(T_m^{(r)}, K_{n-(r-2)}^{(r)}; r) \geq (m-1)\left(\frac{n}{r-1} - 1\right) + 1.$$

If $T_m^{(r)}$ is n-good, then the Ramsey number in Inequality (2) also satisfies

$$R(T_m^{(r)}, K_n^{(r)}; r) \leq (m-1)\left(\frac{n}{r-1} - 1\right) + r - 1.$$

Lemma 3 then implies

$$
\begin{aligned}
R(T_m^{(r)}, K_{n-1}^{(r)}; r) &\leq (m-1)\left(\frac{n}{r-1} - 1\right) + r - 2 \\
R(T_m^{(r)}, K_{n-2}^{(r)}; r) &\leq (m-1)\left(\frac{n}{r-1} - 1\right) + r - 3 \\
&\vdots \\
R(T_m^{(r)}, K_{n-(r-2)}^{(r)}; r) &\leq (m-1)\left(\frac{n}{r-1} - 1\right) + 1,
\end{aligned}
$$

resulting in the statement of the theorem. □

The following corollary is an immediate consequence of Theorem 10. While it does not provide a complete solution to the conjecture in [6] that all r-uniform trees are n-good, it does substantially reduce the work necessary to prove the conjecture.

Corollary 4. *Let $T_m^{(r)}$ be an r-uniform tree of order m. If $T_m^{(r)}$ is n-good for all $(r-1)|n$, then $T_m^{(r)}$ is n-good for all $n \geq r$.*

We conclude by considering how the conjecture that all r-uniform trees are n-good may be resolved. One possible approach is to consider how Hook [14] went about proving the value of the star-critical Ramsey number for trees versus complete graphs (see also Proposition 2.4 of [15]). Specifically, she used an inductive argument to prove the uniqueness of critical colorings for the corresponding Ramsey numbers. If one could prove the uniqueness of the suspected critical colorings for $R(T_m^{(r)}, K_n^{(r)}; r)$ (or at least classify all such colorings) in the cases where $(r-1)|n$, then they may serve as a tool for a completing the proof of the conjecture in [6].

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