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PARTIAL DYCK PATHS WITH AIR POCKETS

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Abstract

Dyck paths with air pockets are obtained from ordinary Dyck paths by compressing maximal runs of down-steps into giant down-steps of arbitrary size. Using the kernel method, we consider partial Dyck paths with air pockets, both from left to right, and from right to left. In a last section, this concept is combined with the concept of skew Dyck paths.

1. Introduction

In a recent paper [1], Baril et al. introduced a new family of Dyck-like paths, called *Dyck paths with air pockets.* Many of the usual parameters that one could think of are investigated in this paper. The paths have the usual up-steps $(1, 1)$ and down-steps $(1, -k)$ for any $k = 1, 2, \dots,$ but no such down-steps may follow each other. Otherwise, they cannot go into negative territory, and must end at the x-axis, as usual. One could just think about ordinary Dyck paths, and each (maximal) run of down-steps is condensed into one (giant) downstep.

Figure 1 explains the actions readily: up-steps can follow up-steps, in black; a down-step of any step is dashed, but must be followed immediately by an up-step (curved). Paths that lead to state i on the top layer correspond to partial Dyck paths ending at level i with an up-step, and paths that lead to state i on the bottom layer correspond to partial Dyck paths ending at level i with a down-step.

We introduce generating functions $f_k(z)$ and $g_k(z)$ where the coefficient of z^n in any of these functions counts paths ending in the respective state according to the number of steps. The function $f_0(z)+g_0(z)$ counts the Dyck paths with air pockets, as the zero in the index just means that they returned to the x-axis.

In this short paper, we will enumerate *partial* Dyck paths with air pockets, namely we allow the paths to end at level k. In other words, we compute all $f_k(z)$ and $g_k(z)$.

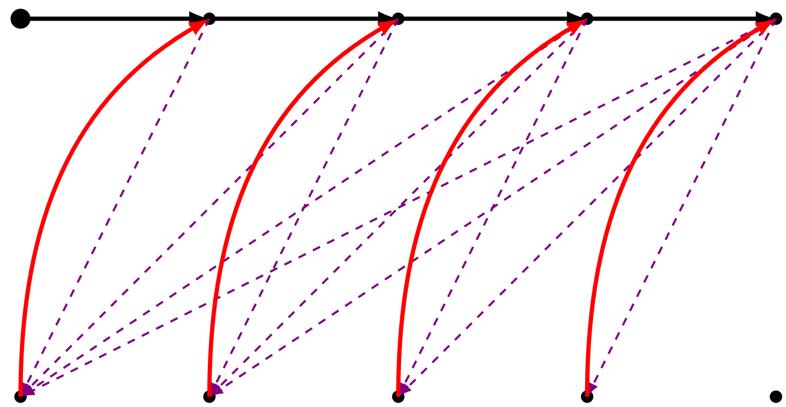

Figure 1: Graphical description of Dyck paths with air pockets. Top layer describes the situation after an up-step, bottom layer after a down-step.

Our instrument of choice is the kernel method, as can be found in the popular account [3].

2. Generating Functions

Just looking at Figure 1, we find the following recursion, where we write f_k for $f_k(z)$ for simplicity:

$$\begin{aligned} f_0 &= 1, \\ f_k &= zf_{k-1} + zg_{k-1}, \quad k \geq 1, \\ g_k &= zf_{k+1} + zf_{k+2} + zf_{k+3} + \cdots, \end{aligned}$$

and now we introduce bivariate generating functions

$$F(u,z) = F(u) = \sum_{k\geq 0} u^k f_k(z), \quad G(u,z) = G(u) = \sum_{k\geq 0} u^k g_k(z).$$

Summing the recursions,

$$F(u) = 1 + zuF(u) + zuG(u)$$

and

$$G(u) = \sum_{k\geq 0} u^k z \sum_{j>k} f_j = z\sum_{j>0} f_j \sum_{k=0}^{j-1} u^k = z\sum_{j>0} f_j \frac{1-u^k}{1-u} = \frac{z}{1-u}(F(1) - F(u)).$$

Eliminating one function, we are left to analyze

$$F(u) = 1 + zuF(u) + \frac{z^2u}{1-u}(F(1) - F(u)).$$

Solving, we find

$$F(u) = \frac{1 - u + z^2uF(1)}{-zu + zu^2 + z^2u + 1 - u} = \frac{1 - u + z^2uF(1)}{z(u - s_1)(u - s_2)},$$

with

$$s_1 = \frac{1 + z - z^2 + \sqrt{-z^2 - 2z^3 - 2z + z^4 + 1}}{2z},$$
$$s_2 = \frac{1 + z - z^2 - \sqrt{-z^2 - 2z^3 - 2z + z^4 + 1}}{2z}.$$

Note that $s_1s_2 = \frac{1}{z}$. We still need to compute $F(1)$. Before we can plug in $u = 1$ and compute it, we must cancel the bad factor of numerator and denominator. In this case, this is the factor $u - s_2$, since the reciprocal of it would not allow a Taylor expansion around $u = 1$. The result is

$$F(u) = \frac{-1 + z^2F(1)}{zs_2 - z + z^2 - 1 + zu},$$

from which we now can compute $F(1)$ by plugging in $u = 1$. We get

$$F(1) = \frac{-1 + z^2F(1)}{zs_2 + z^2 - 1} = \frac{1}{1 - zs_2}$$

and therefore

$$F(u) = \frac{1 - s_1}{1 - zs_2}\frac{1}{u - s_1} = -\frac{1}{s_1}\frac{1 - s_1}{1 - zs_2}\frac{1}{1 - u/s_1}.$$

Reading off the coefficient of u^k, we further get

$$f_k = -\frac{1}{s_1^{k+1}}\frac{1 - s_1}{1 - zs_2} = -z^{k+1}s_2^{k+1}\frac{1 - 1/(zs_2)}{1 - zs_2} = -z^k s_2^k\frac{zs_2 - 1}{1 - zs_2} = z^k s_2^k.$$

Since $G(u) = \dfrac{F(u) - 1 - zuF(u)}{zu}$, we also find

$$g_k = \frac{1}{z}f_{k+1} - f_k = z^k(s_2^{k+1} - s_2^k).$$

We can also compute $\textsc{total}(z) = F(1, z) + G(1, z)$ which counts paths that end anywhere, and the result is

$$\textsc{total}(z) = \frac{1 - z - z^2 - \sqrt{-z^2 - 2z^3 - 2z + z^4 + 1}}{2z^3} = \frac{1}{z^2}g_0.$$

In retrospect, this is not surprising, since if we consider paths that end at state 0 in the bottom layer, and we go back two steps, we could have been indeed in any state.

It is worthwhile to notice that

$$f_0 + g_0 = 1 + z^2 + z^3 + 2z^4 + 4z^5 + 8z^6 + 17z^7 + 37z^8 + 82z^9 + 185z^{10} + 423z^{11} + \cdots$$

and the coefficients $1, 1, 2, 4, 8, 17, \ldots$ are sequence A004148 in [5].

Theorem 1. *The generating functions describing partial Dyck paths with air pockets, landing in state k of the upper layer (f_k) resp. lower layer (g_k), are given by*

$$f_k = z^k s_2^k, \quad g_k = z^k (s_2^{k+1} - s_2^k).$$

In particular, $f_k + g_k = z^k s_2^{k+1}$ is the generating function of partial paths ending at level k.

3. Right to Left Model

Reading Dyck paths with air pockets from right to left means to consider paths that have arbitrarily long up-steps, but not two of them following each other immediately. While the enumeration for those paths that end at the x-axis is the same as before, this is not the case for *partial* paths.

Figure 3 explains the concept. The generating functions a_k refer to the top layer and b_k to the bottom layer.

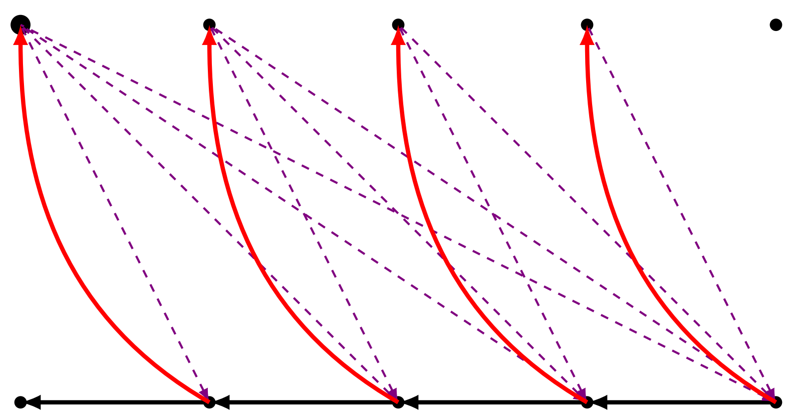

Figure 2: Graphical description of Dyck paths with air pockets. Top layer describes the situation after a down-step, bottom layer after an up-step.

The recursions are[1]

$$a_k = [k = 0] + zb_{k+1},$$
$$b_k = zb_{k+1} + z \sum_{0 \le j < k} a_j.$$

With bivariate generating functions analogous to those defined before, by summing we find the following:

$$A(u) = 1 + \frac{z}{u}(B(u) - b_0)$$

and

$$B(u) = \frac{z}{u}(B(u) - b_0) + z \sum_{0 \le j < k} a_j u^k = \frac{z}{u}(B(u) - b_0) + \frac{zu}{1-u} A(u).$$

One variable can be eliminated:

$$B(u) = \frac{z}{u}(B(u) - b_0) + \frac{zu}{1-u} + \frac{z^2}{1-u}(B(u) - b_0).$$

Solving

$$B(u) = \frac{z\left(B(0) - B(0)u - u^2 + zB(0)u\right)}{z - zu + z^2u - u + u^2}.$$

The denominator factors as $(u - s_1^{-1})(u - s_2^{-1})$. The bad factor this time is $(u - s_1^{-1})$. We obtain, after dividing it out,

$$B(u) = \frac{z\,(-us_1 - B(0)s_1 + B(0)s_1 z - 1)}{us_1 - zs_1 + z^2 s_1 - s_1 + 1}$$

and, further,

$$B(0) = b_0 = \frac{z}{s_1 - 1} = s_2 - 1.$$

Thus, after some simplifications,

$$B(u) = -z + \frac{zs_1}{(s_1 - 1)(1 - \frac{u}{zs_1})},$$

or

$$B(u) = -z + \frac{1}{s_2(s_1 - 1)(1 - s_2 u)} = -z + \frac{s_2 - 1}{zs_2(1 - s_2 u)}$$

and then

$$b_k = \frac{s_2 - 1}{z} s_2^{k-1}, \quad k \ge 1.$$

The functions a_k could be computed from here as well, but for the partial paths only the functions b_k are of relevance, if we don't consider the empty path.

[1] Iverson's notation is used here [2].

Theorem 2. *The generating functions of partial Dyck paths with air pockets in the right to left model are*

$$1 + b_0 = s_2$$

and

$$b_k = \frac{s_2 - 1}{z} s_2^{k-1}, \quad k \geq 1.$$

To consider the total does not make sense, since in just 1 or 2 steps, every state can be reached, so a sum over b_k would not converge.

4. Skew Dyck Paths with Air Pockets

The walks according to Figure 3 are related to skew Dyck paths [4]; the red down-steps are modeled to stand for south-west steps, and the way they are arranged, there are no overlaps of such a path. See [4] and the references cited there.

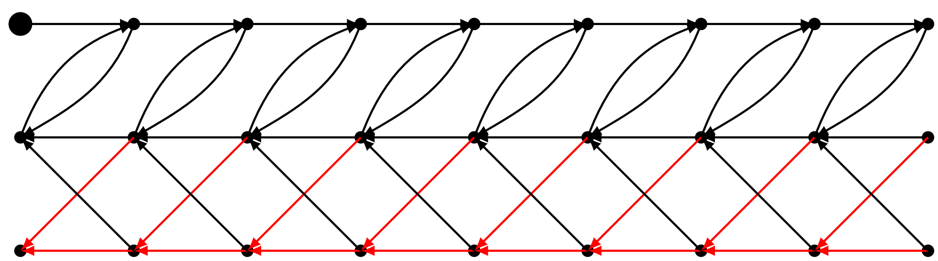

Figure 3: Three layers of states according to the type of steps leading to them (up, down-black, down-red).

Now we combine this model with air pockets. Each maximal sequence of black down-steps is condensed into one giant down-step, depicted in dashed grey in Figure 4.

Note that in Figure 4 we could alternatively and equivalently draw the backward arrows that appear in the third layer in the first layer. We decided against it since now the top layer is linked to partial paths ending with an up-step, and the other two ending with a down-step.

Introducing generating functions, according to the three layers, we find the fol-

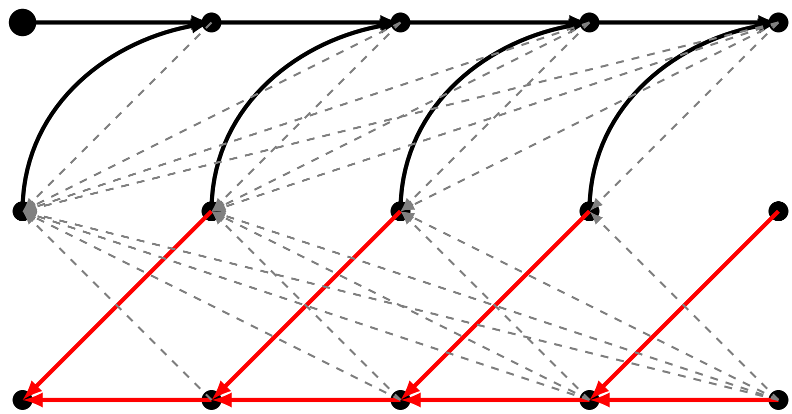

Figure 4: Three layers of states according to the type of steps leading to them (up, down-black, down-red). Black down-steps are condensed into giant grey down-steps.

lowing recursions by inspection,

$$\begin{aligned} &a_0 = 1, \quad a_{k+1} = za_k + zb_k,\ k \geq 0,\\ &b_k = z\sum_{j>k} a_j + z\sum_{j>k} c_j,\\ &c_k = zb_{k+1} + zc_{k+1}. \end{aligned}$$

Translating these into bivariate generating functions, we further have

$$A(u) = 1 + zuA(u) + zuB(u),$$
$$B(u) = \frac{z}{1-u}[A(1) - A(u)] + \frac{z}{1-u}[C(1) - C(u)],$$
$$C(u) = zuB(u) + zuC(u).$$

Solving,

$$\begin{aligned} A(u) &= \frac{z^3u^2A(1) + z^3u^2C(1) - zu^2 - z^2uC(1) - z^2u - z^2uA(1) + zu + u - 1}{(-1+zu)\,(zu^2 + 2z^2u - zu - u + 1)},\\ B(u) &= -\frac{(-A(1) - C(1) + zuA(1) + zuC(1) + 1)\,z}{zu^2 + 2z^2u - zu - u + 1},\\ C(u) &= \frac{z^2u\,(-A(1) - C(1) + zuA(1) + zuC(1) + 1)}{(-1+zu)\,(zu^2 + 2z^2u - zu - u + 1)}. \end{aligned}$$

We factor $zu^2 + 2z^2u - zu - u + 1 = (u - s_1)(u - s_2)$ with

$$s_2 = \frac{-2z^2 + z + 1 - \sqrt{4z^4 - 4z^3 - 3z^2 - 2z + 1}}{2z}, \quad s_1 = \frac{1}{zs_2}.$$

Since $A(u) - C(u) = \frac{1}{1-zu}$, we have $A(1) - C(1) = \frac{1}{1-z}$, and we only need to compute one of them. Dividing the (bad) factor $(u - s_2)$ out, plugging in $u = 1$ and solving leads to

$$A(1) = \frac{-s_2 z + 2 - z}{2\,(1 - s_2 z)\,(1 - z)} = \frac{1}{2(1 - z)} + \frac{1}{2(1 - z s_2)}$$

and

$$C(1) = -\frac{1}{2(1 - z)} + \frac{1}{2(1 - z s_2)}.$$

Using these values, we find

$$A(u) + B(u) + C(u) = \frac{s_2(1 - z^2 - z s_2)}{(1 - z s_2)(1 - u z s_2)}$$

and furthermore

$$[u^k](A(u) + B(u) + C(u)) = \frac{z^k s_2^{k+1}(1 - z^2 - z s_2)}{(1 - z s_2)}.$$

These functions describe all skew Dyck paths with air pockets, ending at level k. For $k = 0$, this yields

$$1 + z^2 + z^3 + 3z^4 + 7z^5 + 17z^6 + 45z^7 + 119z^8 + 323z^9 + 893z^{10} + 2497z^{11} + \cdots .$$

References

[1] J.-L. Baril, S. Kirgizov, R. Maréchal, and V. Vajnovszki, Enumeration of Dyck paths with air pockets, arXiv:2202.06893.

[2] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, MA, 1999.

[3] H. Prodinger, The kernel method: A collection of examples, *Séminaire Lothar. Comb.* **B50f** (2004), 19 pp.

[4] H. Prodinger, Partial skew Dyck paths: a kernel method approach, *Graphs and Combinatorics* **38** (2022), Article #135, 11 pp.

[5] N. J. A. Sloane et al., The On-line Encyclopedia of Integer Sequences, 2022.