Some summation and transformation formulas from inversion techniques

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Abstract
With the help of Gould–Hsu inversions, we find a generalization of Hagen and Rothe’s identity and give a new proof for some generalizations of Simons’ identity and Bruckman’s identity. Furthermore, we derive several Bruckman-type identities via the derivative operator.

1. Introduction
For any complex number $x$ and nonnegative integer $n$, define the shifted-factorial to be

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)},$$

where $\Gamma(x)$ is the well-known Gamma function. There are a lot of combinatorial identities in the literature. Thereinto, Hagen and Rothe’s identity (cf. [4]) reads

$$\sum_{k=0}^{n} \binom{a+\lambda k}{k} \frac{c-\lambda n}{c-\lambda k} \frac{c-\lambda k}{n-k} = \binom{a+c}{n},$$

(1.1)

where $c-\lambda k \neq 0$. Replacing $k$ by $n-k$ in Equation (1.1), we have

$$\sum_{k=0}^{n} \binom{a+\lambda n-\lambda k}{n-k} \frac{c-\lambda n}{c-\lambda n+\lambda k} \frac{c-\lambda n+\lambda k}{k} = \binom{a+c}{n}.$$

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Replacing $a$ and $c$ by $c-\lambda n$ and $a+\lambda n$, respectively, in the last equation, we obtain

$$
\sum_{k=0}^{n} \frac{a}{a+\lambda k} \binom{a+\lambda k}{k} \binom{c-\lambda k}{n-k} = \binom{a+c}{n},
$$

(1.2)

where $a+\lambda k \neq 0$. The linear combination of Equation (1.1) and Equation (1.2) produces

$$
\sum_{k=0}^{n} \frac{a}{a+\lambda k} \binom{a+\lambda k}{k} \frac{c-\lambda kn}{c-\lambda k} \frac{(c-\lambda k)(a+c)}{a+c} = \binom{a+c}{n},
$$

where $a+\lambda k \neq 0$ and $c-\lambda k \neq 0$.

Following Andrews, Askey, and Roy [1], define the hypergeometric series by

$$
1+\text{rF}_{s}\left[a_0, a_1, \ldots, a_r\mid b_1, \ldots, b_s; z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} z^k,
$$

where $\{a_i\}_{i \geq 0}$ and $\{b_j\}_{j \geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right-hand side. In this paper, we shall establish the following generalization of Equation (1.1).

**Theorem 1.** Let $a, b, c, d, \lambda$ be complex numbers subject to $\min \{\Re(1+a-b), \Re(\lambda - 1)\} > 0$. Then

$$
\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} (x+1)^k = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k.
$$

(1.3)

When $b = 0$, Theorem 1 reduces to Equation (1.1) exactly. When $n = 0$, Theorem 1 becomes Gauss’ summation formula:

$$
\text{2F}_1 \left[ a, b \mid c ; 1 \right] = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)},
$$

where $\Re(c-a-b) > 0$.

In 2001, Simons [16] proved a curious identity. Prodinger [15] displayed another attractive proof of this identity via Cauchy’s integral formula. We refer the reader to [17, 18] for two other proofs of Equation (1.3). Some related conclusions can be seen in the paper [12]. Hirschhorn [7] pointed out that Simons’ identity can be deduced by specifying the parameters in Pfaff’s transformation formula (cf. [1, P. 79]), which can be stated as the following theorem.
Theorem 2. Let $b, c, x$ be complex numbers. Then

$$2F_1\left[\begin{array}{c}
-n/b \\
c
\end{array} ; x \right] = \frac{(c-b)_n}{(c)_n} 2F_1\left[\begin{array}{c}
-n/b \\
1+b-c-n \end{array} ; 1-x \right].$$

By means of Cauchy’s integral formula, Munarini [13] discovered the nice generalization of Equation (1.3):

$$\sum_{k=0}^{n} \binom{\alpha}{n-k} \binom{\beta+k}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{\beta - \alpha + n}{n-k} \binom{\beta+k}{k} (-1)^{n-k} (x+y)^k y^{n-k}.$$ 

Here we point out that this result is an equivalent form of Theorem 2.

In a letter to Henry Gould on 15 April 2008, Paul Bruckman posed the following problem:

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{k}{(1-2k)(k+r)} = \frac{1+2n}{1+2r},$$

where $r$ is a positive integer with $1 \leq r \leq n$. Through Zeilberger’s Algorithm (cf. [14, Chapter 6]), Gould [5] established an interesting generalization of Bruckman’s identity (1.4), which can be expressed the following theorem.

Theorem 3. Let $x$ be a complex number. Then

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{k}{(1-2k)(k+x)} = \frac{1+2n}{1+2x} - \frac{1}{1+2x} \frac{1}{1+2x}.$$

(1.5)

It should be mentioned that Xu and Cen [18] also proved Theorem 3 in terms of the contour integral method.

For a differentiable function $f(x)$, define the derivative operator $D_x$ as

$$D_x f(x) = \frac{d}{dx} f(x).$$

For any complex number $x$ and positive integer $\ell$, define the generalized harmonic numbers of $\ell$-order to be

$$H_0^{(\ell)}(x) = 0 \quad \text{and} \quad H_n^{(\ell)}(x) = \sum_{k=1}^{n} \frac{1}{(x+k)^\ell} \quad \text{when} \quad n \in \mathbb{Z}^+.$$

Setting $\ell = 1$ in $H_0^{(\ell)}(x)$ and $H_n^{(\ell)}(x)$, we get generalized harmonic numbers

$$H_0(x) = 0 \quad \text{and} \quad H_n(x) = \sum_{k=1}^{n} \frac{1}{x+k}.$$
When \( x = 0 \), they reduce to classical harmonic numbers

\[
H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.
\]

Applying the derivative operator \( D_x \) to both sides of Equation (1.5), we arrive at the following result.

**Theorem 4.** Let \( x \) be a complex number. Then

\[
\sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{k}{(1-2k)(k+x)^2}
= \frac{2 + 4n}{(1+2x)^2} - \frac{1}{(1+2x)(1+x)} \left\{ \frac{2}{1+2x} + H_n(x) + H_n(-x) \right\}.
\]

Fixing \( x = r \) in Theorem 4 and utilizing the relation

\[
(1 - x)_n H_n(-x) = (1 - x)_{r-1}(r - x)(1 + r - x)_{n-r}
\]

\[
\times \left\{ H_{r-1}(-x) + \frac{1}{r-x} + H_{n-r}(r - x) \right\},
\]

we can derive the following formula.

**Corollary 5.** Let \( r \) be a positive integer with \( 1 \leq r \leq n \). Then

\[
\sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{k}{(1-2k)(k+r)^2}
= \frac{2 + 4n}{(1+2r)^2} + \frac{(-1)^r}{(1+2r)} \frac{(1)_r(1)_{n-r}}{(1+r)_n}.
\]

Applying the derivative operator \( D_x \) to both sides of Equation (1.6), we are led to the following conclusion.

**Theorem 6.** Let \( x \) be a complex number. Then

\[
\sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{k}{(1-2k)(k+x)^3}
= \frac{4 + 8n}{(1+2x)^3} - \frac{1}{(2+4x)(1+x)} \left\{ \frac{8}{(1+2x)^2} + \Omega_n(x) \right\},
\]

where

\[
\Omega_n(x) = \left[ \frac{4}{1+2x} + H_n(x) + H_n(-x) \right] [H_n(x) + H_n(-x)]
+ [H_n^{(2)}(x) - H_n^{(2)}(-x)].
\]
Choosing \( x = r \) in Theorem 6, we can deduce the following formula.

**Corollary 7.** Let \( r \) be a positive integer with \( 1 \leq r \leq n \). Then

\[
\sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{k}{(1-2k)(k+r)^3} = 4 + 8n \left( \frac{1}{1+2r} \right) + \left( \frac{-1}{1+2r} \right) \frac{(1-r)_{n-r}}{(1+r)n} \left\{ H_{n+r} + H_{n-r} - 2H_r + \frac{1+4r}{r(1+2r)} \right\}.
\]

For a complex variable \( x \) and two complex sequences \( \{a_k, b_k\}_{k \geq 0} \), define a polynomial sequence by

\[
\phi(x; 0) \equiv 1 \quad \text{and} \quad \phi(x; n) = \prod_{i=0}^{n-1} (a_i + xb_i) \quad \text{when} \quad n \in \mathbb{Z}^+.
\]

Then a pair of inverse series relations due to Gould and Hsu [6] can be written as

\begin{align*}
  f(n) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \phi(k; n) g(k), \\  g(n) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a_k + kb_k}{\phi(n; k+1)} f(k). \tag{1.8}
\end{align*}

Inversion techniques are very useful for dealing with combinatorial identities. Some nice results can be seen in the papers [3, 8–11].

Inspired by the works just mentioned, we shall prove Theorems 1-3 by using Gould–Hsu inversions (1.7) and (1.8). The corresponding details will be provided in Sections 2-4, respectively.

### 2. Proof of Theorem 1

In order to prove Theorem 1, we require the following transformation formula (cf. [1, P. 142]):

\[
3F2 \left[ \begin{array}{c} a, b, c \\ d, e \end{array} : 1 \right] = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)} 3F2 \left[ \begin{array}{c} a, d-b, d-c \\ d, d+e-b-c \end{array} : 1 \right]. \tag{2.1}
\]

**Proof of Theorem 1.** Performing the replacements \( a \rightarrow -n, b \rightarrow -a-c, c \rightarrow d-b, d \rightarrow 1-b-c+\lambda n-n, e \rightarrow d \) in Equation (2.1), we obtain...
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(-b - c + \lambda k)(d - b)_k}{(-b - c + \lambda n - n)_k} \times 3F_2 \left[ \begin{array}{c} -k, 1 + a - b + \lambda k - k, 1 - c - d + \lambda k - k \\ 1 - b - c + \lambda k - k, 1 + a + \lambda k - k \end{array} ; 1 \right] = \frac{(1 + a + \lambda n - n)_n}{(-b - c + \lambda n - n)(d)_n} 3F_2 \left[ \begin{array}{c} -n, 1 + a - b + \lambda n - n, 1 - c - d + \lambda n - n \\ 1 - b - c + \lambda n - n, 1 + a + \lambda n - n \end{array} ; 1 \right]. \]

It satisfies Equation (1.8) and Equation (1.7) brings out the dual relation:

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(-b - c + \lambda k)(d - b)_k}{(-b - c + \lambda n - n)_k} \times 3F_2 \left[ \begin{array}{c} -k, 1 + a - b + \lambda k - k, 1 - c - d + \lambda k - k \\ 1 - b - c + \lambda k - k, 1 + a + \lambda k - k \end{array} ; 1 \right] = \frac{(-a - c)_n (d - b)_n}{(-b - c + \lambda n)(d)_n}. \] (2.2)

The iteration of Equation (2.1) results in the transformation formula (cf. [1, P. 143]):

\[ 3F_2 \left[ \begin{array}{c} a, b, c \\ d, e \end{array} ; 1 \right] = \frac{\Gamma(d)\Gamma(e)\Gamma(d + e - a - b - c)}{\Gamma(a)\Gamma(d + e - a - b)\Gamma(d + e - a - c)} \times 3F_2 \left[ \begin{array}{c} d - a, e - a, d + e - a - b - c \\ d + e - a - b, d + e - a - c \end{array} ; 1 \right]. \] (2.3)

Employing the replacements \( a \rightarrow 1 + a - b + \lambda k - k \), \( b \rightarrow -k \), \( c \rightarrow 1 - c - d + \lambda k - k \), \( d \rightarrow 1 - b - c + \lambda k - k \), \( e \rightarrow 1 + a + \lambda k - k \) in Equation (2.3), we have

\[ 3F_2 \left[ \begin{array}{c} -k, 1 + a - b + \lambda k - k, 1 - c - d + \lambda k - k \\ 1 - b - c + \lambda k - k, 1 + a + \lambda k - k \end{array} ; 1 \right] = \frac{\Gamma(1 - b - c + \lambda k - k)\Gamma(1 + a + \lambda k - k)\Gamma(d + k)}{\Gamma(1 + a - b + \lambda k - k)\Gamma(1 + c + \lambda k)\Gamma(d)} \times 3F_2 \left[ \begin{array}{c} -a - c, b, d + k \\ 1 - c + \lambda k, d \end{array} ; 1 \right]. \] (2.4)

Substituting Equation (2.4) into Equation (2.2), we get Theorem 1 after some simplifications. \( \square \)

3. Proof of Theorem 2

For the sake of proving Theorem 2, we draw support from the binomial theorem and Chu–Vandermonde convolution (cf. [1, P. 67]):
\begin{equation}
\genfrac{}{}{0pt}{}{1}{F_0} \left[ \begin{array}{c} -n \\ x \end{array} \right] = (1 - x)^n,
\end{equation}
\begin{equation}
\genfrac{}{}{0pt}{}{2}{F_1} \left[ \begin{array}{c} -n, b \\ c \end{array} ; 1 \right] = \frac{(c - b)_n}{(c)_n}.
\end{equation}

Proof of Theorem 2. According to Equations (3.1) and (3.2), it is routine to verify that

$$\sum_{i=0}^{n} \frac{(-n)_i(b)_i}{i!(1 + c - b - n)_i} (-x)^i \sum_{k=0}^{n-i} \frac{(-n + i)_k(c + i)_k}{k!(1 + c - b - n + i)_k} = \frac{(b)_n}{(b - c)_n} (1 - x)^n.$$ 

Shifting the summation index \( k \to k - i \) and then interchanging the summation order in the last equation, we arrive at

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (b - c - k)_n \frac{(c)_k}{(1 + c - b)_k} \sum_{i=0}^{k} \frac{(-k)_i(b)_i}{i!(c)_i} x^i = (b)_n (1 - x)^n.$$ 

It suits to Equation (1.7), and Equation (1.8) gives the dual relation

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{b - c}{(b - c - n)_{k+1}} (b)_k (1 - x)^k = \frac{(c)_n}{(1 + c - b)_n} \sum_{i=0}^{n} \frac{(-n)_i(b)_i}{i!(c)_i} x^i.$$ 

This completes the proof of Theorem 2. \( \square \)

4. Proof of Theorem 3

For the purpose of proving Theorem 3, we need the following lemma.

Lemma 8. Let \( n \) be a nonnegative integer and let \( x \) be a complex number. Then

\begin{equation}
\sum_{k=0}^{n} \frac{(-n)_k}{(2 + n)_k} (1 + 2k)^2 = \frac{1 + n}{1 - 2n},
\end{equation}
\begin{equation}
\sum_{k=0}^{n} \frac{(-n)_k(1 - x)_k}{(2 + n)_k(1 + x)_k} (1 + 2k) = \frac{x(1 + n)}{x + n}.
\end{equation}
Proof. It is not difficult to show that
\[ \sum_{k=0}^{n} \frac{(-n)_k (b)_k}{k!(2 + n)_k} (1 + 2k)^2 \]
\[ = \sum_{k=0}^{n} \frac{(-n)_k (b)_k}{k!(2 + n)_k} + 4 \sum_{k=1}^{n} \frac{(-n)_k (b)_k}{k!(2 + n)_k} k + 4 \sum_{k=1}^{n} \frac{(-n)_k (b)_k k^2}{k!(2 + n)_k} \]
\[ = \sum_{k=0}^{n} \frac{(-n)_k (b)_k}{k!(2 + n)_k} - \frac{8bn}{2 + n} \sum_{k=0}^{n-1} \frac{(1 - n)_k (1 + b)_k}{k!(3 + n)_k} \]
\[ - \frac{4b(1 + b)n(1 - n)}{(2 + n)(3 + n)} \sum_{k=0}^{n-2} \frac{(2 - n)_k (2 + b)_k}{k!(4 + n)_k}. \]

Taking \( b = 1 \) in the last equation and then evaluating the three series on the right-hand side by Equation (3.2), we obtain Equation (4.1).

Setting \( a = 1, b = 1, c = 1 - x \) in Dougall’s summation formula for \( 5F_4 \) series (cf. [1, P. 71]):
\[ 5F_4 \left[ \frac{a, 1 + \frac{a}{2}, b, c, -n}{\frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a + n; 1} \right] = \frac{(1 + a)n(1 + a - b - c)_n}{(1 + a - b)_n(1 + a - c)_n}, \]
we get Equation (4.2). \( \square \)

Now we prove Theorem 3.

Proof of Theorem 3. In accordance with Lemma 8, it is easy to see that
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1 + 2k}{(1 + n)_k + 1} k! \left\{ \frac{1 + 2k}{1 + 2x} - \frac{1}{1 + 2x} (1 - x)_k \right\} = \frac{n}{(1 - 2n)(n + x)}. \]
The last equation fits into Equation (1.8), and Equation (1.7) produces the dual relation
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1 + k)^n \frac{k}{(1 - 2k)(k + x)} = n! \left\{ \frac{1 + 2n}{1 + 2x} - \frac{1}{1 + 2x} (1 - x)_n \right\}. \]
This completes the proof. \( \square \)

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