SOME SUMMATION AND TRANSFORMATION FORMULAS FROM INVERSION TECHNIQUES

Chuanan Wei
School of Biomedical Information and Engineering, Hainan Medical University, Haikou, China

Yi Wei[*]
College of Traditional Chinese Medicine, Hainan Medical University, Haikou, China
ythainmc@163.com

Chun Li
School of Mathematics and Statistics, Hainan Normal University, Haikou, China

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Abstract
With the help of Gould–Hsu inversions, we find a generalization of Hagen and Rothe's identity and give a new proof for some generalizations of Simons' identity and Bruckman's identity. Furthermore, we derive several Bruckman-type identities via the derivative operator.

1. Introduction

For any complex number x and nonnegative integer n, define the shifted-factorial to be

$$(x)_n = \Gamma(x+n)/\Gamma(x),$$

where $\Gamma(x)$ is the well-known Gamma function. There are a lot of combinatorial identities in the literature. Thereinto, Hagen and Rothe's identity (cf. [4]) reads

$$\sum_{k=0}^{n} \binom{a+\lambda k}{k} \frac{c-\lambda n}{c-\lambda k} \binom{c-\lambda k}{n-k} = \binom{a+c}{n}, \tag{1.1}$$

where $c - \lambda k \neq 0$. Replacing k by $n-k$ in Equation (1.1), we have

$$\sum_{k=0}^{n} \binom{a+\lambda n-\lambda k}{n-k} \frac{c-\lambda n}{c-\lambda n+\lambda k} \binom{c-\lambda n+\lambda k}{k} = \binom{a+c}{n}.$$

[*]The corresponding author.

Replacing a and c by $c-\lambda n$ and $a+\lambda n$, respectively, in the last equation, we obtain

$$\sum_{k=0}^{n}\frac{a}{a+\lambda k}\binom{a+\lambda k}{k}\binom{c-\lambda k}{n-k}=\binom{a+c}{n}, \tag{1.2}$$

where $a+\lambda k\neq 0$. The linear combination of Equation (1.1) and Equation (1.2) produces

$$\sum_{k=0}^{n}\frac{a}{a+\lambda k}\binom{a+\lambda k}{k}\frac{c-\lambda n}{c-\lambda k}\binom{c-\lambda k}{n-k}=\frac{a+c-\lambda n}{a+c}\binom{a+c}{n},$$

where $a+\lambda k\neq 0$ and $c-\lambda k\neq 0$.

Following Andrews, Askey, and Roy [1], define the hypergeometric series by

$${}_{1+r}F_s\left[\begin{matrix}a_0, & a_1, & \ldots, & a_r\\ & b_1, & \ldots, & b_s\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(a_0)_k(a_1)_k\cdots(a_r)_k}{(1)_k(b_1)_k\cdots(b_s)_k}z^k,$$

where $\{a_i\}_{i\geq 0}$ and $\{b_j\}_{j\geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right-hand side. In this paper, we shall establish the following generalization of Equation (1.1).

Theorem 1. *Let a,b,c,d,λ be complex numbers subject to* $\min\{\Re(1+a-b),\Re(\lambda-1)\}>0$. *Then*

$$\sum_{k=0}^{n}\frac{(-n)_k}{k!}\frac{\Gamma(1+a+\lambda k)\Gamma(n-b-c+\lambda k-k)}{\Gamma(1+a-b+\lambda k-k)\Gamma(1-c+\lambda k)}{}_3F_2\left[\begin{matrix}-a-c,b,d+k\\ 1-c+\lambda k,d\end{matrix};1\right]$$
$$=\frac{(-a-c)_n(d-b)_n}{(-b-c+\lambda n)(d)_n}.$$

When $b=0$, Theorem 1 reduces to Equation (1.1) exactly. When $n=0$, Theorem 1 becomes Gauss' summation formula:

$${}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

where $\Re(c-a-b)>0$.

In 2001, Simons [16] proved a curious identity:

$$\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\binom{n+k}{k}(x+1)^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k. \tag{1.3}$$

Chapman [2] gave an interesting combinatorial proof of it. Prodinger [15] displayed another attractive proof of this identity via Cauchy's integral formula. We refer the reader to [17, 18] for two other proofs of Equation (1.3). Some related conclusions can be seen in the paper [12]. Hirschhorn [7] pointed out that Simons' identity can be deduced by specifying the parameters in Pfaff's transformation formula (cf. [1, P. 79]), which can be stated as the following theorem.

Theorem 2. *Let b, c, x be complex numbers. Then*

$$ {}_2F_1\left[\begin{matrix} -n, b \\ c \end{matrix}; x\right] = \frac{(c-b)_n}{(c)_n}\, {}_2F_1\left[\begin{matrix} -n, b \\ 1+b-c-n \end{matrix}; 1-x\right]. $$

By means of Cauchy's integral formula, Munarini [13] discovered the nice generalization of Equation (1.3):

$$ \sum_{k=0}^{n} \binom{\alpha}{n-k}\binom{\beta+k}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k}(-1)^{n-k}(x+y)^k y^{n-k}. $$

Here we point out that this result is an equivalent form of Theorem 2.

In a letter to Henry Gould on 15 April 2008, Paul Bruckman posed the following problem:

$$ \sum_{k=1}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} \frac{k}{(1-2k)(k+r)} = \frac{1+2n}{1+2r}, \tag{1.4} $$

where r is a positive integer with $1 \leq r \leq n$. Through Zeilberger's Algorithm (cf. [14, Chapter 6]), Gould [5] established an interesting generalization of Bruckman's identity (1.4), which can be expressed the following theorem.

Theorem 3. *Let x be a complex number. Then*

$$ \sum_{k=1}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} \frac{k}{(1-2k)(k+x)} = \frac{1+2n}{1+2x} - \frac{1}{1+2x}\frac{(1-x)_n}{(1+x)_n}. \tag{1.5} $$

It should be mentioned that Xu and Cen [18] also proved Theorem 3 in terms of the contour integral method.

For a differentiable function $f(x)$, define the derivative operator $\mathcal{D}_x$ as

$$ \mathcal{D}_x f(x) = \frac{d}{dx} f(x). $$

For any complex number x and positive integer ℓ, define the generalized harmonic numbers of ℓ-order to be

$$ H_0^{(\ell)}(x) = 0 \quad \text{and} \quad H_n^{(\ell)}(x) = \sum_{k=1}^{n} \frac{1}{(x+k)^\ell} \quad \text{when} \quad n \in \mathbb{Z}^+. $$

Setting $\ell = 1$ in $H_0^{\langle \ell \rangle}(x)$ and $H_n^{\langle \ell \rangle}(x)$, we get generalized harmonic numbers

$$ H_0(x) = 0 \quad \text{and} \quad H_n(x) = \sum_{k=1}^{n} \frac{1}{x+k}. $$

When $x = 0$, they reduce to classical harmonic numbers

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Applying the derivative operator $\mathcal{D}_x$ to both sides of Equation (1.5), we arrive at the following result.

Theorem 4. *Let x be a complex number. Then*

$$\sum_{k=1}^{n}(-1)^k \binom{n}{k}\binom{n+k}{k}\frac{k}{(1-2k)(k+x)^2}$$
$$= \frac{2+4n}{(1+2x)^2} - \frac{1}{(1+2x)}\frac{(1-x)_n}{(1+x)_n}\left\{\frac{2}{1+2x} + H_n(x) + H_n(-x)\right\}. \qquad (1.6)$$

Fixing $x = r$ in Theorem 4 and utilizing the relation

$$(1-x)_n H_n(-x) = (1-x)_{r-1}(r-x)(1+r-x)_{n-r}$$
$$\times\left\{H_{r-1}(-x) + \frac{1}{r-x} + H_{n-r}(r-x)\right\},$$

we can derive the following formula.

Corollary 5. *Let r be a positive integer with $1 \le r \le n$. Then*

$$\sum_{k=1}^{n}(-1)^k \binom{n}{k}\binom{n+k}{k}\frac{k}{(1-2k)(k+r)^2} = \frac{2+4n}{(1+2r)^2} + \frac{(-1)^r}{(1+2r)}\frac{(1)_{r-1}(1)_{n-r}}{(1+r)_n}.$$

Applying the derivative operator $\mathcal{D}_x$ to both sides of Equation (1.6), we are led to the following conclusion.

Theorem 6. *Let x be a complex number. Then*

$$\sum_{k=1}^{n}(-1)^k \binom{n}{k}\binom{n+k}{k}\frac{k}{(1-2k)(k+x)^3}$$
$$= \frac{4+8n}{(1+2x)^3} - \frac{1}{(2+4x)}\frac{(1-x)_n}{(1+x)_n}\left\{\frac{8}{(1+2x)^2} + \Omega_n(x)\right\},$$

where

$$\Omega_n(x) = \left[\frac{4}{1+2x} + H_n(x) + H_n(-x)\right]\left[H_n(x) + H_n(-x)\right]$$
$$+ \left[H_n^{(2)}(x) - H_n^{(2)}(-x)\right].$$

Choosing $x = r$ in Theorem 6, we can deduce the following formula.

Corollary 7. *Let r be a positive integer with $1 \leq r \leq n$. Then*

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{k}{(1-2k)(k+r)^3}$$
$$=\frac{4+8n}{(1+2r)^3}+\frac{(-1)^r}{(1+2r)}\frac{(1)_{r-1}(1)_{n-r}}{(1+r)_n}\left\{H_{n+r}+H_{n-r}-2H_r+\frac{1+4r}{r(1+2r)}\right\}.$$

For a complex variable x and two complex sequences $\{a_k, b_k\}_{k\geq 0}$, define a polynomial sequence by

$$\phi(x;0)\equiv 1 \quad \text{and} \quad \phi(x;n)=\prod_{i=0}^{n-1}(a_i+xb_i) \quad \text{when} \quad n\in\mathbb{Z}^+.$$

Then a pair of inverse series relations due to Gould and Hsu [6] can be written as

$$f(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\phi(k;n)\,g(k), \tag{1.7}$$

$$g(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a_k+kb_k}{\phi(n;k+1)}\,f(k). \tag{1.8}$$

Inversion techniques are very useful for dealing with combinatorial identities. Some nice results can be seen in the papers [3, 8–11].

Inspired by the works just mentioned, we shall prove Theorems 1-3 by using Gould–Hsu inversions (1.7) and (1.8). The corresponding details will be provided in Sections 2-4, respectively.

2. Proof of Theorem 1

In order to prove Theorem 1, we require the following transformation formula (cf. [1, P. 142]):

$${}_3F_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};1\right]=\frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)}{}_3F_2\left[\begin{matrix}a,d-b,d-c\\ d,d+e-b-c\end{matrix};1\right]. \tag{2.1}$$

Proof of Theorem 1. Performing the replacements $a \to -n$, $b \to -a-c$, $c \to d-b$, $d \to 1-b-c+\lambda n-n$, $e \to d$ in Equation (2.1), we obtain

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{-b-c+\lambda k}{(-b-c+\lambda n-n)_{k+1}}\frac{(-a-c)_k(d-b)_k}{(-b-c+\lambda k)(d)_k}$$
$$=\frac{(1+a+\lambda n-n)_n}{(-b-c+\lambda n-n)(d)_n}{}_3F_2\left[\begin{matrix}-n,\,1+a-b+\lambda n-n,\,1-c-d+\lambda n-n\\1-b-c+\lambda n-n,\,1+a+\lambda n-n\end{matrix};1\right].$$

It satisfies Equation (1.8) and Equation (1.7) brings out the dual relation:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(-b-c+\lambda k-k)_n\frac{(1+a+\lambda k-k)_k}{(-b-c+\lambda k-k)(d)_k}$$
$$\times\,{}_3F_2\left[\begin{matrix}-k,\,1+a-b+\lambda k-k,\,1-c-d+\lambda k-k\\1-b-c+\lambda k-k,\,1+a+\lambda k-k\end{matrix};1\right]$$
$$=\frac{(-a-c)_n(d-b)_n}{(-b-c+\lambda n)(d)_n}. \tag{2.2}$$

The iteration of Equation (2.1) results in the transformation formula (cf. [1, P. 143]):

$${}_3F_2\left[\begin{matrix}a,b,c\\d,e\end{matrix};1\right]=\frac{\Gamma(d)\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(a)\Gamma(d+e-a-b)\Gamma(d+e-a-c)}$$
$$\times\,{}_3F_2\left[\begin{matrix}d-a,e-a,d+e-a-b-c\\d+e-a-b,d+e-a-c\end{matrix};1\right]. \tag{2.3}$$

Employing the replacements $a\to 1+a-b+\lambda k-k$, $b\to -k$, $c\to 1-c-d+\lambda k-k$, $d\to 1-b-c+\lambda k-k$, $e\to 1+a+\lambda k-k$ in Equation (2.3), we have

$${}_3F_2\left[\begin{matrix}-k,\,1+a-b+\lambda k-k,\,1-c-d+\lambda k-k\\1-b-c+\lambda k-k,\,1+a+\lambda k-k\end{matrix};1\right]$$
$$=\frac{\Gamma(1-b-c+\lambda k-k)\Gamma(1+a+\lambda k-k)\Gamma(d+k)}{\Gamma(1+a-b+\lambda k-k)\Gamma(1-c+\lambda k)\Gamma(d)}$$
$$\times\,{}_3F_2\left[\begin{matrix}-a-c,b,d+k\\1-c+\lambda k,d\end{matrix};1\right]. \tag{2.4}$$

Substituting Equation (2.4) into Equation (2.2), we get Theorem 1 after some simplifications. □

3. Proof of Theorem 2

For the sake of proving Theorem 2, we draw support from the binomial theorem and Chu–Vandermonde convolution (cf. [1, P. 67]):

$$ {}_1F_0\left[\begin{matrix}-n\\ -\end{matrix};x\right]=(1-x)^n, \tag{3.1}$$

$$ {}_2F_1\left[\begin{matrix}-n,b\\ c\end{matrix};1\right]=\frac{(c-b)_n}{(c)_n}. \tag{3.2}$$

Proof of Theorem 2. According to Equations (3.1) and (3.2), it is routine to verify that

$$\sum_{i=0}^{n}\frac{(-n)_i(b)_i}{i!(1+c-b-n)_i}(-x)^i\sum_{k=0}^{n-i}\frac{(-n+i)_k(c+i)_k}{k!(1+c-b-n+i)_k}=\frac{(b)_n}{(b-c)_n}(1-x)^n.$$

Shifting the summation index $k\to k-i$ and then interchanging the summation order in the last equation, we arrive at

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(b-c-k)_n\frac{(c)_k}{(1+c-b)_k}\sum_{i=0}^{k}\frac{(-k)_i(b)_i}{i!(c)_i}x^i=(b)_n(1-x)^n.$$

It suits to Equation (1.7), and Equation (1.8) gives the dual relation

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{b-c}{(b-c-n)_{k+1}}(b)_k(1-x)^k=\frac{(c)_n}{(1+c-b)_n}\sum_{i=0}^{n}\frac{(-n)_i(b)_i}{i!(c)_i}x^i.$$

This completes the proof of Theorem 2. □

4. Proof of Theorem 3

For the purpose of proving Theorem 3, we need the following lemma.

Lemma 8. *Let n be a nonnegative integer and let x be a complex number. Then*

$$\sum_{k=0}^{n}\frac{(-n)_k}{(2+n)_k}(1+2k)^2=\frac{1+n}{1-2n}, \tag{4.1}$$

$$\sum_{k=0}^{n}\frac{(-n)_k(1-x)_k}{(2+n)_k(1+x)_k}(1+2k)=\frac{x(1+n)}{x+n}. \tag{4.2}$$

Proof. It is not difficult to show that

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{(-n)_k(b)_k}{k!(2+n)_k}(1+2k)^2\\
&=\sum_{k=0}^{n}\frac{(-n)_k(b)_k}{k!(2+n)_k}+4\sum_{k=1}^{n}\frac{(-n)_k(b)_k}{k!(2+n)_k}k+4\sum_{k=1}^{n}\frac{(-n)_k(b)_k}{k!(2+n)_k}k^2\\
&=\sum_{k=0}^{n}\frac{(-n)_k(b)_k}{k!(2+n)_k}-\frac{8bn}{2+n}\sum_{k=0}^{n-1}\frac{(1-n)_k(1+b)_k}{k!(3+n)_k}\\
&\quad-\frac{4b(1+b)n(1-n)}{(2+n)(3+n)}\sum_{k=0}^{n-2}\frac{(2-n)_k(2+b)_k}{k!(4+n)_k}.
\end{aligned}$$

Taking $b=1$ in the last equation and then evaluating the three series on the right-hand side by Equation (3.2), we obtain Equation (4.1).

Setting $a=1, b=1, c=1-x$ in Dougall's summation formula for ${}_5F_4$ series (cf. [1, P. 71]):

$${}_5F_4\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, -n \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a+n \end{matrix};1\right]=\frac{(1+a)_n(1+a-b-c)_n}{(1+a-b)_n(1+a-c)_n},$$

we get Equation (4.2). □

Now we prove Theorem 3.

Proof of Theorem 3. In accordance with Lemma 8, it is easy to see that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1+2k}{(1+n)_{k+1}}k!\left\{\frac{1+2k}{1+2x}-\frac{1}{1+2x}\frac{(1-x)_k}{(1+x)_k}\right\}=\frac{n}{(1-2n)(n+x)}.$$

The last equation fits into Equation(1.8), and Equation (1.7) produces the dual relation

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+k)_n\frac{k}{(1-2k)(k+x)}=n!\left\{\frac{1+2n}{1+2x}-\frac{1}{1+2x}\frac{(1-x)_n}{(1+x)_n}\right\}.$$

This completes the proof. □

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