GENERALIZED SUM-FREE SETS AND CYCLE SATURATED REGULAR GRAPHS

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Received: 9/7/21, Revised: 3/28/22, Accepted: 10/11/22, Published: 11/4/22

Abstract

Gerbner, Patkós, Tuza, and Vizer recently initiated the study of F-saturated regular graphs. One of the essential problems in this line of research is determining when these graphs exist. Using generalized sum-free sets we prove that for any odd integer $k \geq 5$, there is an n-vertex regular C_k-saturated graph for all sufficiently large n. Our proof is based on constructing a special type of sum-free set in $\mathbb{Z}_n$. We prove that for all even $\ell \geq 4$ and integers $n \geq 12\ell^2 + 30\ell + 19$, there is a symmetric complete $(\ell, 1)$-sum-free set in $\mathbb{Z}_n$. We pose the problem of finding the minimum size of such a set, and present some examples found by a computer search.

1. Introduction

A graph G is *F-free* if G does not contain a subgraph that is isomorphic to F. The graph F is often called the *forbidden subgraph*. An important class of F-free graphs are those that are maximal with respect to adding edges. We say that a graph G is *F-saturated* if G is F-free and adding any missing edge to G creates a subgraph that is isomorphic to F. One of the most studied problems in graph saturation is determining the minimum number of edges in an n-vertex F-saturated graph. This minimum is called the *saturation number of F*, denoted $\mathrm{sat}(n, F)$. Kászonyi and Tuza [15] proved that $\mathrm{sat}(n, F) = O(n)$ for any graph F having at least one edge. A famous result of Erdős, Hajnal, Moon [7] and Zykov [20] gives an exact formula for $\mathrm{sat}(n, K_r)$ for $2 \leq r \leq n$, and shows that the join of a clique on $r-2$ vertices with an independent set on $n-r+2$ vertices is the unique extremal graph.

[1]Research is supported in part by Simons Foundation Grant #359419

There has been much research on determining saturation numbers of other graphs. The surveys of Faudree, Faudree, Schmitt [8] and Pikhurko [16] contain a wealth of information on saturation in graphs and hypergraphs.

A common theme in this area is to study F-saturated graphs where some type of degree restriction has been added to the F-saturated graphs under consideration; see [1, 6, 9, 13] to name a few. A recent variant of $\mathrm{sat}(n, F)$ was introduced by Gerbner, Patkós, Tuza, and Vizer [10]. Given a graph F and a positive integer n, one can ask if there exists an n-vertex F-saturated regular graph. When such a graph exists, define

$$\mathrm{rsat}(n, F)$$

to be the minimum number of edges in such a graph. This can be viewed as a regular version of saturation numbers much like the recently studied regular Turán numbers [4, 5, 11, 17].

Gerbner et al. [10] proved that $\mathrm{rsat}(n, K_3)$ exists for all sufficiently large n. This can also be proved using results of Haviv and Levy [14] on symmetric complete sum-free sets in $\mathbb{Z}_n$. Among several other interesting theorems, Gerbner et al. obtained some partial results for K_4. The second author [19] proved that $\mathrm{rsat}(n, K_4)$ and $\mathrm{rsat}(n, K_5)$ exist for all sufficiently large n. However, the general problem for larger cliques remains open. In [19] it was proved that for all $r \geq 6$, $\mathrm{sat}(n, K_r)$ exists for infinitely many n.

Motivated by the approach using symmetric complete sum-free sets, the second author obtained the following theorem on odd cycles.

Theorem 1 ([19]). *For all positive integers α and k, there is a regular $C_{2\alpha+3}$-saturated graph with $2\alpha(\alpha+4)k + 2\alpha + 5$ vertices.*

This theorem implies that for any odd integer $2\alpha + 3 \geq 5$, there are infinitely many n for which an n-vertex $C_{2\alpha+3}$-saturated regular graph exists. However, for fixed α, Theorem 1 requires $n \equiv 2\alpha + 5 \pmod{2\alpha(\alpha+4)}$ which is only one of the possible $2\alpha(\alpha+4)$ residue classes. Our first theorem removes this restriction.

Theorem 2. *Let $\ell \geq 4$ be an even integer. For all $n \geq 12\ell^2 + 30\ell + 19$, there is an n-vertex $C_{\ell+1}$-saturated regular graph.*

Our proof of Theorem 2 is an explicit construction and when combined with Proposition 1, it gives the upper bound

$$\mathrm{rsat}(n, C_{\ell+1}) \leq \frac{n^2}{2(\ell+1)} + \left(\frac{2\ell+1}{2\ell+2}\right) n$$

for all even $\ell \geq 4$ and odd $n \geq 12\ell^2 + 30\ell + 19$. For even $n \geq \ell + 2$, a balanced complete bipartite graph is regular and $C_{\ell+1}$-saturated. Thus, we have a quadratic upper bound on $\mathrm{rsat}(n, C_{\ell+1})$ for all $n \geq 12\ell^2 + 30\ell + 19$ where $\ell \geq 4$ is even. The lower bound $\text{rsat}(n, C_{\ell+1}) = \Omega(n^{1+1/\ell})$, when this regular saturation number exists, is proved in [10]. As discussed in [19], the construction of Haviv and Levy implies $\text{rsat}(n, C_3) = O(n^{3/2})$, and so $\text{rsat}(n, C_3) = \Theta(n^{3/2})$. Determining if $\text{rsat}(n, C_{\ell+1})$ is subquadratic ($\ell \geq 4$ even) is an interesting problem. An answer to the following question is a possible first step towards a solution.

Question 1. Is it true that $\text{rsat}(n, C_5) = o(n^2)$?

Our approach follows that of [19] where the idea is to use Cayley graphs of $(\ell, 1)$-sum-free sets. These sets have been studied in additive combinatorics [2, 3, 12], and are a generalization of classical sum-free sets (see the survey of Tao [18]). For a positive integer ℓ, let $S_1, \dots, S_\ell$ be subsets of an abelian group Γ. Define

$$S_1 + S_2 + \cdots + S_\ell = \{s_1 + s_2 + \cdots + s_\ell : s_j \in S_j \text{ for } 1 \leq j \leq \ell\}.$$

In the special case that $S_1, \dots, S_\ell$ are all the same set S, we write

$$\ell S = \{s_1 + s_2 + \cdots + s_\ell : s_i \in S\}.$$

The set S is (k, ℓ)-*sum-free* if $(kS) \cap (\ell S) = \emptyset$. A (k, ℓ)-sum-free set is *complete* if kS and ℓS form a partition of Γ. Finally, S is *symmetric* if $s \in S$ implies $-s \in S$. The connection between $(\ell, 1)$-sum-free sets and regular $C_{\ell+1}$-saturated graphs is discussed in detail in [19]. Roughly speaking, one can use a symmetric complete $(\ell, 1)$-sum-free set to construct a Cayley graph that will be $C_{\ell+1}$-saturated. This will be made more precise later, but for now, we state our main result on $(\ell, 1)$-sum-free sets. It is the key ingredient in the proof of Theorem 2.

Theorem 3. *Let $\ell \geq 4$ be even, $t \geq 1$ be an integer, and γ be the unique integer for which*

$$1 \leq \ell + 3 - 4(t + 2) + \gamma(2\ell + 2) \leq 2\ell + 2.$$

For any integer $k \geq 4t + 2\ell + 2 + |\gamma|$, there is a symmetric complete $(\ell, 1)$-sum-free set $S \subseteq \mathbb{Z}_{(2\ell+2)k+r}$ where $r = \ell + 3 - 4(t + 2) + \gamma(2\ell + 2)$.

The following corollary is simpler to state and is a byproduct of our proof of Theorem 2 using Theorem 3.

Corollary 1. *Let $\ell \geq 4$ be even. If $n \geq 12\ell^2 + 30\ell + 19$, then there is a symmetric complete $(\ell, 1)$-sum-free set $S \subseteq \mathbb{Z}_n$.*

Finding the smallest size of a symmetric complete $(\ell, 1)$-sum-free set in $\mathbb{Z}_n$ appears to be a challenging problem. Write $\psi_\ell(n)$ for this minimum. Note that ℓ must be even for this function to be well-defined. Indeed, if S is a non-empty symmetric set in $\mathbb{Z}_n$ and $s \in S$, then

$$\underbrace{s - s + s - s + \cdots + s - s}_{2\ell' \text{ terms in all}} + s \equiv s (\text{mod } n).$$

This shows that we cannot have $((2\ell'+1)S)\cap S=\emptyset$ and therefore, ℓ must be even. Furthermore, even when restricting ℓ to be even there may be integers $n\geq 1$ for which $\mathbb{Z}_n$ does not contain a symmetric complete $(\ell,1)$-sum-free set. For these n, $\psi_\ell(n)$ is undefined.

The case $\ell=2$ corresponds to classical sum-free sets and $\psi_2(n)=\Theta(n^{1/2})$ where the upper bound is due to Haviv and Levy [14]. If $S\subset\mathbb{Z}_n$ is a complete $(\ell,1)$-sum-free set, then $\mathbb{Z}_n$ is the disjoint union of ℓS and S so

$$n\leq\binom{|S|+\ell-1}{\ell}+|S|.$$

This inequality gives the lower bound $\psi_\ell(n)=\Omega(n^{1/\ell})$. Corollary 1 shows that $\psi_\ell(n)$ exists for large enough n which allows us to conclude that $\psi_\ell(n)=O(n)$ (this upper bound is trivial once it has been established that $\psi_\ell(n)$ exists for all sufficiently large n).

The value of $\psi_4(n)$ was computed for small values of n. Our results from 41 to 80 are summarized in the table below. For $81\leq n\leq 140$, our program found that $\psi_4(n)=8$ with the exception of $n\in\{113,116,117,125\}$, where $\psi_4(n)=10$ for these n.

n	$\psi_4(n)$	Example	n	$\psi_4(n)$	Example
41	6	1,5,11,30,36,40	61	8	1,3,5,22,39,56,58,60
42	6	1,5,18,24,37,41	62	7	1,5,18,31,44,57,61
43	6	1,6,8,35,37,42	63	6	1,24,28,35,39,62
44	6	1,7,18,26,37,43	64	8	1,5,9,30,34,55,59,63
45	6	1,6,8,37,39,44	65	10	1,3,5,22,24,41,43,60,62,64
46	8	1,3,5,22,24,41,43,45	66	8	1,3,9,32,34,57,63,65
47	6	1,3,13,34,44,46	67	8	1,3,24,28,39,43,64,66
48	6	1,10,21,27,38,47	68	7	1,3,13,34,55,65,67
49	6	1,3,19,30,46,48	69	8	1,3,5,19,50,64,66,68
50	7	1,3,14,25,36,47,49	70	8	1,3,26,30,40,44,67,69
51	6	1,12,23,28,39,50	71	8	1,3,7,26,45,64,68,70
52	6	2,10,13,39,42,50	72	8	1,6,8,35,37,64,66,71
53	10	1,3,5,7,11,42,46,48,50,52	73	8	1,3,15,17,56,58,70,72
54	6	1,10,24,30,44,53	74	8	1,7,13,30,44,61,67,73
55	6	1,5,21,34,50,54	75	8	1,3,5,29,46,70,72,74
56	8	1,3,7,26,30,49,53,55	76	8	1,3,14,18,58,62,73,75
57	8	1,5,18,22,35,39,52,56	77	8	1,3,13,23,54,64,74,76
58	8	1,3,7,26,32,51,55,57	78	6	1,12,17,61,66,77
59	8	1,5,11,17,42,48,54,58	79	8	1,3,13,29,50,66,76,78
60	7	1,3,19,30,41,57,59	80	8	1,3,13,34,46,67,77,79

Small Values of $\psi_4(n)$

While the initial motivation was to obtain a result in graph saturation, perhaps the most interesting open problem related to this work is finding the order of magnitude of $\psi_\ell(n)$ for $\ell \geq 4$. For instance, analogous to Question 1, one can ask if $\psi_\ell(n) = o(n)$ for even $\ell \geq 4$.

1.1. Organization and Notation

In Section 2 we prove Theorem 3. The starting point will be a key lemma which is stated and proved in Section 2.1. Using this lemma, we prove Theorem 3 in Section 2.2. In Section 3 we prove Theorem 2 and Corollary 1.

Given even integers m_1 and m_2 with $m_1 < m_2$, let

$$[m_1, m_2]_e := \{m_1, m_1 + 2, m_1 + 4, \ldots, m_2\}.$$

Similarly, if $m_1 < m_2$ and both are odd,

$$[m_1, m_2]_o := \{m_1, m_1 + 2, m_1 + 4, \ldots, m_2\}.$$

If Γ is a group and $S \subseteq \Gamma$ is an inverse closed subset of Γ, then $\mathrm{Cay}(\Gamma, S)$ denotes the corresponding Cayley graph. This is the graph with vertex set Γ. Two distinct vertices x and y are adjacent if $xy^{-1} \in S$. When Γ is written additively, this last condition is $x - y \in S$. In this paper Γ will always be the cyclic group $\mathbb{Z}_n$.

2. Key Lemma and Theorem 3

2.1. Statement and Proof of Key Lemma

In this section we prove a lemma that gives a formula for a particular ℓ-fold sumset. The initial set will be the union of three intervals of consecutive odd numbers, the last interval being a singleton. We will show that the corresponding ℓ-fold sumset is an interval of consecutive even numbers, but with one interval of evens and one singleton removed (see Equation (1) below).

Let $\ell \geq 4$ be even, and $\alpha \geq 1$ and $t \geq 1$ be integers. Define

$$I_1 = [1, 2\alpha + 1]_o, \quad I_2 = [2\alpha + 5, 2\alpha + 5 + 2t]_o, \quad I_3 = \{2\alpha + 4t + 9\},$$

and $S^+ = I_1 \cup I_2 \cup I_3$.

Lemma 1 (Key Lemma)**.** *Suppose that $\alpha \geq 2t + 2\ell - 2$ and let $M = \ell(2\alpha + 4t + 9)$. For any integer $n > M$,*

$$\ell S^+ = [\ell, M]_e \backslash ([M - 2t - 2, M - 2]_e \cup \{M - 4t - 6\}) \tag{1}$$

in the abelian group $\mathbb{Z}_n$.

Proof. Given integers $0 \le \beta_1, \beta_2, \beta_3 \le \ell$ with $\beta_1 + \beta_2 + \beta_3 = \ell$, let

$$J_{\beta_1,\beta_2,\beta_3} = \beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3.$$

By definition,

$$\begin{aligned} J_{\beta_1,\beta_2,\beta_3} &= \{\beta_1 + 2(\gamma_1 + \cdots + \gamma_{\beta_1}) + \beta_2(2\alpha+5) + 2(\theta_1 + \cdots + \theta_{\beta_2}) \\ &\qquad + \beta_3(2\alpha + 4t + 9) \ : \ 0 \le \gamma_i \le \alpha, 0 \le \theta_i \le t\} \\ &= \{\beta_1 + \beta_2(2\alpha+5) + \beta_3(2\alpha+4t+9) + 2\tau : 0 \le \tau \le \beta_1\alpha + \beta_2 t\}. \end{aligned}$$

This expression simplifies to

$$J_{\beta_1,\beta_2,\beta_3} = [M - \beta_1(2\alpha+4t+8) - \beta_2(4t+4), M - \beta_1(4t+8) - \beta_2(2t+4)]_e. \quad (2)$$

We will use formula (2) often in the remainder of the proof.

Observe that

$$\ell S^+ = \bigcup_{\substack{0 \le \beta_1,\beta_2,\beta_3 \le \ell \\ \beta_1+\beta_2+\beta_3 = \ell}} J_{\beta_1,\beta_2,\beta_3}.$$

For $\beta_1 \in \{0, 1, \ldots, \ell\}$, let $K_{\beta_1} = \bigcup_{\beta_2=0}^{\ell-\beta_1} J_{\beta_1,\beta_2,\ell-\beta_1-\beta_2}$. First we examine K_0. By (2),

$$J_{0,0,\ell} = \{M\},\ J_{0,1,\ell-1} = [M-4t-4, M-2t-4]_e,$$
$$J_{0,2,\ell-2} = [M-8t-8, M-4t-8]_e.$$

Thus,

$$J_{0,0,\ell} \cup J_{0,1,\ell-1} \cup J_{0,2,\ell-2} = [M-8t-8, M]_e \backslash ([M-2t-2, M-2]_e \cup \{M-4t-6\}).$$

Now let $2 \le \beta_2 \le \ell - 1$ and consider the two sets

$$J_{0,\beta_2,\ell-\beta_2} = [M - \beta_2(4t+4), M - \beta_2(2t+4)]_e$$

and

$$J_{0,\beta_2+1,\ell-\beta_2-1} = [M - \beta_2(4t+4) - 4t - 4, M - \beta_2(2t+4) - 2t - 4]_e.$$

The inequality $\max J_{0,\beta_2+1,\ell-\beta_2-1} \ge \min J_{0,\beta_2,\ell-\beta_2} - 2$ is equivalent to $2t\beta_2 \ge 2t+2$. This inequality is true since $t \ge 1$ and $\beta_2 \ge 2$. Therefore,

$$\begin{aligned} J_{0,\beta_2,\ell-\beta_2} \cup J_{0,\beta_2+1,\ell-\beta_2-1} &= [\min J_{0,\beta_2+1,\ell-\beta_2-1}, \max J_{0,\beta_2,\ell-\beta_2}] \\ &= [M - \beta_2(4t+4) - 4t - 4, M - \beta_2(2t+4)]. \end{aligned}$$

Combining this with our expression for $J_{0,0,\ell} \cup J_{0,1,\ell-1} \cup J_{0,2,\ell-2}$ and the fact that $\min J_{0,\ell,0} = M - \ell(4t+4)$ gives

$$K_0 = [M - \ell(4t+4), M]_e \backslash ([M-2t-2, M-2]_e \cup \{M-4t-6\}).$$

Let $\beta_1 \in \{1, 2, \dots, \ell - 1\}$ and now we determine K_{β_1}. Let $0 \le \beta_2 \le \ell - \beta_1 - 1$ and consider the sets $J_{\beta_1,\beta_2,\ell-\beta_1-\beta_2}$ and $J_{\beta_1,\beta_2+1,\ell-\beta_1-\beta_2-1}$. The inequality

$$\max J_{\beta_1,\beta_2+1,\ell-\beta_1-\beta_2-1} \ge \min J_{\beta_1,\beta_2,\ell-\beta_1-\beta_2} - 2$$

is equivalent to

$$\alpha\beta_1 + t\beta_2 \ge t + 1. \tag{3}$$

Since $\beta_1 \ge 1$ and $\beta_2 \ge 0$, we have $\alpha\beta_1 + t\beta_2 \ge \alpha$, and $\alpha \ge t + 1$ by our assumption. Hence, (3) holds and we may conclue that

$$\begin{aligned} K_{\beta_1} &= [\min J_{\beta_1,\ell-\beta_1,0}, \max J_{\beta_1,0,\ell-\beta_1}]_e \\ &= [M - \beta_1(2\alpha + 4t + 8) - (\ell - \beta_1)(4t + 4), M - \beta_1(4t + 8)]_e \\ &= [M - \beta_1(2\alpha + 4) - \ell(4t + 4), M - \beta_1(4t + 8)]_e. \end{aligned}$$

Lastly, $K_\ell = J_{\ell,0,0} = [M - \ell(2\alpha + 4t + 8), M - \ell(4t + 8)]_e = [\ell, M - \ell(4t + 8)]_e$.

Having formulas for $K_0, K_1, \dots, K_\ell$, we now examine how consecutive K_β's intersect. Let $0 \le \beta_1 \le \ell - 1$ and consider K_{β_1} and K_{β_1+1}. The inequality

$$\max K_{\beta_1+1} \ge \min K_{\beta_1} - 2 \tag{4}$$

is equivalent to

$$M - (\beta_1 + 1)(4t + 8) \ge M - \beta_1(2\alpha + 4) - \ell(4t + 4) - 2$$

which simplifies to

$$\beta_1(\alpha - 2t - 2) + 2\ell(t + 1) \ge 2t + 3.$$

Since $\alpha - 2t - 2 \ge 0$ and $2\ell(t + 1) \ge 2t + 3$, this inequality holds and so (4) is true.

From (4) and our formulas for $K_0, K_1, \dots, K_\ell$ we get

$$\begin{aligned} \ell S^+ &= \bigcup_{\beta_1=0}^{\ell} K_{\beta_1} = [\ell, M - 4t - 8]_e \cup [M - 4t - 4, M - 2t - 4]_e \cup \{M\} \\ &= [\ell, M]_e \setminus ([M - 2t - 2, M - 2]_e \cup \{M - 4t - 6\}). \end{aligned}$$

This completes the proof of Lemma 1. □

2.2. Proof of Theorem 3

In this section we use Lemma 1 to prove Theorem 3. We start by defining several parameters that depend on the even integer $\ell \ge 4$ and the integer $t \ge 1$.

Let γ be the unique integer for which

$$1 \le \ell + 3 - 4(t + 2) + \gamma(2\ell + 2) \le 2\ell + 2.$$

Let $r = \ell + 3 - 4(t + 2) + \gamma(2\ell + 2)$. From the definition of γ and the fact that ℓ is even, we get

$$r \in \{1, 3, 5, \dots, 2\ell + 1\}.$$

Proof of Theorem 3. Let k be any integer with $k \geq 4t + 2\ell + 2 + |\gamma|$. Define

$$\alpha = k + \gamma - 2t - 4, \quad M = \ell(2k + 2\gamma + 1), \quad \text{and} \quad n = (2\ell + 2)k + r.$$

Since $k \geq 4t + 2\ell + 2 - \gamma$, we have $\alpha \geq 2t + 2\ell - 2$ and we apply Lemma 1 with these parameters. Note that

$$M = \ell(2k + 2\gamma + 1) = \ell(2\alpha + 4t + 9)$$

from the definition of α.

By Lemma 1, if $I_1 = [1, 2\alpha + 1]_o$, $I_2 = [2\alpha + 5, 2\alpha + 5 + 2t]_o$, $I_3 = \{2\alpha + 4t + 9\}$, and $S^+ = I_1 \cup I_2 \cup I_3$, then

$$lS^+ = [\ell, M]_e \backslash ([M - 2t - 2, M - 2]_e \cup \{M - 4t - 6\}). \tag{5}$$

This holds in $\mathbb{Z}_n$ because $n > M$ is true if and only if $2k > 2\ell\gamma + \ell - r$. Substituting for r and rearranging gives the equivalent inequality $2k > 4t + 5 - 2\gamma$ which is true because our assumed lower bound on k implies $k > 2t + \frac{5}{2} - \gamma$.

Claim 1. With α, n and M defined as above,

$$n - 2\alpha = M + 3.$$

Proof of Claim 1. We have

$$\begin{aligned} n - 2\alpha &= (2\ell + 2)k + r - 2(k + \gamma - 2t - 4) \\ &= [2\ell k + 2k] + [\ell + 3 - 4t - 8 + 2\ell\gamma + 2\gamma] + [-2k - 2\gamma + 4t + 8] \\ &= 2\ell k + 2\ell\gamma + \ell + 3 = M + 3. \end{aligned}$$

□

Define

$$S^- = \{n - s : s \in S^+\} \quad \text{and} \quad S = S^+ \cup S^-.$$

The set S is a symmetric subset of $\mathbb{Z}_n$ and we will now prove that

$$\ell S = \mathbb{Z}_n \backslash S.$$

First we find the intervals that make up S^-. The set S^- is the disjoint union of $n - I_1$, $n - I_2$, and $n - I_3$ where $n - I_i = \{n - s : s \in I_i\}$. By Claim 1,

$$n - I_3 = \{n - (2\alpha + 4t + 9)\} = \{M - 4t - 6\}.$$

Similarly,

$$n - I_2 = n - [2\alpha + 5, 2\alpha + 5 + 2t]_o = [M - 2t - 2, M - 2]_e$$

and

$$n - I_1 = n - [1, 2\alpha + 1]_o = [M + 2, n - 1]_e.$$

Therefore,

$$\begin{aligned} S^- &= (n - I_3) \cup (n - I_2) \cup (n - I_1) \\ &= \{M - 4t - 6\} \cup [M - 2t - 2, M - 2]_e \cup [M + 2, n - 1]_e. \end{aligned}$$

Returning to (5) we now see that

$$\ell S^+ = [\ell, M]_e \backslash ((n - I_2) \cup (n - I_3)). \tag{6}$$

Let $\mathcal{E} = \{0, 2, 4, \ldots, n-1\} \subset \mathbb{Z}_n$. Using (6) and the equation $n - I_1 = [M+2, n-1]_e$, we can write $\mathcal{E}$ as the disjoint union

$$\mathcal{E} = [0, \ell - 2]_e \cup \ell S^+ \cup (n - I_1) \cup (n - I_2) \cup (n - I_3) = [0, \ell - 2]_e \cup \ell S^+ \cup S^-.$$

By symmetry, if $\mathcal{O} = \{1, 3, \ldots, n - 2\} \subset \mathbb{Z}_n$, then

$$\mathcal{O} = [n - \ell + 2, n - 2]_o \cup \ell S^- \cup S^+.$$

Putting these two together gives

$$\mathbb{Z}_n = [0, \ell - 2]_e \cup S^+ \cup \ell S^- \cup \ell S^+ \cup S^- \cup [n - \ell + 2, n - 2]_o. \tag{7}$$

We will now show that

- $([0, \ell - 2]_e \cup \ell S^- \cup \ell S^+ \cup [n - \ell + 2, n - 2]_o) \subseteq \ell S$ and
- $S \cap \ell S = \emptyset$.

Since $S = S^+ \cup S^-$, we have $(\ell S^+ \cup \ell S^-) \subseteq \ell S$. Now S contains 1, 3, and $n - 1$. Given $2z \in [0, \ell - 2]_e$, we can write $2z$ as

$$2z \equiv \underbrace{3 + \cdots + 3}_{z \text{ terms}} + \underbrace{1 + \cdots + 1}_{l/2 - z \text{ terms}} + \underbrace{(n-1) + \cdots + (n-1)}_{l/2 \text{ terms}} (\text{mod } n)$$

and this is an element in ℓS. Therefore, $[0, \ell - 2]_e \subseteq \ell S$. By symmetry, $[n - \ell + 2, n - 2]_o \subseteq \ell S$.

Moving on to showing that $S \cap \ell S = \emptyset$, suppose for contradiction that there exists some $s \in S \cap \ell S$. By symmetry we may assume that $s \in S^+$. Let

$$s \equiv s_1 + s_2 + \cdots + s_\tau + t_{\tau+1} + \cdots + t_\ell \equiv (\text{mod } n) \tag{8}$$

where $\tau \in \{0, 1, \ldots, \ell\}$, $s_j \in S^+$, and $t_j \in S^-$. Defining $s_j = n - t_j$ for $\tau + 1 \le j \le \ell$, we can rewrite (8) as

$$s + s_{\tau+1} + \cdots + s_\ell \equiv s_1 + \cdots + s_\tau (\text{mod } n). \tag{9}$$

Now all terms are in S^+ and we will consider two cases.

`Case 1:` $\tau \in \{1, 2, \ldots, \ell\}$

Observe that for any $x_1, \ldots, x_\ell \in S^+$,

$$\ell \leq |x_1 + \cdots + x_\ell| \leq \ell(2\alpha + 4t + 9) = M = n - 2\alpha - 3 < n.$$

Therefore, since the number of terms on each side of (9) is at most ℓ, congruence (9) holds as an equation in $\mathbb{Z}$ so that

$$s + s_{\tau+1} + \cdots + s_\ell = s_1 + \cdots + s_\tau.$$

Taking this equation modulo 2 gives $1 + (\ell - \tau) \equiv \tau (\text{mod } 2)$. Because ℓ is even, this congruence simplifies to $1 \equiv 0 (\text{mod } 2)$ which is a contradiction.

`Case 2:` $\tau = 0$

In this case (9) is

$$s + s_1 + \cdots + s_\ell \equiv 0 (\text{mod } n). \tag{10}$$

Since

$$\ell + 1 \leq |s + s_1 + \cdots + s_\ell| \leq (\ell + 1)(2\alpha + 4t + 9) = M + 2\alpha + 4t + 9 = n + 4t + 6 < 2n,$$

for (10) to hold we must be able to write n as a sum of $\ell + 1$ elements in S^+. Now

$$\begin{aligned} (\ell + 1)S^+ &= \ell S^+ + S^+ \\ &= ([\ell, M - 4t - 8]_e \cup [M - 4t - 4, M - 2t - 4]_e \cup \{M\}) \\ &+ ([1, 2\alpha + 1]_o \cup [2\alpha + 5, 2\alpha + 5 + 2t]_o \cup \{2\alpha + 4t + 9\}). \end{aligned}$$

A sum of the form

$$M + s' \text{ with } s' \in S^+$$

cannot give n since S^+ does not contain $2\alpha + 3$ and $m = n + 2\alpha + 3$ by Lemma 1. Likewise, a sum of the form

$$t' + 2\alpha + 4t + 9 \text{ with } t' \in \ell S^+$$

cannot give n since ℓS^+ does not contain $M - 4t - 6$. Therefore, any sum from $(\ell + 1)S^+$ giving n cannot use M from ℓS^+ and cannot use $2\alpha + 4t + 9$ from S^+. All other sums have absolute value at most $(M - 2t - 4) + (2\alpha + 5 + 2t) = M + 2\alpha + 1$ which is less than n. We conclude that $0 \notin (\ell + 1)S$. This provides the needed contradiction and completes the proof of Theorem 3. □

3. Proof of Theorem 2 and Corollary 1

In this section we use Theorem 3 to prove that for any even integer $\ell \geq 4$, there is an n-vertex $C_{\ell+1}$-saturated graph that is regular for all $n \geq 12\ell^2 + 30\ell + 19$. Given $S \subseteq \Gamma$ where Γ is an abelian group and $\ell \geq 2$, let

$$\mathcal{R}_\ell(S) = \{s_1 + \cdots + s_\ell : s_i \in S \text{ and } s_i + s_{i+1} + \cdots + s_j \neq 0 \text{ for all } 1 \leq i < j \leq \ell\}.$$

We will combine Theorem 3 with the following proposition which is proved in [19].

Proposition 1 ([19])**.** *Let $\ell \geq 2$ be even and let Γ be an abelian group with $|\Gamma| = n$. If there is a symmetric subset $S \subseteq \Gamma$ with*

$$\mathcal{R}_\ell(S) = \Gamma \backslash (S \cup \{0\}) \quad \text{and} \quad 0 \notin (\ell+1)S,$$

then the Cayley graph $\mathrm{Cay}(\Gamma, S)$ is an $|S|$-regular n-vertex $C_{\ell+1}$-saturated graph.

For our application to graphs we need to consider $\mathcal{R}_\ell(S)$ instead of ℓS. The reason for this is that $C_{\ell+1}$-saturation requires at least one path of length ℓ between each pair of nonadjacent vertices. The sums $s_1 + \cdots + s_\ell$ in ℓS will be used to find these paths. When a consecutive subsum is 0, we do not get a path of length ℓ (the path we are constructing returns to the initial vertex creating a cycle). Having made this remark, let us turn to the proof of Theorem 2.

Proof of Theorem 2. For even n, the graph $K_{n/2,n/2}$ is $C_{\ell+1}$-saturated whenever $n \geq \ell+2$. Let n be an odd integer with $n \geq 12\ell^2+30\ell+19$. Write $n = (2\ell+2)k+r$ where $k \geq 0$ is an integer and $r \in \{1, 3, \ldots, 2\ell+1\}$. From the lower bound on n and upper bound on r we get

$$k = \frac{n-r}{2\ell+2} \geq \frac{12\ell^2 + 30\ell + 19 - (2\ell+1)}{2\ell+2} = 6\ell + 8 + \frac{1}{\ell+1} > 6\ell + 8.$$

Hence, $k \geq 6\ell + 9$ and this lower bound on k will be used when applying Theorem 3.

Claim 2. Given an $r \in \{1, 3, \ldots, 2\ell+1\}$, there is an integer $t \in \{1, 2, \ldots, \ell+1\}$ such that

$$\ell + 3 - 4(t+2) \equiv r (\mathrm{mod}\ 2\ell+2).$$

Proof of Claim 2. Suppose that $\ell+3-4(t_1+2) \equiv \ell+3-4(t_2+2) (\mathrm{mod}\ 2\ell+2)$ where $t_1, t_2 \in \{1, 2, \ldots, \ell+1\}$. This congruence is equivalent to $2t_1 \equiv 2t_2 (\mathrm{mod}\ \ell+1)$. Since $\ell+1$ is odd, we have $t_1 \equiv t_2 (\mathrm{mod}\ \ell+1)$. Since t_1, t_2 are integers in $\{1, 2, \ldots, \ell+1\}$, it must be the case that $t_1 = t_2$. Thus,

$$\{\ell + 3 - 4(t+2) (\mathrm{mod}\ 2\ell+2) : t \in \{1, 2, \ldots, \ell+1\}\} = \{1, 3, \ldots, 2\ell+1\}.$$

$\square$

By Claim 2 we can choose an integer $t \in \{1, 2, \ldots, \ell + 1\}$ such that

$$\ell + 3 - 4(t + 2) \equiv r (\text{mod } 2\ell + 2).$$

Having ℓ, t, and r fixed, we define γ by

$$r = \ell + 3 - 4(t + 2) + \gamma(2\ell + 2).$$

Since $r \in \{1, 3, \ldots, 2\ell + 1\}$, the integer γ is the unique integer for which

$$1 \leq \ell + 3 - 4(t + 2) + \gamma(2\ell + 2) \leq 2\ell + 2.$$

From the definition of t, we know that $t \leq \ell + 1$. The assumption on γ implies that

$$0 \leq \gamma \leq \frac{4(t + 2) - \ell - 3}{2\ell + 2} + 1 \leq \frac{5\ell + 11}{2\ell + 2} < \frac{8\ell}{2\ell} = 4. \tag{11}$$

Thus, $\gamma \leq 3$ and using our upper bound on t, we have

$$4t + 2\ell + 2 + |\gamma| \leq 4(\ell + 1) + 2\ell + 2 + 3 = 6\ell + 9 \leq k.$$

Hence, k satisfies the inequality required by Theorem 3 and we have a symmetric subset $S \subseteq \mathbb{Z}_n$ for which $\mathbb{Z}_n$ is the disjoint union of S and ℓS. To apply Proposition 1, we need to prove that for this S,

$$\mathcal{R}_\ell(S) = \mathbb{Z}_n \backslash (S \cup \{0\}) \tag{12}$$

and $0 \notin (\ell+1)S$. For the latter, if $0 \in (\ell+1)S$, then there are elements $s_1, \ldots, s_{\ell+1}$ such that $s_1 + \cdots + s_{\ell+1} \equiv 0 (\text{mod } n)$. Because S is symmetric, $-s_{\ell+1} \in S$ and therefore, this congruence implies $-s_{\ell+1} \in \ell S \cap S$, a contradiction.

We will complete the proof of Theorem 2 by showing that (12) holds. To prove this, it is enough to show that every element $x \in \ell S \backslash \{0\}$ can be written as

$$x \equiv s_1 + \cdots + s_\ell (\text{mod } n), \;\; s_i \in S$$

where $s_i + \cdots + s_j \not\equiv 0 (\text{mod } n)$ for all $1 \leq i < j \leq \ell$. Critical to this is the structure of the sets S^+ and S^- in the proof of Theorem 3. From (7) we know

$$\mathbb{Z}_n = \big([2, \ell - 2]_e \cup \ell S^- \cup \ell S^+ \cup [n - \ell + 2, n - 2]_o\big) \cup (S \cup \{0\}). \tag{13}$$

An element $s_1 + \cdots + s_\ell$ in ℓS^+ is an element of $\mathcal{R}_\ell(S)$ because $S^+ \subseteq \{1, 3, \ldots, M/\ell\}$ and $M < n$ (note $M/\ell = 2\alpha + 4t + 9 = \max S^+$). Therefore, all consecutive subsums of an element in ℓS^+ are contained in $\{1, 2, \ldots, M\}$ which implies that no subsum is 0. By symmetry there is no consecutive subsum of $s_1 + \cdots + s_\ell \in \ell S^-$ that is 0 modulo n. Thus,

$$(\ell S^+ \cup \ell S^-) \subseteq \mathcal{R}_\ell(S).$$

By (13) it remains to express each element in $[2, \ell-2]_e \cup [n-\ell+2, n-2]_o$ as a sum in $\mathcal{R}_\ell(S)$. Let $2u \in [2, \ell-2]_e$. We can write $2u$ as

$$2u \equiv \underbrace{-M/\ell - \cdots - M/\ell}_{\frac{\ell-2u}{2}\text{ times}} + \underbrace{1 + \cdots + 1}_{2u\text{ times}} + \underbrace{M/\ell + \cdots + M/\ell}_{\frac{\ell-2u}{2}\text{ times}} (\text{mod } n). \tag{14}$$

The first $\frac{\ell-2u}{2}$ terms are in S^- and the remaining terms are in S^+. Suppose, for contradiction, that this sum is not in $\mathcal{R}_\ell(S)$. This means that (14) contains a consecutive subsum that is 0. Since $2u \leq \ell - 2 < M/\ell$, we have

$$0 < 2u + \left(\frac{\ell-2u}{2}\right)(M/\ell) < \ell - 2 + M/2 < M < n.$$

Therefore, any 0 consecutive subsum of (14) must contain at least one of the $-M/\ell$ terms from S^-. A similar argument, using the inequality $0 < 2u < M/\ell$, shows that a 0 consecutive subsum must contain at least one the M/ℓ terms from S^+. Assume that this 0 consecutive subsum is

$$-\theta_1(M/\ell) + 2u + \theta_2(M/\ell) \equiv 0(\text{mod } n)$$

where $1 \leq \theta_1, \theta_2 \leq \frac{\ell-2u}{2}$. If $\theta_2 \geq \theta_1$, then

$$2 \leq 2u + (\theta_2 - \theta_1)(M/\ell) \leq M < n.$$

When $\theta_1 > \theta_2$, we have

$$(\theta_1 - \theta_2)(M/\ell) \equiv 2u(\text{mod } n).$$

This is a contradiction because now the left hand side is in $[M/\ell, M]$ while the right hand side is in $\{2, 4, \ldots, \ell-2\}$, and $\ell - 2 < M/\ell$. The conclusion is that $[2, \ell-2]_e \subseteq \mathcal{R}_\ell(S)$ and by symmetry, $[n-\ell+2, n-2]_o$ is also a subset of $\mathcal{R}_\ell(S)$. Recalling that $(\ell S^+ \cup \ell S^-) \subseteq \mathcal{R}_\ell(S) \subseteq \ell S$ and (13), we see that $\ell S \cap S = \emptyset$ implies

$$\mathcal{R}_\ell(S) = \mathbb{Z}_n \backslash (S \cup \{0\}).$$

By Proposition 1, $\text{Cay}(\mathbb{Z}_n, S)$ is a $C_{\ell+1}$-saturated graph with n vertices and is $|S|$-regular. □

We end this section by demonstrating the upper bound on $\text{rsat}(n, C_{\ell+1})$ that our construction gives. Recalling that n is odd and $n = (2\ell+2)k + r$, a short computation gives $|S| = 2(k + \gamma - t - 1)$. The inequalities $\gamma \leq 3$ (see (11)) and $t \geq 1$ imply that

$$|S| = 2(k + \gamma - t - 1) \leq 2(k+1) = 2\left(\frac{n-r}{2(\ell+1)} + 1\right) \leq \frac{n + 2\ell + 1}{\ell + 1}.$$

Using this upper bound, we have for any even $\ell \geq 4$ and odd $n \geq 12\ell^2 + 30\ell + 19$,

$$\mathrm{rsat}(n, C_{\ell+1}) \leq \frac{n}{2}|S| \leq \frac{n^2}{2(\ell+1)} + \left(\frac{2\ell+1}{2\ell+2}\right) n.$$

Finally, this approach used here can be adapted to the $\ell = 2$ case with several steps becoming easier or no longer necessary. However, the results of [14] encompass this case.

Acknowledgment. The authors would like to thank the anonymous reviewer for feedback that led to significant simplifications in our proof of Theorem 3 and improved the clarity of our arguments.

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