LINEAR COMBINATIONS OF DIRICHLET SERIES ASSOCIATED WITH THE THUE-MORSE SEQUENCE

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Abstract

In this paper we study sums of Dirichlet series whose coefficients are terms of the Thue-Morse sequence and variations thereof. We find closed-form expressions for such sums in terms of known constants and functions including the Riemann zeta function and the Dirichlet eta function using elementary methods.

– *Dedicated to Walter Alexandre.*

1. Introduction

Let t_n denote the binary digit sum of the positive integer n modulo 2, in other words, the n^{th} element of the Thue-Morse sequence $(t_n)_{n\geq 0}$, which begins

$$0, 1, 1, 0, 1, 0, 0, 1, \dots.$$

This sequence was first considered by Prouhet [10] and has applications in many different fields of mathematics (see for instance Allouche and Shallit [8] for a detailed overview). A popular variant is the ± 1 Thue-Morse sequence, denoted by ε_n and defined as $((-1)^{t_n})_{n\geq 0}$. Dirichlet series associated with ε_n have been widely studied during the past years, in particular $\sum_{n\geq 1} \frac{\varepsilon_n}{n^s}$ and $\sum_{n\geq 0} \frac{\varepsilon_n}{(n+1)^s}$, which converge for $\Re(s) > 1$. Allouche and Cohen [3] continued the series analytically and gave the functional equation

$$\sum_{n\geq 0} \frac{\varepsilon_n}{(n+1)^s} = \sum_{k\geq 1} 2^{-s-k} \binom{s+k-1}{k} \sum_{n\geq 0} \frac{\varepsilon_n}{(n+1)^{s+k}},$$

and showed that it admits non-trivial zeros at $s = (2k\pi i)/\log 2$ for any integer k. The series $\sum_{n\geq 1} \frac{\varepsilon_n}{n^s}$ is continued in a similar manner, yielding a set of non-trivial zeros at $s = i\pi(2k+1)/\log 2$ (although as noted by Allouche [2], the question of whether these are *all* the non-trivial zeros is still an open one). These results were then further extended by Allouche, Mendès France and Peyrière [5] to d-automatic sequences, for $d \geq 2$. Furthermore, Allouche and Cohen [3] and Alkauskas [1] have noticed that the two series above are related through the identity

$$\sum_{n\geq 0} \frac{\varepsilon_n}{(n+1)^s} = \frac{1-2^s}{1+2^s} \sum_{n\geq 1} \frac{\varepsilon_n}{n^s}.$$

Since then, these series have been used in various contexts. For instance, Allouche and Cohen [3] used the derivative of $\sum_{n\geq 0} \frac{\varepsilon_n}{(n+1)^s}$ at $s = 0$ to give an alternative proof of the Woods-Robbins product,

$$\prod_{n\geq 0} \left(\frac{2n+1}{2n+2}\right)^{\varepsilon_n} = \frac{\sqrt{2}}{2},$$

which has been extended to many different types of infinite products (see Allouche, Cohen, Mendès France and Shallit [4], Allouche and Sondow [9], Allouche, Riasat and Shallit [6] and Tóth [12] for example).

Series involving the b-ary sum-of-digits function $s_b(n)$ have also been studied. A classic example is

$$\sum_{n\geq 1} \frac{s_b(n)}{n(n+1)} = \frac{b}{b-1} \log b,$$

which was proved by Shallit [11], while other identities such as

$$\sum_{n\geq 1} \frac{s_2(n)(2n+1)}{n^2(n+1)^2} = \frac{\pi^2}{9}$$

are due to Allouche and Shallit [7].

1.1. Scope of This Paper

No closed-form expression is currently known for Dirichlet series involving only t_n and t_{n-1} and combinations thereof in terms of known constants and functions. In this paper, we provide such closed forms and give a formula for generating similar linear combinations. A few examples are

$$\sum_{n\geq 1} \frac{5t_{n-1} + 3t_n}{n^2} = \frac{2\pi^2}{3},$$
$$\sum_{n\geq 1} \frac{9t_{n-1} + 7t_n}{n^3} = 8\zeta(3).$$

Then, we extend our results to combinations of sequences involving different alphabets $\{a, b\}$ along the Thue-Morse sequence, that is, sequences with the substitutions

$0 \to a$ and $1 \to b$ over t_n, with $a, b \in \mathbb{R}$. An example is

$$\sum_{n\geq 1} \frac{17q_{n-1} + 15r_n}{n^4} = 16\eta(4),$$

where $\eta(s)$ denotes the alternating zeta function $\sum_{n\geq 1} \frac{(-1)^{n-1}}{n^s}$ and q_n and r_n denote the sequences defined respectively by $\{-\sqrt{2}, 1-\sqrt{2}\}$ and $\{\frac{17\sqrt{2}-2}{15}, \frac{17\sqrt{2}+13}{15}\}$ along the Thue-Morse sequence.

2. Linear Combinations

Throughout the remainder of this paper, we shall use the following notation:

$$f(s) = \sum_{n\geq 1} \frac{\varepsilon_{n-1}}{n^s}, \quad g(s) = \sum_{n\geq 1} \frac{\varepsilon_n}{n^s}, \quad \varphi(s) = \sum_{n\geq 1} \frac{t_{n-1}}{n^s}, \quad \gamma(s) = \sum_{n\geq 1} \frac{t_n}{n^s}.$$

We begin this section by recalling a result on Dirichlet series whose sign alternates following the Thue-Morse sequence, originally due to Allouche and Cohen [3] and already mentioned in the introduction.

Lemma 1. *For all $s \in \mathbb{C}$ with $\Re(s) > 1$,*

$$f(s) = \frac{1-2^s}{1+2^s} g(s).$$

While Allouche and Cohen's proof involves the analytic continuations of f and g, a simpler proof was proposed by Alkauskas [1] in 2001 that we cannot resist replicating here. This proof relies only on the fact that every positive integer n can be uniquely represented as the product $2^k(2m+1)$ for $k \geq 0$ and $m \geq 0$.

Proof of Lemma 1 ([1]). We have

$$f(s) = \sum_{k\geq 0, m\geq 0} \frac{\varepsilon_{2^k(2m+1)-1}}{2^{ks}(2m+1)^s} = \sum_{k\geq 0, m\geq 0} \frac{(-1)^k \varepsilon_m}{2^{ks}(2m+1)^s} = \frac{2^s}{2^s+1} \sum_{m\geq 0} \frac{\varepsilon_m}{(2m+1)^s}.$$

Now we know that

$$\sum_{m\geq 0} \frac{\varepsilon_m}{(2m+1)^s} = \sum_{m\geq 1} \frac{\varepsilon_m}{(2m)^s} - \sum_{m\geq 1} \frac{\varepsilon_m}{m^s},$$

by splitting $\sum_{m\geq 1} \frac{\varepsilon_m}{m^s}$ into even and odd indexes and using the fact that $\varepsilon_{2m} = \varepsilon_m$ and $\varepsilon_{2m+1} = -\varepsilon_m$. The proof follows naturally. □

We note that one could also evaluate $g(s)$ in the same manner, thus yielding a unified proof of Lemma 1. This is left as an exercise for the reader.

Corollary 1. *For any holomorphic functions u and v, we have*

$$u(s)\varphi(s)+v(s)\gamma(s)=\frac{u(s)+v(s)}{2}\zeta(s)-\frac{f(s)}{2}\left(u(s)+v(s)\frac{1+2^s}{1-2^s}\right).$$

Proof. Using Lemma 1 together with $\varepsilon_n = 1-2t_n$, we have

$$\varphi(s)=\frac{1}{2}\zeta(s)-\frac{1}{2}f(s), \qquad \gamma(s)=\frac{1}{2}\zeta(s)-\frac{1+2^s}{2(1-2^s)}f(s).$$

Now taking two holomorphic functions u and v, we quickly obtain the above expression for $u(s)\varphi(s)+v(s)\gamma(s)$. □

We now have all the tools to establish our first result in this paper.

Theorem 1. *Let $\zeta(s)=\sum_{n\geq 1} 1/n^s$ denote the Riemann zeta function defined for complex s with $\Re(s)>1$. Then*

$$(2^s+1)\varphi(s)+(2^s-1)\gamma(s)=2^s\zeta(s).$$

Note that both Dirichlet series on the left-hand side converge for $\Re(s)>1$ since the sequence $(t_n)_{n\geq 0}$ takes only finitely many values.

We shall now give two proofs of this identity. The first uses Corollary 1, while the second relies on index-splitting (used also within the context of the Woods-Robbins product by Allouche, Mendès France and Peyrière [5], for instance).

Proof 1 of Theorem 1. We take Corollary 1 with $u(s)=2^s+1$ and $v(s)=2^s-1$. The proof immediately follows. □

Of course, we could also prove the Theorem above without having recourse to any of the results above. For instance, we could simply use the relations $t_{2n}=t_n$ and $t_{2n+1}=1-t_n$ as follows.

Proof 2 of Theorem 1. We begin by splitting $\gamma(s)$ and $\varphi(s)$ into odd and even indexes. On the one hand, we have

$$\gamma(s)=2^{-s}\gamma(s)-\sum_{n\geq 0}\frac{t_n}{(2n+1)^s}+(1-2^{-s})\zeta(s).$$

On the other hand,

$$\varphi(s)=2^{-s}\zeta(s)-2^{-s}\varphi(s)+\sum_{n\geq 0}\frac{t_n}{(2n+1)^s}.$$

Now taking the sum of these two equations yields

$$\gamma(s)+\varphi(s)=2^{-s}(\gamma(s)-\varphi(s))+\zeta(s),$$

and a simple rearrangement of the terms concludes this proof. □

Here we note that whenever $t_{n-1} = t_n = 0$, the corresponding n^{th} term disappears from both Dirichlet series on the left-hand side. The first few missing terms are thus $n = 6, 10, 18, 24, \ldots$ (sequence A248056 in the OEIS). Our result above implies several interesting examples, which we have already mentioned in the introduction.

Example 1. We have the following equalities:

$$(a) \qquad \sum_{n\geq 1} \frac{5t_{n-1} + 3t_n}{n^2} = \frac{2\pi^2}{3},$$

$$(b) \qquad \sum_{n\geq 1} \frac{9t_{n-1} + 7t_n}{n^3} = 8\zeta(3).$$

There are several ways to extend these results, which we will do in the following sections.

2.1. Generalization to Different Alphabets Along the Thue-Morse Sequence

In the following paragraphs, we extend Theorem 1 to linear combinations of series involving various alphabets $\{a, b\}$ along the Thue-Morse sequence, i.e., sequences with the substitutions $0 \to a$ and $1 \to b$, with $a, b \in \mathbb{R}$, over the t_n sequence. In particular, we show that there exist alphabets which give rise to identities involving the alternating zeta function $\eta(s) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^s}$, others that give rise to identities involving the Riemann zeta function and, finally, alphabets that produce simple linear combinations with other Dirichlet series.

Theorem 2. *Let $k, \ell > 0$ be real numbers and define the sequences $q_n = t_n - k$ and $r_n = t_n + \ell$ for all $n > 0$. Furthermore, consider the function $\lambda(s; k, \ell) = 2^s - (2^s(k - \ell) + (k + \ell))$ for some complex s with $\Re(s) > 1$. We then have*

$$(1) \quad \sum_{n\geq 1} \frac{q_{n-1}}{n^s} = \frac{1 - 2^s}{1 + 2^s} \sum_{n\geq 1} \frac{r_n}{n^s} \qquad \text{if } \lambda(s; k, \ell) = 0,$$

$$(2) \quad (2^s + 1) \sum_{n\geq 1} \frac{q_{n-1}}{n^s} + (2^s - 1) \sum_{n\geq 1} \frac{r_n}{n^s} = 2^s\zeta(s) \qquad \text{if } \lambda(s; k, \ell) = 2^s,$$

$$(3) \quad (2^s + 1) \sum_{n\geq 1} \frac{q_{n-1}}{n^s} + (2^s - 1) \sum_{n\geq 1} \frac{r_n}{n^s} = 2^s\eta(s) \qquad \text{if } \lambda(s; k, \ell) = 2^s - 2.$$

Proof. Define the sequences $q_n = t_n - k$ and $r_n = t_n + \ell$ for all $n > 0$ and real $k, \ell > 0$. We immediately have

$$\sum_{n\geq 1} \frac{q_{n-1}}{n^s} = \varphi(s) - k\zeta(s), \qquad \sum_{n\geq 1} \frac{r_n}{n^s} = \gamma(s) + l\zeta(s).$$

Thus,

$$(2^s+1)\sum_{n\geq 1}\frac{q_{n-1}}{n^s}+(2^s-1)\sum_{n\geq 1}\frac{r_n}{n^s}=\zeta(s)\left(2^s-(2^s(k-l)+(k+l))\right).$$

Inspired by the coefficient of $\zeta(s)$ on the right-hand side, we define the function $\lambda(s;k,\ell)=2^s-(2^s(k-\ell)+(k+\ell))$ for real $k,\ell>0$ and complex s with $\Re(s)>1$. It is obvious that if $\lambda(s;k,\ell)=0$, the $\zeta(s)$ term on the right-hand side vanishes, thus proving (1). If $\lambda(s;k,\ell)=2^s$, we have identities of the same type as in Theorem 1, proving (2), and if $\lambda(s;k,\ell)=2^s-2$ then the identity $\eta(s)=\left(1-2^{1-s}\right)\zeta(s)$ quickly establishes (3). □

Solutions to each of these cases lead to interesting illustrative examples. A first set of "simple" solutions – in terms of k and ℓ only – are easy to find. For the case (1) above, we have $k=\frac{1}{2},\ell=-\frac{1}{2}$, which leads to the classic Woods-Robbins product by differentiation of the resulting series identity at $s=0$, as already noted by Allouche and Cohen [3]. The solution of the second case, $k=0,\ell=0$, results in our identity in Theorem 1, and finally that of the third case ($k=1,\ell=1$) results in

$$(2^s+1)\sum_{n\geq 1}\frac{q_{n-1}}{n^s}+(2^s-1)\sum_{n\geq 1}\frac{r_n}{n^s}=2^s\eta(s),$$

where $q_n\to\{-1,0\}$ and $r_n\to\{1,2\}$ along the Thue-Morse sequence.

Another, perhaps more interesting set of solutions can be found by taking s into account as well.

Proposition 1. *Define the sequences $q_n=t_n-k$ and $r_n=t_n+\ell$ for all $n>0$ and real $k,\ell>0$. We have the following equalities:*

$$(a)\qquad 5\sum_{n\geq 1}\frac{q_{n-1}}{n^2}=-3\sum_{n\geq 1}\frac{r_n}{n^2},\qquad q_n\to\{-1,0\},r_n\to\{\frac{1}{3},\frac{4}{3}\}$$

$$(b)\qquad \sum_{n\geq 1}\frac{9q_{n-1}+7r_n}{n^3}=8\zeta(3),\qquad q_n\to\{-1,0\},r_n\to\{\frac{9}{7},\frac{16}{7}\}$$

$$(c)\qquad \sum_{n\geq 1}\frac{17q_{n-1}+15r_n}{n^4}=16\eta(4),\qquad q_n\to\{-\sqrt{2},1-\sqrt{2}\},$$

$$r_n\to\{\frac{17\sqrt{2}-2}{15},\frac{17\sqrt{2}+13}{15}\}.$$

Proof. The general solution of the equation $\lambda(s;k,\ell)=0$ (i.e., case (1) in Theorem 2) in the reals is

$$s=\frac{\log(\frac{k+\ell}{-k+\ell+1})}{\log 2},$$

with $k\neq\ell+1$ and $k+\ell\neq 0$. So for instance we can take $k=1$ and $\ell=\frac{1}{3}$, yielding the sequences $q_n=t_n-1$ and $r_n=t_n+\frac{1}{3}$, for all $n\geq 0$, i.e., the sequences defined respectively by $\{-1,0\}$ and $\{\frac{1}{3},\frac{4}{3}\}$ along the Thue-Morse sequence. A simple substitution above gives $s=2$, thereby proving our statement (a).

We now turn our attention to the solution of $\lambda(s;k,\ell)=2^s$ (i.e., case (2) in Theorem 2), which in the reals is

$$s=\frac{\log(-\frac{k+\ell}{k-\ell})}{\log 2},$$

with $k-\ell\neq 0$ and $k+\ell\neq 0$. Let now $q_n=t_n-1$ and $r_n=t_n+\frac{9}{7}$, for all $n\geq 0$, in other words the sequences defined respectively by $\{-1,0\}$ and $\{\frac{9}{7},\frac{16}{7}\}$ along the Thue-Morse sequence. This means that we have $k=1$ and $\ell=\frac{9}{7}$, which yields $s=3$ in the equation above and thus proves statement (b).

Finally, the solution of the equation $\lambda(s;k,\ell)=2^s-2$ in the reals, corresponding to case (3) in Theorem 2 is

$$s=\frac{\log(-\frac{k+\ell-2}{k-\ell})}{\log 2},$$

with $k-\ell\neq 0$ and $k+\ell\neq 2$, which allows us to choose $k=\sqrt{2}$ and $\ell=\frac{17\sqrt{2}-2}{15}$. This gives the sequences $q_n=t_n-\sqrt{2}$ and $r_n=t_n+\frac{17\sqrt{2}-2}{15}$, for all $n\geq 0$, i.e., the sequences defined respectively by $\{-\sqrt{2},1-\sqrt{2}\}$ and $\{\frac{17\sqrt{2}-2}{15},\frac{17\sqrt{2}+13}{15}\}$ along the Thue-Morse sequence. Thus we have $s=4$ and identity (c) as claimed. □

3. Conclusion and Further Work

In this paper we have found closed forms for certain linear combinations of Dirichlet series associated with the Thue-Morse sequence in terms of known constants and functions. However, closed forms for the individual series remain elusive.

Question 1. Do the series $\sum_{n\geq 1}\frac{t_n}{n^s}$ and $\sum_{n\geq 1}\frac{t_{n-1}}{n^s}$ for $s\in\mathbb{C}$ with $\Re(s)>1$ admit closed forms in terms of known constants and functions?

Despite our efforts, we have not been able to find a set of linear combinations allowing us to eliminate either f or g from Theorem 1.

3.1. Extension to Other Sequences

In some of our proofs we used an index-splitting method to find expressions for series involving the Thue-Morse sequence and variations thereof. The same method can possibly be applied to other series whose coefficients are generated by finite automata. A few examples are listed below, and the proofs are left to the reader.

Example 2. Let $\delta_n = t_n - t_{n-1}$ for all $n \geq 1$ and $\varepsilon_n = (-1)^{t_n}$ the ± 1 Thue-Morse sequence. Then for all s with $\Re(s) > 1$,

$$\sum_{n \geq 1} \frac{\delta_n}{n^s} = \frac{4^s}{4^s - 1} \sum_{n \geq 0} \frac{\varepsilon_n}{(2n+1)^s}.$$

Example 3. Let σ_n denote the "period-doubling sequence" (A096268 in the OEIS), defined by the recurrence $\sigma_{2n} = 0, \sigma_{4n+1} = 1, \sigma_{4n+3} = \sigma_n$. Then for all s with $\Re(s) > 1$,

$$\sum_{n \geq 1} \frac{\sigma_n \left((4n+3)^s - n^s\right)}{\left(4n^2 + 3n\right)^s} = 4^{-s} \zeta\left(s, \frac{1}{4}\right),$$

where $\zeta(s, a)$ denotes the Hurwitz zeta function.

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