DIFFUSION AND POLYOMINOES

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Abstract

In diffusion, a chip-firing variant, chips flow from places of high concentration to places of low concentration (or equivalently, from the rich to the poor). It was proved by Long and Narayanan that this process is periodic with period 1 or 2. In this paper, we enumerate the number of period configurations on complete graphs using connections to polyominoes.

1. Introduction

Diffusion involves vertices sending chips, one at a time, to those adjacent vertices that have fewer chips. This model was introduced in [3], and in [6], it was shown that the diffusion process ended either when all the vertices have the same number of chips or the process alternated between two states. See Figure 1 for an example.

In this paper, with Lemma 1, we show that there is an equivalence relation on the states, with only a finite number of equivalence classes. Consequently, the question of "how many non-equivalent ways can the diffusion process terminate?" can be raised.

We are interested in counting the number of configurations (distributions of chips) on K_n, $n \geq 1$ (up to equivalence). Configurations either exist inside the pre-period (meaning that they never repeat) or inside the period (meaning that they will repeat infinitely). We will show, with Theorem 3, a bijection between the number of period

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configurations (up to equivalence) that exist on unlabelled complete graphs of order n and the number of board-pile n-ominoes (or sets of stacked rectangles of height 1 with a total area of n). An example of a board-pile n-omino is given in Figure 2, and an example of a configuration on K_n is given in Figure 3.

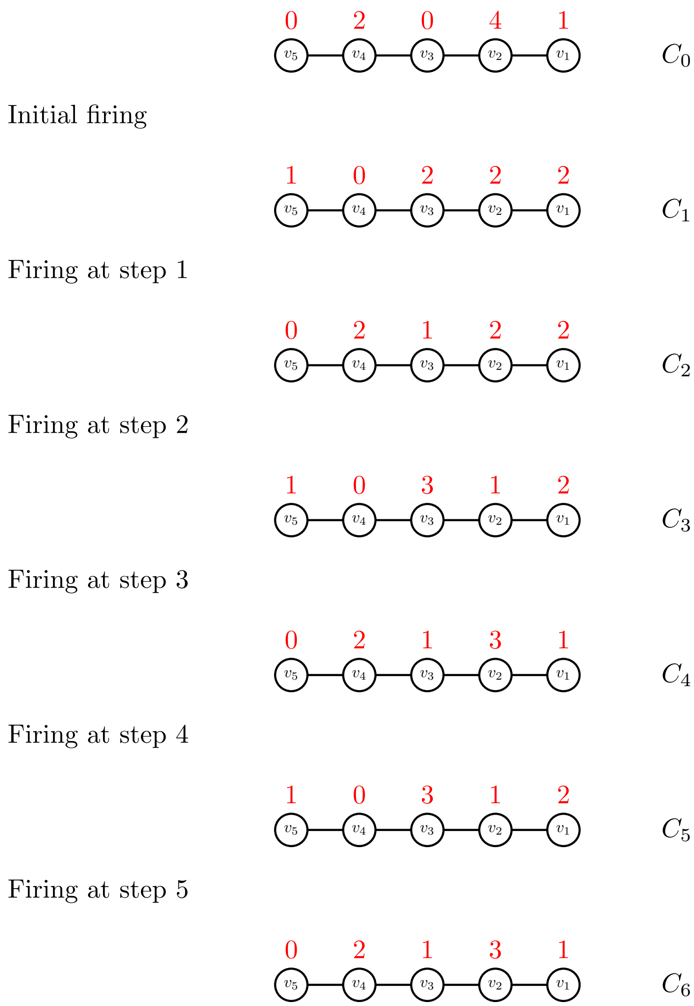


Figure 1: Several steps in a diffusion game on P_5. The period begins with C_3. This is the first configuration that is repeated.

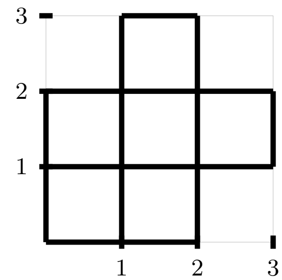
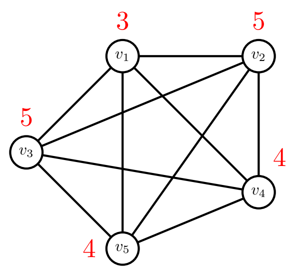


Figure 2: Board-pile 6-omino X.



$$= \{(v_1, 3), (v_2, 5), (v_3, 5), (v_4, 4), (v_5, 4)\}$$

Figure 3: Configuration on a complete graph, K_5.

Unlike some other chip-firing processes, like the original chip-firing game [1] and brushing [7], in diffusion it is possible for a vertex to initially have a positive number of chips but for that number to become negative as time goes on. For example if some vertex v with n chips, $n \in \mathbb{N}$, is adjacent to $n+1$ vertices, each of which having 0 chips, then after firing, v would have -1 chips. However, diffusion is such that an addition of some constant k chips, $k \in \mathbb{Z}$, to each vertex will have no effect on determining when and if a chip will move from one vertex to another (see Lemma 1). So if one wanted to view diffusion as a process in which vertices never had negative amounts of chips, one would only need to add a sufficient constant k, $k \in \mathbb{N}$, to each vertex. Some results pertaining to locating an appropriate k value for any given graph can be found in [2].

2. Diffusion Background

We begin with some necessary terminology.

Each vertex is assigned a *stack size* which is an integral number. This number represents the number of chips a vertex has. At each time step, the chips are redistributed by *firing* the vertices of the graph such that the following rule is obeyed: if a vertex is adjacent to a vertex with fewer chips, it sends a chip to that vertex, meaning a chip is taken from its stack and added it to the stack of the neighbor with fewer chips. Note that when a vertex fires but has no neighbors with fewer chips, it does not send any chips. An assignment of stack sizes to the vertices of a graph G is referred to as a *configuration* and is denoted $C = \{(v, |v|^C) : v \in V(G)\}$, where $|v|^C$ is the stack size of v in C. We omit the superscript when the configuration is clear. A vertex v is said to be *richer* than another vertex u in configuration C if $|v|^C > |u|^C$. In this instance, u is said to be *poorer* than v in C.

In diffusion, given a graph G and a configuration C on G, to *fire* C is to decrease the stack size of every vertex $v \in V(G)$ by the number of poorer neighbors v has and increase the stack size of v by the number of richer neighbors v has. More formally, for all v, let $Z_-^C(v) = \{u \in N(v) : |v|^C > |u|^C\}$ and let $Z_+^C(v) = \{u \in N(v) : |u|^C > |v|^C\}$. Firing results in every vertex v changing from a stack size of $|v|^C$ to a stack size of $|v|^C + |Z_+^C(v)| - |Z_-^C(v)|$.

The diffusion process occurs in *steps*, which correspond to the discrete time increments. A step consists of both a configuration and the subsequent firing of the vertices in that configuration, which yields the configuration for the next step. We refer to the configuration at step 0 as the *initial configuration.* The firing of vertices at step 0 which yields the configuration at step 1 is called the *initial firing.*

Given a graph G and an initial configuration, as time progresses in the diffusion process, we may want to refer to the configuration at a given step or the stack size of a particular vertex. The configuration at step t is denoted by C_t, so the initial configuration is C_0. The stack size of vertex v at time t is denoted $|v|_t^{C_0}$. We omit the superscript when the initial configuration is clear from context. This means that the configuration corresponding to step t can be expressed as $C_t = \{(v, |v|_t^{C_0}) : v \in V(G)\}$. When step t occurs, the vertices of our graph fire according to their stack sizes in C_t and the configuration corresponding to step $t+1$, C_{t+1}, is obtained. The *configuration sequence* $Seq(C_0) = (C_0, C_1, C_2, \dots)$ is the sequence of configurations that arises as the steps of the diffusion process occur. The configuration sequence clearly depends on both the initial configuration and the graph G. However, G is omitted from the notation since it will always be clear to which graph we are referring.

For a configuration sequence $Seq(C_0)$, a positive integer p is a *period length* if $C_t = C_{t+p}$ for all $t \geq N$ for some natural number N. In this case, N is a *preperiod length.* For such a value N, if $k \geq N$, then we say that the configuration C_k is *inside* the period. For the purposes of this paper, all references to period length will refer to the *minimum period length* p in a given configuration sequence. Also, all references to preperiod length will refer to the *least preperiod length* that yields that minimum period length p in a given configuration sequence. Given two configurations, C and D, of a graph G, in which the vertices are labelled, C and D are *equal* if $|v|^C = |v|^D$

for all $v \in V(G)$. In Figure 1, the period length is 2 and the preperiod length is 3.

In their paper [6], Long and Narayanan prove that the period length of every configuration sequence is either 1 or 2. Let $\overline{Seq}(C_0)$ denote the singleton or ordered pair of configurations contained within the period of a configuration sequence $Seq(C_0)$.

Let C be a configuration on a graph G. Let $C+k$ denote the configuration created by adding an integer k to every stack size in the configuration C. Two configuration sequences, $Seq(C)$ and $Seq(D)$, are *equivalent* if $\overline{Seq}(C+k) = \overline{Seq}(D)$ for some integer k. For all configurations C and all integers k, we say that C and $C+k$ are *equivalent*.

We see an example of equivalent configuration sequences in Figure 4.

A configuration D on a graph G is a *period configuration* if D is in $\overline{Seq}(C)$ for some configuration C.

We conclude this section with the following lemma which will prove useful. It was first stated in [3]. Here we include the proof from [8].

Lemma 1. *Let C and D be configurations on a graph G. Let k be an integer. Suppose that for all $v \in V(G)$, $|v|^C = |v|^D + k$. Then for all t, $|v|^C_t = |v|^D_t + k$.*

Proof. We will prove this by induction on t. Let C and D be configurations on a graph G. Let k be an integer. Suppose that for all $v \in V(G)$, $|v|^C = |v|^D + k$. So for all $u, v \in V(G)$, $|u|^C > |v|^C$ if and only if $|u|^C = |u|^D + k > |v|^D + k = |v|^C$. Thus after the first firing, we get that $|u|^C_1 = |u|^D_1 + k$ for all $u \in V(G)$. We will consider this as the base case of an induction. Our induction hypothesis is that $|v|^C_t = |v|^D_t + k$ for all $v \in V(G)$. So for all $u, v \in V(G)$, $|u|^C_t > |v|^C_t$ if and only if $|u|^C_t = |u|^D_t + k > |v|^D_t + k = |v|^C_t$. Thus, after the firing at step t, we get that $|u|^C_{t+1} = |u|^D_{t+1} + k$ for all $u \in V(G)$. Thus, we conclude that for all t, $|v|^C_t = |v|^D_t + k$. □

3. Polyominoes

We now introduce polyominoes. The following definitions are from David Klarner's paper [4], reworded slightly to improve the clarity of our results.

Definition 1. A *polyomino* is a plane figure composed of a number of connected unit squares joined edge on edge. A polyomino with exactly n unit squares is called an *n-omino*.

Definition 2. In a polyomino X, a *horizontal strip*, or *h-strip*, is a maximal rectangle of height one.

By convention, we will set each h-strip in the plane so that its height spans from an integer k to $k+1$.

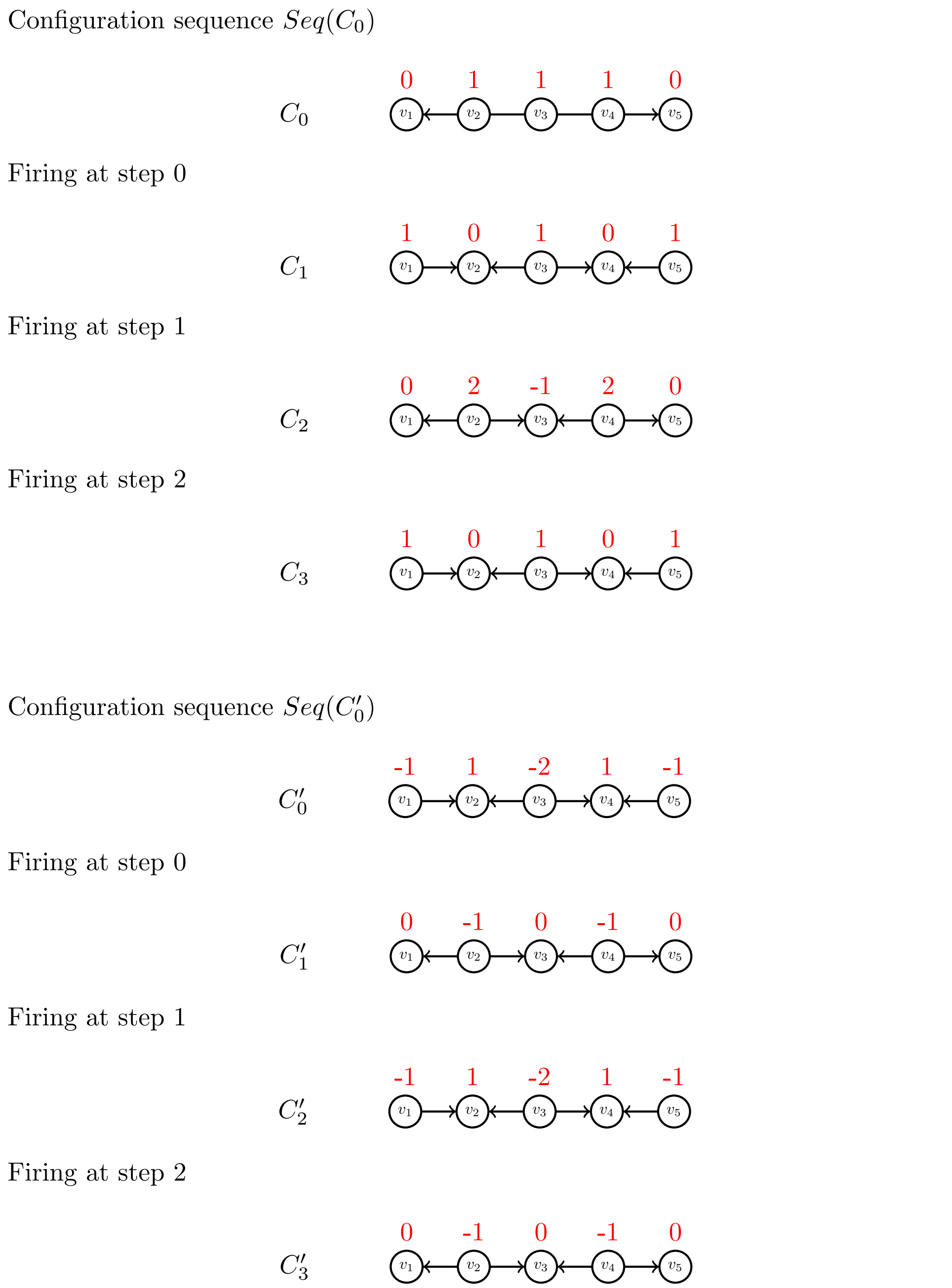


Figure 4: Two equivalent configuration sequences.

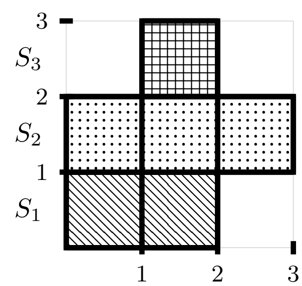
Definition 3. The infinite area enclosed by the lines $y = k$ and $y = k + 1$ is called a *row*.

Definition 4. A *board-pile polyomino* is a polyomino which has a finite number of h-strips, with one h-strip per row. A board-pile polyomino with n unit squares is called a *board-pile n-omino*.

With this, we can now begin to prove that there exists a bijection between the number of board-pile n-ominoes and the number of period configurations of an unlabelled complete graph on n vertices. To accomplish this, we first develop a notation for polyominoes that will eliminate the necessity of a pictorial representation. Then we define a mapping from the set of all board-pile polyominoes on n unit squares to the set of all period configurations of (an unlabelled) K_n up to equivalence, and then show that mapping to be a bijection.

Given a polyomino X, we will use the convention of labelling the h-strips from bottom to top as S_1, S_2, …, S_N, where N is the number of h-strips in X.

A board-pile polyomino X can be represented as a list of ordered pairs of the form $X = [(d_1, |S_1|), (d_2, |S_2|), (d_3, |S_3|), \ldots, (d_N, |S_N|)]$, where $|S_i|$ is the number of unit squares in the h-strip S_i, and d_i is the difference between the greatest x-coordinate in S_i and the least x-coordinate in S_{i-1}. By convention, $d_1 = 0$. Note that since polyominoes are connected edge on edge, $d_i \geq 1$ for all $i \geq 2$. See Figure 5 for an example of a board-pile 6-omino with h-strips S_1, S_2, S_3.



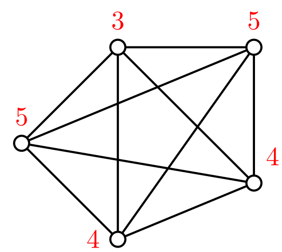
$$X = \{(0, 2), (3, 3), (2, 1)\}$$

Figure 5: Board-pile 6-omino X with shading differentiating between S_1, S_2, and S_3.

4. Period Configurations on Unlabelled Complete Graphs

In this section, we will be counting the number of period configurations that exist on unlabelled complete graphs.

A configuration C of an unlabelled K_n, $n \geq 1$, is represented by a set of cardinality N, where N is the number of distinct stack sizes. We will use the notation a^k to represent k instances of stack size a in C. An example is shown in Figure 6. Note the difference between this notation and that which is used in the first section for labelled graphs. As K_n is a vertex transitive graph, we need not label the vertices. Thus in our notation, we do not need to use a set of ordered pairs to keep vertex labels and corresponding stack sizes together.



$$C = \{3, 4^2, 5^2\}$$

Figure 6: Configuration on an unlabelled complete graph.

We will show that given $n \in \mathbb{N}$, we can define a map f from the set of all board-pile n-ominoes to a set of period configurations on K_n, and show this map to be a bijection, thus giving a way to enumerate all period configurations on K_n.

Let $\mathcal{B}_n$ be the set of all board-pile n-ominoes, $\mathcal{P}_n$ a set of configurations on K_n and $f : \mathcal{B}_n \to \mathcal{P}_n$ be such that for a board-pile n-omino $X = \{(d_1, |S_1|), (d_2, |S_2|), (d_3, |S_3|), \ldots (d_N, |S_N|)\}$,

$$f(X) = f(\{(d_1, |S_1|), (d_2, |S_2|), (d_3, |S_3|), \ldots (d_n, |S_N|)\}) \tag{1}$$

$$= \Big\{0^{|S_1|}, \Big(\sum_{i=1}^{2} d_i\Big)^{|S_2|}, \Big(\sum_{i=1}^{3} d_i\Big)^{|S_3|}, \ldots \Big(\sum_{i=1}^{N} d_i\Big)^{|S_N|}\Big\}. \tag{2}$$

It will be shown in Theorem 3 that $\mathcal{P}_n$ is the set of period configurations of K_n (up to the equivalence that the least stack size is 0). As can be seen from f, the board-pile polyomino X is mapped to a configuration with least stack size 0. From Lemma 1, we know that any configuration is equivalent to one with minimum stack size 0.

The configuration $f(X)$ corresponds to a complete graph that is such that its number of vertices is equal to the number of unit squares in X. The number of unique stack sizes in $f(X)$ is equal to N, the number of h-strips in X. The unit squares of X are partitioned by the h-strips into S_1, S_2, …, S_N. Similarly, the vertices of K_n with configuration $f(X)$ are partitioned into N sets of vertices, each with a common stack size.

The complete graph configuration $f(X)$ has $\sum_{k=1}^{N} |S_k|$ vertices. We denote the set of vertices in $f(X)$ corresponding to the h-strip S_k to be V_k for all $k \leq N$. Each $v \in V_k$ has a stack size of $\sum_{i=1}^{k} d_i$. An example of this mapping is shown in Figure 7.

We will now use f to show that the number of period configurations of K_n is equal to the number of board-pile polyominoes containing exactly n unit squares.

Lemma 2. *Let* $X = \{(d_1, |S_1|), (d_2, |S_2|), (d_3, |S_3|), \dots (d_N, |S_N|)\}$ *be a board-pile polyomino with exactly* N *h-strips. If* $1 \leq i \leq N-1$*, then* $1 \leq d_{i+1} \leq |S_i|+|S_{i+1}|-1$.

Proof. Suppose $1 \leq i \leq N-1$. Since S_i and S_{i+1} are adjacent h-strips, and polyominoes, by definition, are joined edge on edge, the distance from the least x-coordinate of S_i to the greatest x-coordinate of S_{i+1} must be less than the sum of the two lengths ($|S_i| + |S_{i+1}|$). So, d_{i+1} must be less than $|S_i| + |S_{i+1}|$. Since d_{i+1} is equal to the difference between the greatest x-coordinate in S_i and the least x-coordinate in S_{i-1}, and since S_i and S_{i-1} are connected edge on edge, $d_{i+1} \geq 1$. Thus, we conclude $1 \leq d_{i+1} \leq |S_i| + |S_{i+1}| - 1$. □

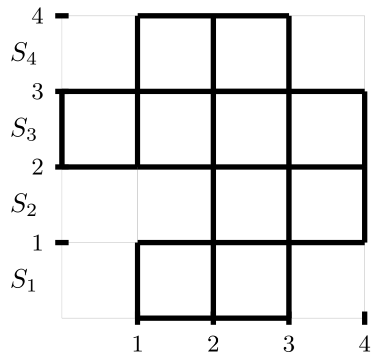
Theorem 3. *For all* $n \geq 1$*, the set of all board-pile* n*-ominoes has the same cardinality as the set of all period configurations on unlabelled complete graphs up to equivalence.*

Proof. This proof will amount to proving a list of three statements. Consider the map f from Equation 1. We will show that $\mathcal{P}_n$ is the set of period configurations of K_n (up to the equivalence that the least stack size is 0) (i), and that f is a bijection by showing that the map is surjective (ii) and injective (iii). This will prove that the set of all board-pile n-ominoes has the same cardinality as the set of all period configurations on an unlabelled copy of K_n up to equivalence. The statements are as follows.

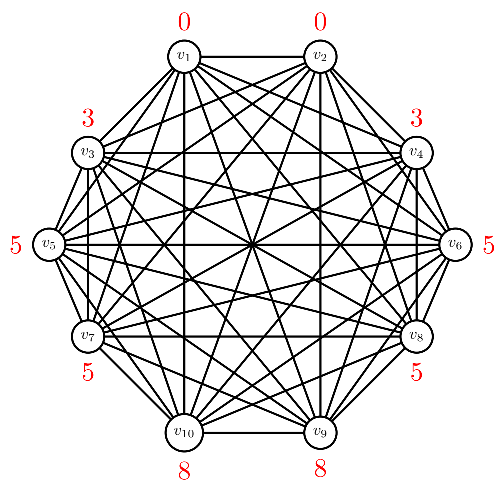
(i) For any board-pile polyomino X on n unit squares, $n \geq 1$, $f(X)$ is a period configuration on K_n.

(ii) For any period configuration C of K_n, there is some board-pile polyomino X on n unit squares such that $C = f(X)$.

(iii) For any two board-pile polyominos X and Y, if $X \neq Y$, then $f(X) \neq f(Y)$.



$$X = \{(0,2),(3,2),(2,4),(3,2)\}$$



$$\begin{aligned} f(X) &= \{0^2, (0+3)^2, (0+3+2)^4, (0+3+2+3)^2\} \\ &= \{0^2, 3^2, 5^4, 8^2\} \end{aligned}$$

Figure 7: Mapping a board-pile 10-omino to its corresponding configuration on K_{10}.

(i) We will first suppose that X is a board-pile n-omino and reach that $f(X)$ is a period configuration on the complete graph K_n. We will prove this by inducting on the number of h-strips in the board-pile n-omino. We begin with showing the basis cases of a polyomino having one h-strip or two h-strips hold true.

A board-pile polyomino with exactly one h-strip maps trivially to a period configuration on K_n. In this case, every stack size is equal to 0, as $f([(d_1, S_1)]) = \{0^{S_1}\}$, which is a period configuration with a period length of 1.

We now show that any board-pile polyomino with exactly two h-strips maps to a period configuration. Let $X = [(d_1, S_1), (d_2, S_2)]$ be such a board-pile polyomino. Recall that, by convention, $d_1 = 0$. So $f(X) = \{0^{|S_1|}, d_2^{|S_2|}\}$. Let V_1 be the $|S_1|$ vertices with stack size 0 and let V_2 be the $|S_2|$ vertices with stack size d_2.

By Lemma 2, in $f(X)$ the vertices of V_2 must have a common stack size such that $1 \leq d_2 \leq |S_1| + |S_2| - 1$. Without loss of generality, assume that the initial configuration for K_n is $C_0 = f(X)$. After the initial firing, the vertices of V_1 will each have $|S_2|$ chips, having just received from $|S_2|$ richer neighbors, and the vertices of V_2 will each have $d_2 - |S_1|$ chips, having just sent to $|S_1|$ poorer neighbors. Call this configuration C_1. By Lemma 1, we can normalize the resulting configuration by subtracting $d_2 - |S_1|$ from the stack size of every vertex in C_1, leaving $|S_1| + |S_2| - d_2$ chips on each of the vertices in V_1 and leaving 0 chips on each of the vertices in V_2. Note that since the x-distance from the least x-coordinate of S_1 to the greatest x-coordinate of S_2 is d_2, then the x-distance from the least x-coordinate of S_2 to the greatest x-coordinate of S_1 must, when added to d_2, equal the sum of the two h-strip lengths. So, the x-distance from the least x-coordinate of S_2 to the greatest x-coordinate of S_1 is $|S_1| + |S_2| - d_2$. To show that the relative sizes have changed and that in the second firing, the vertices of V_1 will send chips back to the vertices of V_2, we must show that $|S_2| + |S_1| - d_2 > 0$. We know that the maximum value that d_2 can take on is $|S_1| + |S_2| - 1$. So,

$$
\begin{aligned}
&|S_1| + |S_2| - d_2 \geq |S_1| + |S_2| - (|S_1| + |S_2| - 1), \text{ so} \\
&|S_1| + |S_2| - d_2 \geq |S_1| + |S_2| - |S_1| - |S_2| + 1, \text{ thus} \\
&|S_1| + |S_2| - d_2 \geq 1, \text{ therefore} \\
&|S_1| + |S_2| - d_2 > 0.
\end{aligned}
$$

Thus we can conclude that in C_1, the vertices of V_1 are richer than the vertices of V_2.

This gives us that C_1 is itself equal to $f(X')$ for some board-pile n-omino X'. In fact, since in $f(X')$, the difference between the two stack sizes is $|S_1| + |S_2| - d_2$ and every pair of vertices with common stack size in C_0 also have common stack size in C_1, X' is the board-pile n-omino created by reflecting X about the horizontal axis (see Figure 8).

Call the configuration after the second firing C_2. In C_2, the vertices of V_2 have $|S_1|$ chips and the vertices of V_1 have $|S_1| - d_2$ chips. By adding $d_2 - |S_1|$ to both totals (to counteract our subtracting of $d_2 - |S_1|$ chips previously), we get back where we started with the vertices of V_1 having 0 chips and the vertices of V_2 having d_2 chips. So $C_2 = C_0$ and we have that $f(X)$ is a period configuration.

Thus, the basis cases of our induction hold.

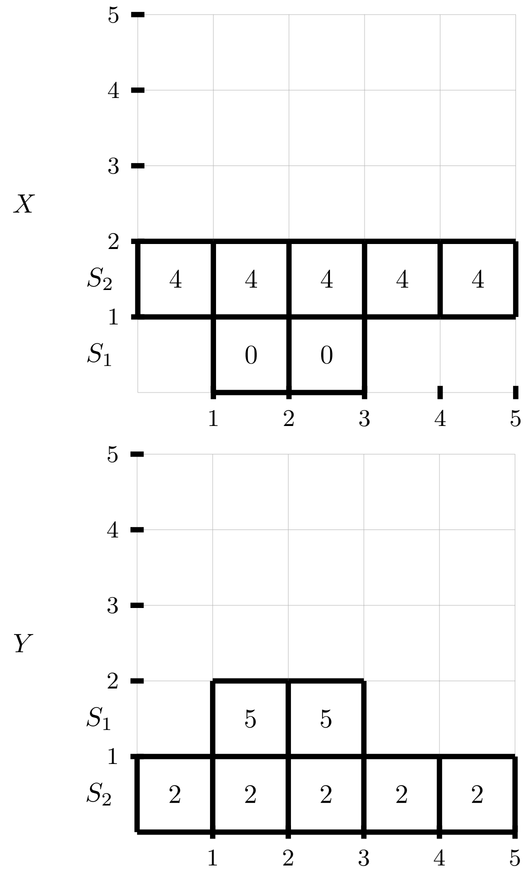


Figure 8: Board-pile polyominoes X and Y are reflections of each other over the x-axis. The complete graph configurations $f(X)$ and $f(Y)$ contain one vertex for every unit square and the vertices are given the stack sizes shown. The complete graph configuration $f(X)$ yields the configuration $f(Y)$ upon firing.

We now continue our induction on the number of h-strips to show that for all board-pile n-ominoes, X, the initial firing of $C_0 = f(X)$ yields $C_1 = f(X')$ (up to an addition of a constant to each of the stack sizes) where X' is the board-pile n-omino created by reflecting X about the horizontal axis. When $f(X')$ is fired, we will show that we again obtain $f(X)$ and this will imply that $f(X)$ is a period configuration of some complete graph.

Our inductive hypothesis is that for all board-pile polyominoes X with at most k h-strips, the initial firing of $f(X)$ yields $f(X')$ (up to an addition of a constant to each of the stack sizes) where X' is the board-pile polyomino created by reflecting X about the horizontal axis.

Now suppose we have a board-pile n-omino, $Y = [(d_1, S_1), \ldots, (d_{k+1}, S_{k+1}]$, with exactly $k+1$ h-strips. Then $f(Y)$ is some configuration on K_n. We know by our inductive hypothesis that if S_{k+1} were removed from Y, then the resulting polyomino would map to a period configuration on some complete graph.

Let $C_0 = f(Y)$. Recall that V_j is the set of $|S_j|$ vertices in C_0 with stack size $(\sum_{i=1}^{j} d_i)$, where $1 \leq j \leq k+1$.

Let W be the board-pile polyomino created by removing the h-strip, S_k and S_{k+1} from Y.

By our induction hypothesis, $f(W)$ when fired is equal to the configuration $f(W')$, where W' is obtained by reflecting the polyomino W about the horizontal axis.

Now consider adding S_k and S_{k+1} back to obtain Y and configuration $C_0 = f(Y)$. Consider the initial firing to obtain C_1.

The vertices of V_{k+1} and V_k in C_0 have greater stack size than all other vertices in V_i, $i < k$, and thus send a chip to all of those vertices. By Lemma 1, the addition of such vertices to a configuration cannot affect the relative stack sizes of the other vertices upon firing because each of the vertices in W will receive a chip from each vertex in V_k and V_{k+1}. We know from our base cases that the board-pile polyomino Z, composed of just the squares of the top two h-strips of Y, namely S_k and S_{k+1}, is such that the configuration $f(Z)$, when fired, obtains a new configuration $f(Z')$ where Z' is the polyomino Z reflected about the horizontal axis. That is, the vertices in V_k become richer than those in V_{k+1} after the initial firing. Thus in Y, after the initial firing when we obtain C_1, all vertices in V_k and V_{k+1} lose a chip to each vertex in V_i, $i < k$, which by Lemma 1 does not affect the diffusion process. Thus in C_1, the vertices of V_k are richer than those of V_{k+1}.

So in C_1, for every pair of vertices u and v, if $|u|_0 > |v|_0$, then $|u|_1 < |v|_1$. Therefore, $C_1 = f(Y')$ where Y' is the polyomino Y reflected about the horizontal axis.

(ii) We now present the second portion of the proof where we begin by supposing that C is a period configuration on K_n and reach that there exists some board-pile n-omino X such that $C = f(X)$.

Suppose the vertices of C have N distinct stack sizes, listed in increasing order,

$$\left\{0^{a_1}, \Big(\sum_{i=1}^{2} d_i\Big)^{a_2}, \ldots, \Big(\sum_{i=1}^{N} d_i\Big)^{a_N}\right\}$$

for some set $a_1, a_2, \ldots, a_N \in \mathbb{N}$ and some set $d_1 = 0, d_2, d_3, \ldots, d_N$ with $d_2, \ldots, d_N \in \mathbb{N}$. By Lemma 1, we can suppose at least one vertex in C has a stack size of 0 and that every stack size is non-negative.

For j, $1 \leq j \leq N$, let V_j be the set of all vertices in C with stack size $(\sum_{i=1}^{j} d_i)$. Let X be a collection of N h-strips on a plane with exactly one h-strip per row for $y = 1, 2, \ldots, N$. Call these h-strips $S_1, S_2, \ldots, S_N$, with S_i spanning y-coordinates $i-1$ to i for all $i \leq N$ and containing $|S_i|$ unit squares. Arrange these h-strips so that d_j is equal to the x-distance from the leftmost coordinate of S_{j-1} to the rightmost coordinate of S_j.

By the definition of board-pile polyomino, since X is a plane figure with a finite number of h-strips and exactly one h-strip per row, all that remains to be shown is that the unit squares of X are connected edge on edge in a single connected plane figure.

In particular, we must prove that $d_{j+1} \leq |V_j| + |V_{j+1}| - 1$ for all j such that $1 \leq j \leq N-1$. By Lemma 2, this will imply that every pair of h-strips $|S_j|$ and $|S_{j+1}|$ are connected edge on edge, proving that X is a single board-pile polyomino rather than a number of disconnected board-pile polyominoes in the plane.

By contradiction, suppose $d_{j+1} > |V_j| + |V_{j+1}| - 1$ for some j such that $1 \leq j \leq N-1$. Let $C = C_0$ be the initial configuration on K_n. Following the initial firing, the vertices of V_j each have $(\sum_{i=1}^{j} d_i) + |V_{j+1}| + \delta$ chips, where δ represents the difference between the number of vertices richer than those in V_{j+1} and the number of vertices poorer than those in V_j. Also, following the initial firing, the vertices of V_{j+1} each have $(\sum_{i=1}^{j+1} d_i) - |V_j| + \delta$ chips. Since C is a period configuration, we get that

$$
\begin{aligned}
(\sum_{i=1}^{j} d_i) + |V_{j+1}| + \delta &> (\sum_{i=1}^{j+1} d_i) - |V_j| + \delta \\
|V_{j+1}| &> d_{j+1} - |V_j|. \qquad (1)
\end{aligned}
$$

Together with $d_{j+1} > |V_j| + |V_{j+1}| - 1$, we will reach a contradiction. We have

$$
\begin{aligned}
d_{j+1} &> |V_j| + |V_{j+1}| - 1 \\
d_{j+1} - |V_j| &> |V_j| + |V_{j+1}| - 1 - |V_j| \\
d_{j+1} - |V_j| &> |V_{j+1}| - 1. \qquad (2)
\end{aligned}
$$

Combining inequalities (1) and (2), we get that some integer, $d_{j+1} - |V_j|$, is less than the integer $|V_{j+1}|$ and greater than $|V_{j+1}| - 1$. This is a contradiction. Thus for all period configurations C, $C = f(X)$, for some board-pile polyomino X.

(iii) For the final portion of the proof, we begin by supposing that X and Y are board-pile polyominoes and that $X \neq Y$. Suppose, by way of contradiction, that $f(X) = f(Y)$ and call this configuration C.

Let $X = \{(d_1, |S_1|), (d_2, |S_2|), \ldots (d_N, |S_N|)\}$, and let
$Y = \{(d'_1, |S'_1|), (d'_2, |S'_2|), \ldots (d'_{N'}, |S'_{N'}|)\}$.

Since $C = f(X) = f(Y)$, the length of each h-strip in ascending order must be equal in X and Y, that is $|S_i| = |S'_i|$ for $1 \leq i \leq N$. It follows that in order to reach a contradiction, we only need to show that the corresponding d_i values in X and Y are equal for $1 \leq i \leq N$.

By convention, $d_1 = d'_1 = 0$ for both X and Y. By induction, we now suppose that $d_i = d'_i$ for $0 \leq i \leq k-1$ are equal in X and Y. By the definition of f, $d_k = d'_k$ since there are $|S_k| = |S'_k|$ vertices in C with stack size equal to $\sum_{i=0}^{k-1} d_i$. Therefore, the corresponding d_i values in X and Y are equal, which is a contradiction since $X \neq Y$. Therefore $f(X) \neq f(Y)$.

Therefore, f is a bijection and thus the set of all board-pile n-ominoes has the same cardinality as the set of all period configurations on an unlabelled K_n up to equivalence. □

Since we have shown f to be bijective, we can now count the number of period configurations of unlabelled complete graphs with a given number of vertices by using previous results regarding board-pile polyominoes.

Theorem 4 ([5], [9]). The number of board-pile polyominoes with n unit squares follows the recurrence relation $a_n = 5a_{n-1} - 7a_{n-2} + 4a_{n-3}$ for $n \geq 5$ with initial values $a_1 = 1$, $a_2 = 2$, $a_3 = 6$, and $a_4 = 19$.

Thus from Theorem 3 we have the following.

Theorem 5. The number of period configurations of a complete graph on n vertices follows the recurrence relation $a_n = 5a_{n-1} - 7a_{n-2} + 4a_{n-3}$ for $n \geq 5$ with initial values $a_1 = 1$, $a_2 = 2$, $a_3 = 6$, and $a_4 = 19$.

Corollary 1 ([5]). The sequence (a_n), the number of period configurations of complete graphs (up to equivalence), has a growth rate of approximately 3.2.

5. Open Problems

One of the main open problems in this area is the enumeration of pre-period and period configurations. In this paper, we found a relationship between board-pile polyominoes and period configurations of K_n, which allowed us to count the number of period configurations. Perhaps there are other combinatorial objects related to the diffusion process on other families of graphs that will aid in the enumeration of their pre-period and period configurations as well.

Prior to this result using polyominoes, the only results for counting period configurations in diffusion were for paths and stars [8]. It remains an open problem to count all of the period configurations up to equivalence on many other families of graphs like cycles, trees, and complete bipartite graphs. However, there also remain many questions about pre-period configurations that have not been answered.

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