PYRAMID NIM

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Abstract

Pyramid Nim is played on a directed acyclic graph. Players remove vertices of a path of undominated vertices. We determine Grundy-values for some small games of Pyramid Nim, and Grundy-values for a special class of directed acyclic graphs called triangular pyramids. The rules of the game are quite simple, and the analysis in general may be difficult. These two properties make Pyramid Nim an appealing game.

1. Introduction and Preliminaries

Combinatorial game theory (CGT) developed in the context of recreational mathematics. In their seminal work and with a spirit of playfulness, Berlekamp, Conway and Guy [3, 6] established the mathematical framework from which games of complete information could be studied. The power of this theory would soon become apparent and was utilized by many researchers (see Fraenkel's bibliography [7]). Along with its natural appeal, combinatorial game theory has applications to complexity theory, logic, and biology. Literature on the subject continues to increase and the interested reader can find comprehensive introductions to CGT in [2, 3, 6, 13]. Additional research articles with a theoretical flavor can be found in [1, 8, 9, 10, 11].

We first recall some basic concepts from CGT which are used in this paper. Terms which are not explicitly defined can be found in [13]. A *combinatorial game* is one of complete information and no element of chance is involved in gameplay. Each player is aware of the game position at any point in the game. Under *normal play*, two players (P1 and P2) alternate taking turns and a player loses when he cannot make a move. An *impartial* combinatorial game is one where both players have the same options from any position. A *finite* game eventually terminates (with a winner and a loser, no draws allowed). It is understood that P1 makes the first move in any combinatorial game.

For any finite impartial combinatorial game Γ, there is an associated non-negative integer value (*Grundy-value*) $Gr(\Gamma)$. The Grundy-value $Gr(\Gamma)$ immediately tells us if Γ is a $\mathcal{P}$-*position* (previous player win) or an $\mathcal{N}$-*position* (next player win). In particular, $Gr(\Gamma) = 0$ if and only if Γ is a $\mathcal{P}$-position. To compute $Gr(\Gamma)$, we need the following definitions.

Definition 1. The *minimum excluded value* (or *mex*) of a multiset of non-negative integers is the smallest non-negative integer which does not appear in the multiset. This is denoted by $\text{mex}\{t_1, t_2, t_3, \ldots, t_k\}$.

Definition 2. Let Γ be a finite impartial game. Then, the *Grundy-value* of Γ (denoted by $Gr(\Gamma)$) is defined to be

$$Gr(\Gamma) = \text{mex}\{Gr(\Delta) : \Delta \text{ is an option of } \Gamma\}.$$

The *sum* of finite impartial games is the game obtained by placing the individual games, side by side. On a player's turn, a move is made in a single summand. Under normal play, the last person to make a move wins. For any finite impartial game $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_k$, the Grundy-value of Γ is computed in the following way. First, convert $Gr(\gamma_i)$ into binary. Then, compute $\bigoplus Gr(\gamma_i)$, where the sum is BitXor (Nim-addition). Finally, convert this value back into a nonnegative integer.

Example 1. Suppose that γ_1, γ_2 and γ_3 are finite impartial games with $Gr(\gamma_1) = 1$, $Gr(\gamma_2) = 2$ and $Gr(\gamma_3) = 3$. Then the game $\Gamma = \gamma_1 + \gamma_2 + \gamma_3$ has Grundy-value

$$Gr(\Gamma) = 01 \oplus 10 \oplus 11 = 00,$$

and thus has Grundy-value 0. $\diamondsuit$

2. Pyramid Nim

In 1902, Bouton [5] gave a beautiful mathematical analysis and complete solution for Nim. Since then, many variations of Nim have been investigated. Within the literature, analyses on Nim variants with modified rule sets, Nim played on different configurations (circular, triangular and rectangular), and Nim played on graphs can be found. As of this writing, a keyword search for Nim yields 135 entries in the MathSciNet database.

Here is how Pyramid Nim is played. For general graph-theoretic definitions, we refer the reader to [4]. Let D be a directed acyclic graph. A *source* in D is a vertex of indegree zero. A *sink* in D is a vertex with outdegree zero. We say that D is *weakly-connected* if the undirected graph that results from removing the orientations from the arcs of D is a connected graph. If a directed acyclic graph D has more than one vertex, D cannot be strongly-connected. Hence, there will be no confusion to say that D is *connected* if it is weakly-connected. A subdigraph H of D is *undominated* if there are no pairs of vertices x and y with x in $V(H)$ and y in $V(D) - V(H)$ such that yx is an arc of D. Two players play Pyramid Nim on D by alternately removing the vertices of an undominated directed path P, where P has at least one vertex. A player loses when there is no move remaining.

We note the following:

- Pyramid Nim is an impartial game which has to end after at most $|V(D)|$ moves.
- If the digraph D has connected components H_1, H_2, …, H_k, then $Gr(D) = Gr(H_1) \oplus Gr(H_2) \oplus \cdots \oplus Gr(H_k)$.

Definition 3. The *triangular pyramid of height n*, denoted by T_n, is the directed acyclic graph, with n squares in the bottom row, that has vertices representing the squares in a 2-dimensional pyramid with a directed edge from square A to square B if square A sits partially on top of square B.

Example 2. The triangular pyramid T_3 of height three is illustrated in Figure 1. There are three possible moves from T_3, up to isomorphism. These three possible moves from T_3 in Pyramid Nim are shown in Figure 2. The removed squares in each move are shaded in blue. ◊

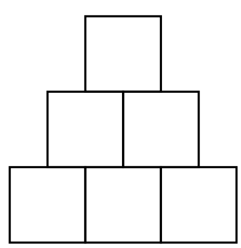

Figure 1: The triangular pyramid T_3 of height three.

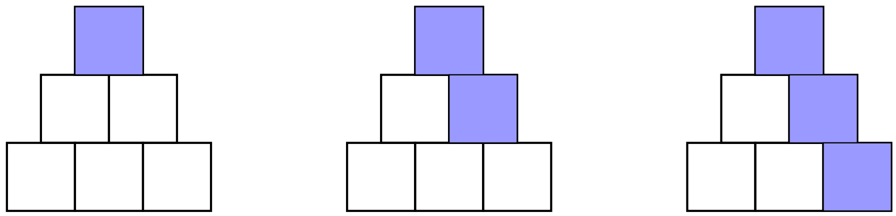

Figure 2: The three possible moves in Pyramid Nim from T_3, up to isomorphism. The removed squares are shaded in blue.

Observation 1. If every connected component of D is a path, then Pyramid Nim on D is equivalent to regular Nim.

Observation 2. If D has a single sink t and every other vertex has indegree at most one, then ignoring t until the very last move results in a game of Nim. When the indegree of t is reduced to one, the next player should take the remaining path.

The strategy for the directional dual is only a bit more complicated and we can determine the Grundy-value of a position in $\mathcal{O}(|V(D)|)$ time.

Observation 3. If D has a single source s, and every other vertex has outdegree at most one, then removing s leaves a game of Nim, and the first player to play from D can leave a $\mathcal{P}$-position by selecting an appropriate directed path. Thus, D is an $\mathcal{N}$-position.

In fact, we may calculate the Grundy-value of a singe source directed acyclic graph such that every other vertex has outdegree at most one.

Theorem 1. *Let D be a directed acyclic graph with a single source such that every other vertex has outdegree at most one. Suppose the maximal paths from the source have a_1+1, a_2+1, $\ldots$, $a_{k-1}+1$, and a_k+1 vertices. Let*

$$s = a_1 \oplus a_2 \oplus \cdots \oplus a_k \quad \textit{and}$$
$$S = \{s\} \cup \{s \oplus a_j \oplus i : 1 \leqslant j \leqslant k, 0 \leqslant i < a_j\}.$$

Then, $Gr(D) = \operatorname{mex}(S)$.

Proof. Let t be the source of D, and, for each $1 \leqslant j \leqslant k$, let P_j denote the maximal path from the source that has a_j+1 vertices. The possible moves from D are either

- remove the source t, or
- for some $1 \leqslant j \leqslant k$ and $1 \leqslant \ell \leqslant a_j$, remove the $\ell+1$ vertices on path P_j that are closest to the source (including the source).

When we remove source t, the resultant Grundy-value is $s = a_1 \oplus a_2 \oplus \cdots \oplus a_k$. Consider the move of removing the $\ell+1$ vertices on path P_j that are closest to the source for some $1 \leqslant j \leqslant k$ and $1 \leqslant \ell \leqslant a_j$. Let $i = a_j - \ell$. Then, we have i vertices left on path P_j after applying this move where $0 \leqslant i < a_j$. Hence, the Grundy-value of this move is

$$a_1 \oplus a_2 \oplus \cdots \oplus a_{j-1} \oplus i \oplus a_{j+1} \oplus \cdots \oplus a_k = s \oplus a_j \oplus i.$$

Therefore,

$$Gr(D) = \text{mex}\big(\{s\} \cup \{s \oplus a_j \oplus i : 1 \leqslant j \leqslant k, 0 \leqslant i < a_j\}\big) = \text{mex}(S).$$

□

3. Balanced Complete Binary Trees

We consider Pyramid Nim on a category of trees called balanced complete binary trees.

Definition 4. The *balanced complete binary tree of height* n is defined recursively by

1. B_0 is a single-vertex directed acyclic graph, and
2. B_{n+1} is constructed from two disjoint copies of B_n by adding a new source with an arc to the source of each copy of B_n.

Remark 1. Alternatively, we may define B_n as the directed acyclic graph with vertex set

$$V(B_n) = \{1, 2, \dots, 2^{n+1} - 1\}$$

and arc set

$$A(B_n) = \{(j, 2j), (j, 2j+1) : 1 \leqslant j \leqslant 2^n - 1\}.$$

We will show that $Gr(B_n)$ is the highest power of 2 that divides $n+1$. In order to demonstrate this result, we introduce the following defining property of the sequence of highest powers of 2 in $n+1$, as n ranges over all non-negative integers.

Definition 5. For any non-negative integer n, let q_n be the highest power of 2 in $n+1$. We write $n+1 = q_n F_n$, where $q_n = 2^{t_n}$ for some non-negative integer t_n and F_n is an odd positive integer.

Lemma 1. *We have* $q_0 = 1$. *Also, let* n *be a non-negative integer. We have*

$$q_{2^n+k} = q_k \text{ for any integer } 0 \leqslant k < 2^n - 1, \text{ and}$$
$$q_{2^n+k} = 2q_k \text{ for } k = 2^n - 1.$$

Proof. Let $0 \leqslant k < 2^n - 1$. Since $k+1 = q_k F_k$, we have $q_{2^n+k} F_{2^n+k} = 2^n + k + 1 = 2^n + q_k F_k$. Thus, $2^{t_{2^n+k}} F_{2^n+k} = 2^n + 2^{t_k} F_k = 2^{t_k}(2^{n-t_k} + F_k)$. Since $n > t_k$, $t_{2^n+k} = t_k$ and $F_{2^n+k} = 2^{n-t_k} + F_k$. Therefore, $q_{2^n+k} = 2^{t_{2^n+k}} = 2^{t_k} = q_k$.

Let $k = 2^n - 1$. Then, $q_{2^n+k} = 2^{t_{2^n+k}} = 2^{n+1} = 2 \times 2^{t_k} = 2q_k$. □

We use Lemma 1 to demonstrate that the Grundy-value of B_k is the highest power of 2 that divides $k+1$.

Theorem 2. *For any non-negative integer k, $Gr(B_k)$ is the highest power of 2 that divides $k+1$.*

Proof. We use a double induction argument. For convenience, we let $\beta_k = Gr(B_k)$. First, we observe that $\beta_0 = 1$. If we remove the source of B_k, the resultant digraph $B_k - s$ has two connected components, each of which is a copy of B_{k-1}. Thus, $Gr(B_k - s) = 0$. If we remove an undominated path P on i vertices, for some $2 \leqslant i \leqslant k$, from B_k, the resultant digraph $B_k - V(P)$ has $i+1$ connected components consisting of one copy of each of $B_{k-1}, B_{k-2}, \ldots, B_{k-i+1}$ and two copies of B_{k-i}. Thus, $Gr(B_k - V(P)) = \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \oplus \beta_{k-i+1}$. If P is an undominated path on $k+1$ vertices, the digraph $B_k - V(P)$ has k connected components consisting of one copy of each of $B_{k-1}, B_{k-2}, \ldots, B_0$. Thus, $Gr(B_k - V(P)) = \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \oplus \beta_0$. Hence, $\beta_k = \text{mex}\{0, \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k-1\}$. In particular, $\beta_{2^0+0} = \beta_1 = \text{mex}\{0, \beta_0\} = 2 = 2\beta_0$. This establishes the base case.

Suppose n is a positive integer and k is an integer where $0 \leqslant k \leqslant 2^n - 1$ such that, for all integers $0 \leqslant n' < n$, we have

- $\beta_{2^{n'}+k'} = \beta_{k'}$ for all integers $0 \leqslant k' < 2^{n'} - 1$,
- $\beta_{2^{n'}+k'} = 2\beta_{k'}$ for $k' = 2^{n'} - 1$, and
- $\beta_{2^n+j} = \beta_j$ for all integers $0 \leqslant j < k$.

We want to show that $\beta_{2^n+k} = \beta_k$ if $0 \leqslant k < 2^n - 1$, and $\beta_{2^n+k} = 2\beta_k$ if $k = 2^n - 1$. Since β_j satisfies the property of q_j in Lemma 1 for all integers $0 \leqslant j < 2^n + k$, β_j is the highest power of 2 that divides $j+1$ for all integers $0 \leqslant j < 2^n + k$. In particular, β_{2^n-1} is the highest power of 2 that divides 2^n. Thus,

$$\beta_{2^n-1} = 2^n. \tag{1}$$

Let $k_n = 2^n - 1$. By (1), we have $\beta_{k_n} = 2^n$. Let

$$S = \{0, \beta_{k_n-1} \oplus \beta_{k_n-2} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k_n - 1\}.$$

Then, $\beta_{k_n} = \text{mex}(S)$. Since S has at most $k_n + 1 = 2^n$ distinct elements and $\beta_{k_n} = \text{mex}(S) = 2^n$, S is a permutation on the set of non-negative integers $\{0, 1, \ldots, 2^n - 1\}$. Thus, $S = \{i : 0 \leqslant i \leqslant 2^n - 1\}$.

Suppose $k = 0$. We need to show that $\beta_{2^n+0} = \beta_{k_n+1} = 1 = \beta_0$. Since $\beta_{k_n} = 2^n$ and $S = \{i : 0 \leqslant i \leqslant 2^n - 1\}$, we have

$$\begin{aligned}\beta_{2^n+0} &= \operatorname{mex}\{0, \beta_{k_n} \oplus \beta_{k_n-1} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k_n\}\\ &= \operatorname{mex}\{0, \beta_{k_n} \oplus \alpha : \alpha \in S\}\\ &= \operatorname{mex}\{0, 2^n \oplus i : 0 \leqslant i \leqslant 2^n - 1\}\\ &= \operatorname{mex}\{0, j : 2^n \leqslant j \leqslant 2^{n+1} - 1\} = 1 = \beta_0.\end{aligned}$$

Suppose $0 < k < 2^n - 1 = k_n$. Let $\widehat{\beta}_k = \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \oplus \beta_0$. Since $2^n \nmid j + 1$ for all $0 \leqslant j \leqslant k_n - 1$, we have $\beta_j = 2^{t_j} < 2^n$ for all $0 \leqslant j \leqslant k_n - 1$. Thus, $\widehat{\beta}_k = \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \oplus \beta_0 < 2^n$. By the induction hypothesis,

$$\beta_{2^n+k-1} \oplus \beta_{2^n+k-2} \oplus \cdots \oplus \beta_{2^n+j} = \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \oplus \beta_j \tag{2}$$

for all $0 \leqslant j < k$ and

$$\begin{aligned}\beta_{2^n+k-1} \oplus \beta_{2^n+k-2} \oplus \cdots \oplus \beta_{2^n} \oplus \beta_{k_n} \oplus \cdots \oplus \beta_j &\\ = \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \beta_0 \oplus \beta_{k_n} \oplus \cdots \oplus \beta_j &\\ = \widehat{\beta}_k \oplus \beta_{k_n} \oplus \cdots \oplus \beta_j &\end{aligned} \tag{3}$$

for all $0 \leqslant j \leqslant k_n$.

By (2), we have

$$\begin{aligned}&\operatorname{mex}\{0, \beta_{2^n+k-1} \oplus \beta_{2^n+k-2} \oplus \cdots \oplus \beta_{2^n+j} : 0 \leqslant j \leqslant k - 1\}\\ &\quad = \operatorname{mex}\{0, \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k - 1\} = \beta_k.\end{aligned} \tag{4}$$

There are at most $k < 2^n - 1$ distinct non-negative integers in the list

$$\beta_{k-1}, \beta_{k-1} \oplus \beta_{k-2}, \ldots, \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \oplus \beta_0. \tag{5}$$

Thus, there exists a positive integer $r < 2^n$ missing from (5). Hence,

$$\beta_k = \operatorname{mex}\{0, \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k - 1\} \leqslant r < 2^n. \tag{6}$$

Since $S = \{i : 0 \leqslant i \leqslant 2^n - 1\}$ and $\beta_{k_n} = 2^n$, we have

$$\begin{aligned}\{\beta_{k_n} \oplus \beta_{k_n-1} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k_n\} &= \{\beta_{k_n} \oplus \alpha : \alpha \in S\}\\ &= \{2^n \oplus i : 0 \leqslant i \leqslant 2^n - 1\}\\ &= \{j : 2^n \leqslant j \leqslant 2^{n+1} - 1\}.\end{aligned}$$

Since $\widehat{\beta}_k < 2^n$,

$$\{\widehat{\beta}_k \oplus \beta_{k_n} \oplus \beta_{k_n-1} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k_n\} = \{\widehat{\beta}_k \oplus j : 2^n \leqslant j \leqslant 2^{n+1} - 1\} \tag{7}$$

is a permutation on the set of integers $\{2^n, 2^n+1\ldots, 2^{n+1}-1\}$. By (3) and (7), we have

$$\{\beta_{2^n+k-1} \oplus \beta_{2^n+k-2} \oplus \cdots \oplus \beta_{2^n} \oplus \beta_{k_n} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k_n\} \\ = \{j : 2^n \leqslant j \leqslant 2^{n+1}-1\}. \tag{8}$$

By (4), (6) and (8), we have

$$\begin{aligned}\beta_{2^n+k} &= \text{mex}\{0, \beta_{2^n+k-1} \oplus \beta_{2^n+k-2} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant 2^n+k-1\} \\ &= \text{mex}\big(\{0, \beta_{2^n+k-1} \oplus \beta_{2^n+k-2} \oplus \cdots \oplus \beta_{2^n+j} : 0 \leqslant j \leqslant k-1\} \\ &\cup \{\beta_{2^n+k-1} \oplus \beta_{2^n+k-2} \oplus \cdots \oplus \beta_{2^n} \oplus \beta_{k_n} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k_n\}\big) \\ &= \text{mex}\big(\{0, \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k-1\} \\ &\cup \{j : 2^n \leqslant j \leqslant 2^{n+1}-1\}\big) \\ &= \text{mex}\{0, \beta_{k-1} \oplus \beta_{k-2} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k-1\} = \beta_k.\end{aligned}$$

Suppose $k = 2^n - 1 = k_n$. By the induction hypothesis, for all $0 \leqslant j < k_n$, $\beta_{2^n+j} = \beta_j$. We want to show that $\beta_{2^n+k_n} = 2\beta_{k_n}$. An argument similar to the one above shows that

$$\begin{aligned}&\{0, \beta_{2^n+k_n-1} \oplus \beta_{2^n+k_n-2} \oplus \cdots \oplus \beta_{2^n+j} : 0 \leqslant j \leqslant k_n-1\} \\ &\quad = \{0, \beta_{k_n-1} \oplus \beta_{k_n-2} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k_n-1\} \\ &\quad = S = \{i : 0 \leqslant i \leqslant 2^n-1\}\end{aligned}$$

and

$$\{\beta_{2^n+k_n-1} \oplus \beta_{2^n+k_n-2} \oplus \cdots \oplus \beta_{2^n} \oplus \beta_{k_n} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k_n\} \\ = \{j : 2^n \leqslant j \leqslant 2^{n+1}-1\}.$$

Hence,

$$\begin{aligned}\beta_{2^n+k_n} &= \text{mex}\{0, \beta_{2^n+k_n-1} \oplus \beta_{2^n+k_n-2} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant 2^n+k_n-1\} \\ &= \text{mex}\big(\{0, \beta_{2^n+k_n-1} \oplus \beta_{2^n+k_n-2} \oplus \cdots \oplus \beta_{2^n+j} : 0 \leqslant j \leqslant k_n-1\} \\ &\cup \{\beta_{2^n+k_n-1} \oplus \beta_{2^n+k_n-2} \oplus \cdots \oplus \beta_{2^n} \oplus \beta_{k_n} \oplus \cdots \oplus \beta_j : 0 \leqslant j \leqslant k_n\}\big) \\ &= \text{mex}\{j : 0 \leqslant j \leqslant 2^{n+1}-1\} = 2^{n+1} = 2\beta_{k_n}.\end{aligned}$$

Since β_k satisfies the property of q_k in Lemma 1, $\beta_k = Gr(B_k)$ is the highest power of 2 that divides $k+1$. □

4. Grundy-values of Truncated T_n

We consider the triangular pyramid of height n with some of the top rows removed.

Definition 6. Let T_n^j denote the triangular pyramid of height n with the top j rows removed.

We determine the Grundy-values of T_n with the top $n-1$ and $n-2$ rows removed.

Theorem 3. *Let n be a positive integer. Then,*

$$Gr(T_n^{n-1}) = 1 \textit{ if } n \textit{ is odd, and}$$
$$Gr(T_n^{n-1}) = 0 \textit{ if } n \textit{ is even.}$$

Proof. Since T_1 is a single vertex, it has Grundy-value 1. Note that T_n^{n-1} corresponds to the graph made up of n disjoint vertices. If n is even, then the BitXoR of $1 \oplus \cdots \oplus 1$ (n terms) is 0, and thus $Gr(T_n^{n-1}) = 0$. If n is odd, we see that $Gr(T_n^{n-1}) = 1$. □

Theorem 4. *Let $n \geqslant 2$ be an integer. Then,*

$$Gr(T_n^{n-2}) = 0 \textit{ if } n \textit{ is odd, and}$$
$$Gr(T_n^{n-2}) = 2 \textit{ if } n \textit{ is even.}$$

Proof. It is straightforward to see that the Grundy-values of T_1 and $T_2^0 = T_2$ are 1 and 2, respectively. Let $n \geqslant 2$. The base case for the induction proof has already been established. Now, assume that the claim of the theorem holds for all $n \leqslant k$. Let us consider $T_{k+1}^{k+1-2} = T_{k+1}^{k-1}$. There are two cases to consider.

Case 1. Assume k is even. Remove an end vertex from the top row. This results in a disjoint vertex, along with a T_k^{k-2}. The Grundy-value of this position is $1 \oplus 2 = 3$ by the inductive hypothesis. Removing an end vertex from the top row and the resultant undominated vertex in the bottom row yields T_k^{k-2}, which has Grundy-value 2. If we remove an interior vertex from the top row of T_{k+1}^{k-1}, the resultant graph is $T_j^{j-2} + T_{k+1-j}^{k-1-j}$ for some integer $2 \leqslant j \leqslant k-1$. Since $j + (k+1-j) = k+1$ is odd, j and $k+1-j$ have opposite parity. Thus, one Grundy-value is 0 and the other is 2. Hence, $Gr(T_j^{j-2}) \oplus Gr(T_{k+1-j}^{k-1-j}) = 0 \oplus 2 = 2$. Thus, for k even, $Gr(T_{k+1}^{k-1}) = \text{mex}\{2, 3\} = 0$.

Case 2. Assume k is odd. Remove an end vertex from the top row. This results in a disjoint vertex, along with a T_k^{k-2}. The Grundy-value of this position is $1 \oplus 0 = 1$ by the inductive hypothesis. Removing an end vertex from the top row and the resultant undominated vertex in the bottom row yields T_k^{k-2}, which has Grundy-value 0. If we remove an interior vertex from the top row of T_{k+1}^{k-1}, the resultant graph is $T_j^{j-2} + T_{k+1-j}^{k-1-j}$ for some integer $2 \leqslant j \leqslant k-1$. Since $j + (k+1-j) = k+1$ is even, j and $k+1-j$ have the same parity. Thus, they each have the same Grundy-value of $\beta = 0$ or $\beta = 2$. Hence, $Gr(T_j^{j-2}) \oplus Gr(T_{k+1-j}^{k-1-j}) = \beta \oplus \beta = 0$. Thus, for k odd, $Gr(T_{k+1}^{k-1}) = \text{mex}\{0, 1\} = 2$. □

We introduce the Pyramid Nim signature of n and k in order to determine the Grundy-value of T_n^{n-3}.

Definition 7. Let $n, k \geqslant 0$ be integers. The *Pyramid Nim signature of* n *and* k is given by

$$\sigma(n,k) = 2\big((n+1) - 2\lfloor (n+1)/2 \rfloor\big) + \big(k - 2\lfloor k/2 \rfloor\big).$$

Note that,

$$\sigma(n,k) = 1 \text{ if } n \text{ is odd and } k \text{ is odd,} \tag{9}$$

$$\sigma(n,k) = 3 \text{ if } n \text{ is even and } k \text{ is odd,} \tag{10}$$

$$\sigma(n,k) = 0 \text{ if } n \text{ is odd and } k \text{ is even, and} \tag{11}$$

$$\sigma(n,k) = 2 \text{ if } n \text{ is even and } k \text{ is even.} \tag{12}$$

Lemma 2. *For all integers $n, k \geqslant 0$, we have*

$$\begin{aligned} \sigma(n,k-1) &= \sigma(n,k) \oplus 1, \\ \sigma(n-1,k) &= \sigma(n,k) \oplus 2, \ \textit{and} \\ \sigma(n-1,k-1) &= \sigma(n,k) \oplus 3. \end{aligned}$$

Proof. When n is odd, we have $\sigma(n,k-1) = \sigma(n,k) \oplus 1$ by (9) and (11). Also, when n is even, we have $\sigma(n,k-1) = \sigma(n,k) \oplus 1$ by (10) and (12). A similar argument shows that $\sigma(n-1,k) = \sigma(n,k) \oplus 2$ and $\sigma(n-1,k-1) = \sigma(n,k) \oplus 3$. □

Definition 8. Let $n \geqslant 2$ and $0 \leqslant k \leqslant n-2$ be integers. We let B_n^k denote the collection of connected subdigraphs D of T_n with the property that

1. all vertices of D are on the bottom 3 rows of T_n,
2. D has k vertices on the third row from the bottom,
3. D has $n-1$ vertices on the second row from the bottom, and
4. D has n vertices on the bottom row.

We will show that any digraph that lies in the set B_n^k share the same Grundy-value. For convenience, we will use the symbol B_n^k to represent any of the digraphs in B_n^k.

Example 3. There are two digraphs that lie in the set B_5^2, up to isomorphism. These two digraphs are illustrated in Figure 3. ♢

Remark 2. We observe that the Pyramid Nim signature of n and k is defined for all non-negative integers n and k in Definition 7, and Lemma 2 holds for all non-negative integers n and k. However, we only make use of the Pyramid Nim signature of n and k for integers $n \geqslant 2$ and $0 \leqslant k \leqslant n-2$ in Proposition 1.

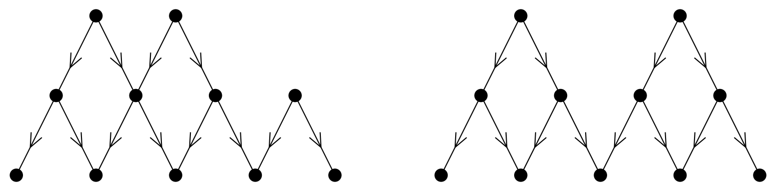

Figure 3: The two digraphs that lie in B_5^2, up to isomorphism.

Proposition 1. *Let $n \geqslant 2$ and $0 \leqslant k \leqslant n-2$ be integers. Then, $Gr(B_n^k) = \sigma(n,k)$.*

Proof. The proof will be by induction on ordered pairs of integers (n,k) from the set $\{(n,k) \in \mathbb{Z} \times \mathbb{Z} : n \geqslant 2 \text{ and } 0 \leqslant k \leqslant n-2\}$ placed into lexicographic order. By Theorem 4, we have $Gr(B_{2m}^0) = Gr(T_{2m}^{2m-2}) = 2$ and $Gr(B_{2m+1}^0) = Gr(T_{2m+1}^{2m-1}) = 0$. This establishes the base case.

Suppose there are integers $n \geqslant 3$ and $0 < k \leqslant n-2$ such that

- for all integers $n' < n$ and $0 \leqslant k' \leqslant n'-2$, $Gr(B_{n'}^{k'}) = \sigma(n',k')$, and
- for all integers $0 \leqslant k' < k$, $Gr(B_n^{k'}) = \sigma(n,k')$.

We want to show that $Gr(B_n^k) = \sigma(n,k)$. In particular, we want to show that

$$Gr(B_n^k) = 1 \text{ if } n \text{ is odd and } k \text{ is odd,}$$
$$Gr(B_n^k) = 3 \text{ if } n \text{ is even and } k \text{ is odd,}$$
$$Gr(B_n^k) = 0 \text{ if } n \text{ is odd and } k \text{ is even, and}$$
$$Gr(B_n^k) = 2 \text{ if } n \text{ is even and } k \text{ is even.}$$

If cell t, $1 < t < n-1$, of the second row is undominated, we can delete this cell, leaving a position P that is a disjoint union of B_t^i and B_{n-t}^j for some i and j with $i+j=k$. If cell t, $1 < t < n-1$, of the second row is dominated by only one cell x in the top row, we can delete the cells x and t, leaving a position P that is a disjoint union of B_t^i and B_{n-t}^j for some i and j with $i+j=k-1$. In either case, $Gr(P) = Gr(B_t^i) \oplus Gr(B_{n-t}^j)$.

If n is even, then t and $n-t$ have the same parity. Thus, $Gr(P) = Gr(B_t^i) \oplus Gr(B_{n-t}^j) \in \{2\oplus 2, 2\oplus 3, 3\oplus 3, 0\oplus 0, 0\oplus 1, 1\oplus 1\} = \{0,1\}$. Since $\sigma(n,k) \in \{2,3\}$, $Gr(P) \neq \sigma(n,k)$.

If n is odd, then t and $n-t$ have opposite parity. Thus, $Gr(P) = Gr(B_t^i) \oplus Gr(B_{n-t}^j) \in \{2\oplus 0, 2\oplus 1, 3\oplus 0, 3\oplus 1\} = \{2,3\}$. Since $\sigma(n,k) \in \{0,1\}$, $Gr(P) \neq \sigma(n,k)$.

Note that when we only take one cell from the top row of B_n^k, we are left with B_n^{k-1}. By the induction hypothesis, $Gr(B_n^{k-1}) = \sigma(n,k-1)$. By Lemma 2, $Gr(B_n^{k-1}) = \sigma(n,k) \oplus 1$.

We let t be the first cell of the second row of B_n^k. We consider the cases t is undominated and t is dominated individually.

First, suppose t is undominated. On one hand, removing t and the first cell in the bottom row leaves a position P with $Gr(P) = Gr(B_{n-1}^k)$. By the induction hypothesis, $Gr(B_{n-1}^k) = \sigma(n-1,k)$. By Lemma 2, $Gr(B_{n-1}^k) = \sigma(n,k) \oplus 2$. On the other hand, removing t leaves a position P that is a disjoint union of B_{n-1}^k and T_1. Thus, $Gr(P) = Gr(B_{n-1}^k) \oplus Gr(T_1) = (\sigma(n,k) \oplus 2) \oplus 1 = \sigma(n,k) \oplus 3$.

Next, suppose t is dominated by a cell y. On one hand, removing y, t, and the first cell in the bottom row leaves a position P with $Gr(P) = Gr(B_{n-1}^{k-1})$. By the induction hypothesis, $Gr(B_{n-1}^{k-1}) = \sigma(n-1,k-1)$. By Lemma 2, $Gr(B_{n-1}^{k-1}) = \sigma(n,k) \oplus 3$. On the other hand, removing t and y leaves a position P that is a disjoint union of B_{n-1}^{k-1} and T_1. Thus, $Gr(P) = Gr(B_{n-1}^{k-1}) \oplus Gr(T_1) = (\sigma(n,k) \oplus 3) \oplus 1 = \sigma(n,k) \oplus 2$.

Thus, we have moves from B_n^k to positions with Grundy-values $\sigma(n,k) \oplus 1$, $\sigma(n,k) \oplus 2$, and $\sigma(n,k) \oplus 3$, but no move to a position with Grundy-value $\sigma(n,k)$. Hence, $Gr(B_n^k) = \sigma(n,k)$, completing the proof by induction. □

Theorem 5. *Let $n \geqslant 3$ be an integer. Then,*

$$Gr(T_n^{n-3}) = 1 \text{ if } n \text{ is odd, and}$$
$$Gr(T_n^{n-3}) = 2 \text{ if } n \text{ is even.}$$

Proof. Since there are $n-2$ cells on the top row of T_n^{n-3}, we have $T_n^{n-3} = B_n^{n-2}$. By Proposition 1, we have $Gr(T_n^{n-3}) = Gr(B_n^{n-2}) = \sigma(n, n-2) = 1$ if n is odd, and $Gr(T_n^{n-3}) = Gr(B_n^{n-2}) = \sigma(n, n-2) = 2$ if n is even. □

We propose the following conjecture.

Conjecture 1. *Let n be a positive integer. If k is an integer such that $0 \leqslant 2k \leqslant n$, then*

$$Gr(T_n^{n-2k}) = 0 \text{ if } n \text{ is odd, and}$$
$$Gr(T_n^{n-2k}) = 2k \text{ if } n \text{ is even.}$$

If k is an integer such that $0 \leqslant 2k+1 \leqslant n$, then

$$Gr(T_n^{n-2k-1}) = 1 \text{ if } n \text{ is odd, and}$$
$$Gr(T_n^{n-2k-1}) = 2k \text{ if } n \text{ is even.}$$

5. Grundy-values of T_n

We first make some general observations about T_n in the following two lemmas.

Lemma 3. *Let $n \geqslant 1$. In Pyramid Nim, T_n is an $\mathcal{N}$-position.*

Proof. Our argument is similar to a proof in [12]. If taking the top element is not a win, then Player 2 makes a move that wins. So, Player 1 (on his first move) steals this move. □

Lemma 4. *Let $n \geqslant 1$. Then, $1 \leqslant Gr(T_n) \leqslant n$.*

Proof. The lower bound follows immediately from Lemma 3. For the upper bound, we note that there are only n possible first moves from T_n, up to isomorphism. Hence, $Gr(T_n) \leqslant n$. □

Definition 9. Let $n \geqslant 3$ be an integer and $0 \leqslant k, \ell \leqslant n-1$ be integers. The digraph $M_n(k, \ell)$ is the digraph T_n with an undominated path on $n-k$ vertices removed from the left side of T_n followed by the removal of an undominated path on $n-1-\ell$ vertices on the right side. The result is a digraph with an undominated path on k vertices on the left side of $M_n(k, \ell)$ and an undominated path on ℓ vertices on the right side.

We make conjectures about $Gr(M_n(k,k))$ when n is odd, and $Gr(M_n(n-1,k))$ when n is even.

Conjecture 2. *Let $n \geqslant 3$ be an odd integer and $0 \leqslant k \leqslant n-2$ be an integer. Then, $Gr(M_n(k,k)) = 1$. Also, $Gr(M_n(n-1,n-1)) = 0$.*

Conjecture 3. *Let $n \geqslant 2$ be an even integer and $0 \leqslant k \leqslant n-1$ be an integer. Then,*

$$Gr(M_n(n-1,k)) = k-1 \text{ if } k \text{ is odd, and}$$
$$Gr(M_n(n-1,k)) = k+1 \text{ if } k \text{ is even.}$$

We use Conjectures 2 and 3 to prove the following theorem.

Theorem 6. *Suppose n is a positive integer, and suppose Conjectures 2 and 3 are true. Then*

$$Gr(T_n) = 1 \quad \text{if } n \text{ is odd, and}$$
$$Gr(T_n) = n \quad \text{if } n \text{ is even.}$$

Proof. Case 1. Assume n is odd. The reachable positions from T_n are $M_n(n-1,k)$ for integers $0 \leqslant k \leqslant n-1$, up to isomorphism. By Lemma 4, $Gr(T_n) \geqslant 1$. By Conjecture 2, $Gr(M_n(n-1,n-1)) = 0 \neq 1$.

Consider the position $M_n(n-1,k)$ for some integer $0 \leqslant k \leqslant n-2$. We remove an undominated path on $n-1-k$ vertices on the left side of $M_n(n-1,k)$ which results in $M_n(k,k)$. By Conjecture 2, $Gr(M_n(k,k)) = 1$. Thus, $Gr(M_n(n-1,k)) \neq 1$. Therefore,

$$Gr(T_n) = \text{mex}\{Gr(M_n(n-1,k)) : 0 \leqslant k \leqslant n-1\} = 1.$$

Case 2. Assume n is even. Again, the reachable positions from T_n are $M_n(n-1,k)$ for integers $0 \leqslant k \leqslant n-1$, up to isomorphism. By Conjecture 3, for all $0 \leqslant k < \frac{1}{2}n$,

$$Gr(M_n(n-1,2k+1)) = 2k \qquad \text{and}$$
$$Gr(M_n(n-1,2k)) = 2k+1.$$

Therefore,

$$Gr(T_n) = \text{mex}\{Gr(M_n(n-1,k)) : 0 \leqslant k \leqslant n-1\} = \text{mex}\{0,1,\ldots,n-1\} = n.$$

□

Note that Theorem 6 also follows from the special cases of Conjecture 1, where k is chosen so that the top index is 0.

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