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THE POLYNOMIAL PROFILE OF DISTANCE GAMES ON PATHS AND CYCLES

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Abstract

Distance games are games played on graphs in which the players alternately color vertices, and which vertices can be colored depends only on the distance to previously colored vertices. The polynomial profile encodes the number of positions with a fixed number of vertices from each player. We extend previous work on finding the polynomial profile of several distance games (Col, Snort, and Cis) played on paths. We give recursions and generating functions for the polynomial profiles of generalizations of these three games when played on paths. We also find the polynomial profile of Cis played on cycles and the total number of positions of Col and Snort on cycles, as well as pose a conjecture about the number of positions when playing Col and Snort on complete bipartite graphs.

1. Introduction

A combinatorial game is a game with two players, Left and Right, with perfect information, no elements of chance, and the players must alternate turns, such as Chess or Checkers. In this paper, we will be enumerating the positions in several distance games. These are games in which the two players place pieces on empty vertices of the board, a finite graph, with the placement of a piece only being restricted by the distance to previously played pieces. Two well-known examples of distance games are Snort and Col, where pieces cannot be placed adjacent to an opponent's piece or one of their own pieces, respectively.

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A relatively new topic of interest in combinatorial game theory is the enumeration of positions. Early work has focused on specific types of positions, such as Go endgame positions [6, 5, 17] and second-player win position for some lesser known games [7, 11]. Recently, all general Domineering positions, as well as specific types of positions, were counted in [9]. Taking a graph theory view, positions of several other games played on grids are enumerated in [12, 13]. Our work is motivated by Brown et al. [2], who introduced the polynomial profile of a game, which is the generating polynomial for the number of positions given the number of Left and Right pieces placed. They found generating functions for the polynomial profiles of several games played on paths, including Snort, Col, and the game Cis. We extend their work and will find the polynomial profiles of generalizations of these three games played on paths (Section 3), and consider the original three games when played on cycles and complete bipartite graphs (Section 4).

In the next section, we provide some background and previous results from this area of combinatorial game theory. This includes a few definitions and descriptions of different placement games. We will conclude this paper with several questions for future work in Section 5.

2. Background

Many combinatorial games are ones in which the players place pieces on a board without moving or removing them later. These are known as placement or pen-and-paper games. We will be studying several games that belong to a subclass of placement games. Note that for placement games, placing a piece is equivalent to coloring the corresponding vertices of the board. For the placement, Left will color a vertex blue and Right will color it red.

Definition 1 ([10])**.** *Distance games* are a subclass of combinatorial games played on finite graphs, with each game uniquely identified by a pair of sets (S, D). On their turn, a player will color in an empty vertex that is not distance $s \in S$ from a piece of the same color or distance $d \in D$ from a piece of the opposite (different) color.

Although the rules of a distance game are deceptively easy to describe, several games have been extensively studied and are still unsolved. For example, their computational complexities have been studied in [14, 3, 8].

Definition 2. The following are some previously studied distance games.

- Col [1] is the distance game where $S = \{1\}$ and $D = \emptyset$. In other words, pieces of different colors can be adjacent, but pieces of the same color cannot.

- SNORT [1] is the distance game where $S = \emptyset$ and $D = \{1\}$. In other words, pieces of the same color can be adjacent, but pieces of different colors cannot.
- CIS [2] is the distance game where $S = D = \{1\}$. In other words, no two pieces can be adjacent, no matter the color.

Note that playing CIS is equivalent to playing NODEKAYLES. If a player can place a piece on a vertex in either game, then so can the other player. The difference between these games is that in NODEKAYLES both players color using the same color, while in CIS they use different colors. This is not relevant to game play, but will be important for enumeration.

We will consider generalizations of the above games, the first two of which were defined in [3].

Definition 3. We give the following definitions.

- ENCOL(k) is the distance game where $S = \{1, \ldots, k\}$ and $D = \emptyset$.
- ENSNORT(k) is the distance game where $S = \emptyset$ and $D = \{1, \ldots, k\}$.
- ENCIS(k) is the distance game where $S = D = \{1, \ldots, k\}$.
- CIS$_2$ is the distance game where $S = D = \{2\}$.

We will count the positions in SNORT, COL, and CIS played on cycles, stars, and complete bipartite graphs, as well as their generalization when played on paths. To enumerate the positions of these distance games we use a generating function called the polynomial profile.

Definition 4 ([2])**.** The *polynomial profile* of the game G played on the board B with n vertices is the bivariate polynomial

$$P_{G,B}(x, y) = \sum_{k=0}^{n} \sum_{j=0}^{k} f_{j,k-j} x^j y^{k-j},$$

where $f_{j,j-k}$ is the number of positions with j Left pieces and $k - j$ Right pieces.

Setting $x = y$ we get the univariate polynomial

$$P_{G,B}(x) := P_{G,B}(x, x) = \sum_{i=0}^{n} c_i x^i,$$

where c_i is the number of positions with exactly i pieces. Finally, the total number of legal positions can be obtained by setting $x = y = 1$.

The polynomial profile counts the number of positions without assuming alternating play. In many combinatorial games, including distance games, the board naturally breaks into smaller, independent components as game play progresses. On their turn, a player then chooses which component to play in and makes their move there. In combinatorial game theory, this is called the *disjunctive sum* of the components. Although game play in the entire game is alternating, in a component it can be non-alternating. In many cases in combinatorial game theory, including enumeration of positions, it helps to assume that a game is a component of a larger game and the condition of alternating play is dropped. If desired, we can find the number of positions restricted to alternating play from the polynomial profile by taking only the terms where the exponents on x and y differ by at most 1. More information on combinatorial game theory and common techniques can be found in [1, 15].

To illustrate these concepts, we will look at a relatively simple example.

Example 1. Consider CIS played on P_4. There is one empty position, four positions with a single Left or Right piece each, three positions with two pieces of the same color, and six positions with one piece each by Left and Right (shown in Figure 1). Thus the polynomial profile is

$$P_{\text{CIS},P_4}(x,y) = 1 + 4x + 4y + 3x^2 + 6xy + 3y^2.$$

The univariate polynomial is

$$P_{\text{CIS},P_4}(x) = 1 + 8x + 12x^2$$

and the total number of positions is $P_{\text{CIS},P_4}(1) = 21$.

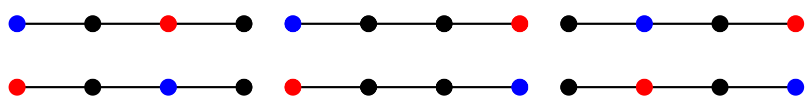

Figure 1: Possible positions of CIS on P_4 with one Left piece and one Right piece played.

Taking only the relevant terms for alternating play from $P_{\text{CIS},P_4}(x,y)$, we get that the positions in alternating play are enumerated by $1 + 4x + 4y + 6xy$.

We can find the polynomial profile of a game by using the auxiliary board.

Definition 5. The *auxiliary board* $\Gamma_{G,B}$ of a distance game G on a board B is the graph that represents all minimal illegal moves. The vertex set of $\Gamma_{G,B}$ is given by $V(\Gamma) = V(B) \times \{1,2\}$ where the vertex $(x_i,1)$ represents Left moving in vertex x_i in B and, similarly, $(x_j,2)$ is a move by Right in vertex x_j. Two vertices (x_i,a) and (x_j,b) are adjacent if the corresponding moves are at an illegal distance.

Note that the auxiliary board can be generalized to many other placement games, and the resulting simplicial complex is known as the illegal complex [4].

Example 2. Consider Col played on C_4. The auxiliary board Γ_{Col,C_4}, shown in Figure 2, has vertex set $V(G_{\text{Col},C_4}) = \{x_1, x_2, x_3, x_4\} \times \{1, 2\}$. Vertices (x_i, p) and (x_j, q) are adjacent if x_i is adjacent to x_j and $p = q$, or if $i = j$ and $p \neq q$.

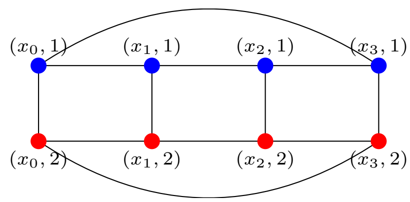

Figure 2: Auxiliary board for Col on C_4.

From Figure 2 we can see that there are eight independent sets of size 1, four of which are for Left and four for Right. This gives us the terms $4x$ and $4y$ in the polynomial profile. We can also see that there are twelve independent sets with one blue and one red piece. This gives us the term $12xy$. We do this for every possible combination of blue and red pieces to get the full polynomial

$$P_{\text{Col},C_4}(x, y) = 1 + 4x + 4y + 2x^2 + 12xy + 2y^2 + 4x^2y + 4xy^2 + 2x^2y^2.$$

Example 3. Consider Snort played on C_4. The auxiliary board $\Gamma_{\text{Snort},C_4}$, shown in Figure 3, has vertex set $V(G_{\text{Snort},C_4}) = \{x_1, x_2, x_3, x_4\} \times \{1, 2\}$. Vertices (x_i, p) and (x_j, q) are adjacent if $i = j$ and $p \neq q$, or if x_i is adjacent to x_j and $p \neq q$.

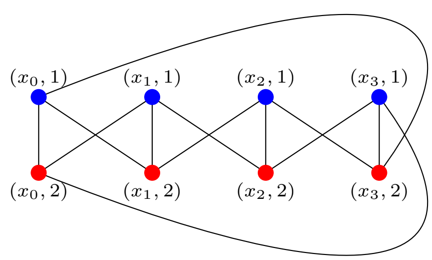

Figure 3: Auxiliary board for Snort on C_4.

From Figure 3 we can see that there are two independent sets of size 4, one for Left and one for Right. This gives us the terms x^4 and y^4 in the polynomial profile. We can also see that there are eight independent sets of size 3, four of which are for Left and four for Right. This gives us the entries $4x^3$ and $4y^3$. We do this for every possible combination of blue and red pieces to get the full polynomial

$$P_{\text{SNORT},C_4} = 1 + 4x + 4y + 6x^2 + 4xy + 6y^2 + 4x^3 + 4y^3 + x^4 + y^4.$$

We will now turn our attention to determining the polynomial profiles in greater generality.

3. Distance Games on Paths

The positions of Col, Snort, and Cis played on paths were enumerated in [2]. The generating functions for these three games were found to be

$$GF_{\text{Col}}(x,y,t) = \frac{(1+xt)(1+yt)}{1-(xyt^2+t(1+xt)(1+yt))},$$

$$GF_{\text{Snort}}(x,y,t) = \frac{1-xyt^2}{1-(xt+yt+xyt^2+t(1-xyt^2))}, \text{ and}$$

$$GF_{\text{Cis}}(x,y,t) = \frac{1+xt+yt}{1-t-xt^2-yt^2},$$

where the coefficient of t^n is the polynomial profile when playing on a path of n vertices. Note that there was a typographical error in the generating function of Cis in [2]. We have given the corrected function here.

We will now generalize this and enumerate the positions of various other distance games when played on paths.

As a step towards finding the generating function we take into account the empty vertices in positions by using an e to represent them. We then set $e = 1$ to get the polynomial profile.

For many cases we will use regular expressions to represent all possible positions in the game. For example, $E^*(B|R)$ represents 0 or more empty vertices followed by a Left (blue) or a Right (red) piece. We will represent the empty word by ϵ. For more information on regular expressions, see for example [16].

When finding the generating function from the regular expression, we use the following operations.

- *Concatenation*: Corresponds to multiplication in the generating function.
- *Union*: Corresponds to addition in the generating function.
- *Star*: The star operator indicates 0 or more repetitions, the generating function for this is a geometric series, and thus corresponds to $\frac{1}{1-\bullet}$. For example B^* is enumerated by

$$1 + xt + x^2t^2 + x^3t^3 + \cdots = \frac{1}{1-xt}.$$

3.1. EnCis(k)

In this section we find the recursions for the polynomial profile and the number of positions of EnCis(k) played on paths, as well as the generating function.

Proposition 1. *For all $k \geq 1$, the polynomial profile for the distance game* EnCis(k) *played on P_n for all $n \leq k$ is*

$$P_{\text{EnCis}(k),P_n}(x,y) = 1 + nx + ny,$$

and for all $n > k$ it is recursively given by

$$P_{\text{EnCis}(k),P_n}(x,y) = P_{\text{EnCis}(k),P_{n-1}}(x,y) + (x+y)P_{\text{EnCis}(k),P_{n-(k+1)}}(x,y).$$

The total number of positions is recursively given by

$$P_{\text{EnCis}(k),P_n}(1) = P_{\text{EnCis}(k),P_{n-1}}(1) + 2P_{\text{EnCis}(k),P_{n-(k+1)}}(1)$$

with $P_{\text{EnCis}(k),P_n}(1) = 2n+1$ for all $n \leq k$.

Proof. When $n \leq k$, no pair of pieces can be placed. Therefore the only positions are the empty board and the n positions each with a single blue or red piece, giving the initial terms.

For the recursion, consider the leftmost vertex of the path. If this vertex is uncolored, the rest of the $n-1$ vertices can be any legal position on P_{n-1}. If this vertex is colored blue or red, then the next k vertices must be empty, which leaves any legal position on $P_{n-(k+1)}$ vertices. This gives the desired recursion. Setting $x = y = 1$ gives the recursion for the total number of positions. □

For large k, this recursion will require a large amount of initial terms. Although we are not able to give a closed form of the polynomial profile, we will now turn to determining the generating function, which is

$$GF_{G,P_n}(x,y,t) = \sum_{n=0}^{\infty} P_{G,P_n}(x,y)t^n.$$

As an in-between step, we will often consider a refined polynomial profile

$$P_{G,P_n}(e,x,y),$$

where the exponent on e indicates the number of empty vertices, i.e. the degree of every term will be n. The generating function for this polynomial is denoted $GF_{G,P_n}(e,x,y,t)$ and we have $GF_{G,P_n}(x,y,t) = GF_{G,P_n}(1,x,y,t)$ and $P_{G,P_n}(x,y) = P_{G,P_n}(1,x,y)$.

Proposition 2. *For all $k \geq 1$, the generating function for the polynomial profile of* EnCis(k) *played on P_n is*

$$GF_{\textsc{EnCis}(k),P_n}(x,y,t) = \frac{1-t+xt+yt-xt^{k+1}-yt^{k+1}}{(1-t)(1-t-xt^{k+1}-yt^{k+1})}.$$

The total number of positions is generated by

$$GF_{\textsc{EnCis}(k),P_n}(1,1,t) = \frac{1+t-2t^{k+1}}{(1-t)(1-t-2t^{k+1})}.$$

Proof. In EnCis(k), no pair of vertices at distance $1, 2, \ldots, k$ can be colored simultaneously. Therefore, the positions on a path are exactly those that satisfy the following pattern:

1. Starts with zero or more empty vertices;
2. followed by repeated patterns taken from
   - a blue vertex followed by k empty
   - a red vertex followed by k empty

   followed by zero or more empty vertices; and
3. ends with
   - a blue vertex followed by 0 to $k-1$ empty,
   - a red vertex followed by 0 to $k-1$ empty, or
   - nothing added.

For example, for EnCis(2) this gives the regular expression

$$E^*[(B|R)EEE^*]^*(B|BE|R|RE|\epsilon).$$

For the general case, we use the notation S_B^{k-1} to represent the string

$$B|BE|BEE|\ldots|BE^{k-1}.$$

Similarly, we set

$$S_R^{k-1} = R|RE|REE|\ldots|RE^{k-1}.$$

The regular expression for EnCis(k) is then

$$E^*[(B|R)E^kE^*]^*[S_B^{k-1}|S_R^{k-1}|\epsilon].$$

The corresponding expression in the generating function to the term S_B^{k-1} is

$$xt\frac{1-e^kt^k}{1-et},$$

and similarly for S_R^{k-1} we get

$$yt\frac{1-e^kt^k}{1-et}.$$

The generating function taking empty vertices into account is then

$$GF_{\text{EnCis}(k)}(e,x,y,t) = \left(\frac{1}{1-et}\right)\left(\frac{1}{1-(\frac{xe^kt^{k+1}}{1-et}+\frac{ye^kt^{k+1}}{1-et})}\right) \times \left(\frac{xt(1-e^kt^k)}{1-et}+\frac{yt(1-e^kt^k)}{1-et}+1\right),$$

and setting $e=1$ gives

$$GF_{\text{EnCis}(k)}(x,y,t) = \frac{1-t+xt+yt-xt^{k+1}-yt^{k+1}}{(1-t)(1-t-xt^{k+1}-yt^{k+1})}.$$

With $x=y=1$ we get that the generating function for the number of positions is

$$GF_{\text{EnCis}(k)}(1,1,t) = \frac{1+t-2t^{k+1}}{(1-t)(1-t-2t^{k+1})}. \qquad \square$$

3.2. EnSnort(k)

To find the generating function for the polynomial profile of EnSnort(k), we will introduce the following shorthand notation for the regular expression:

$$\begin{aligned}
\mathbf{S}_B^{k-1} &= B|BE|BEE|...|BE^{k-1}\\
\mathbf{S}_R^{k-1} &= R|RE|REE|...|RE^{k-1}\\
\mathbf{T}_B^{k-1} &= B(\mathbf{S}_B^{k-1})^*|B(\mathbf{S}_B^{k-1})^*E|\dots|B(\mathbf{S}_B^{k-1})^*E^{k-1}\\
\mathbf{T}_R^{k-1} &= R(\mathbf{S}_R^{k-1})^*|R(\mathbf{S}_R^{k-1})^*E|\dots|R(\mathbf{S}_R^{k-1})^*E^{k-1}.
\end{aligned}$$

The regular expression for EnSnort(k) is then

$$E^*[(B(\mathbf{S}_B^{k-1})^*|R(\mathbf{S}_R^{k-1})^*)E^kE^*]^*[\mathbf{T}_B^{k-1}|\mathbf{T}_R^{k-1}|\epsilon].$$

Example 4. Consider EnSnort(2) played on P_n. The regular expression for EnSnort(2) is

$$E^*[(B(B|BE)^*|R(R|RE)^*)EEE^*]^* \\ [B(B|BE)^*|B(B|BE)^*E|R(R|RE)^*|R(R|RE)^*E|\epsilon],$$

giving that the generating function for the polynomial profile is

$$GF_{\text{EnSnort}(2)}(x,y,t) = \left(\frac{1}{1-t}\right)\left(\frac{1}{1-(\frac{xt}{1-xt-xt^2}+\frac{yt}{1-yt-yt^2})(\frac{t^2}{1-t})}\right) \times\left(\frac{xt}{1-xt-xt^2}+\frac{xt^2}{1-xt-xt^2}+\frac{yt}{1-yt-yt^2}+\frac{yt^2}{1-yt-yt^2}+1\right).$$

The first few polynomial profiles are in Table 1.

n	$P_{\text{EnSnort}(2),P_n}(x,y)$	$P_{\text{EnSnort}(2),P_n}(x)$	$P_{\text{EnSnort}(2),P_n}(1)$
0	1	1	1
1	$1+1x+1y$	$1+2x$	3
2	$1+2x+2y+x^2+y^2$	$1+4x+2x^2$	7
3	$1+3x+3y+3x^2+3y^2+x^3+y^3$	$1+6x+6x^2+2x^3$	15

Table 1: First few initial terms for the recursion of the polynomial profile of EnSnort(2).

In general, we get the following result.

Proposition 3. *For $k \geq 1$, the generating function for the polynomial profile of* EnSnort(k) *on P_n is*

$$GF_{\text{EnSnort(k)},P_n}(x,y,t) = \left(\frac{1}{1-t}\right)\left(\frac{1}{1-\left(\frac{xt}{1-\sum_{i=1}^{k} xt^i}+\frac{yt}{1-\sum_{i=1}^{k} yt^i}\right)\left(\frac{t^k}{1-t}\right)}\right)$$
$$\times\left(\sum_{n=1}^{k}\frac{xt^n}{1-\sum_{i=1}^{k} xt^i}+\sum_{n=1}^{k}\frac{yt^n}{1-\sum_{i=1}^{k} yt^i}+1\right).$$

3.3. Cis$_2$

In this section we find the recursion for the polynomial profile and number of positions of Cis$_2$ played on paths, as well as the generating function for the polynomial profile.

Proposition 4. *The polynomial profile for* Cis$_2$ *played on P_n is given by*

$$P_{\text{Cis}_2,P_n}(x,y) = P_{\text{Cis}_2,P_{n-1}}(x,y) + (x+y)P_{\text{Cis}_2,P_{n-3}}(x,y) + (x^2+y^2+2xy)P_{\text{Cis}_2,P_{n-4}}(x,y),$$

for $n \geq 4$.

The total number of positions[2] *is recursively given by*

$$P_{\text{CIS}_2,P_n}(1) = P_{\text{CIS}_2,P_{n-1}}(1) + 2P_{\text{CIS}_2,P_{n-3}}(1) + 4P_{\text{CIS}_2,P_{n-4}}(1).$$

The initial terms are given in Table 2.

n	$P_{\text{CIS}_2,P_n}(x,y)$	$P_{\text{CIS}_2,P_n}(x)$	$P_{\text{CIS}_2,P_n}(1)$
0	1	1	1
1	$1+x+y$	$1+2x$	3
2	$1+2x+2y+x^2+y^2+2xy$	$1+4x+3x^2$	9
3	$1+3x+3y+2x^2+2y^2+4xy$	$1+6x+8x^2$	15

Table 2: Initial terms for the recursion of the polynomial profile of CIS_2.

Proof. The initial cases can be checked computationally.

For the recursion, consider the leftmost vertex of the path. If this vertex is uncolored, the rest of the $n-1$ vertices can be any legal position for this game on P_{n-1}, which gives $P_{\text{CIS}_2,P_{n-1}}(x,y)$ as a term in the recursion. On the other hand, if this vertex is colored blue or red, then either the next two vertices in the path are both empty or the next vertex is also colored blue or red, followed by the next two empty. The first case leaves any legal position on P_{n-3}, which gives $xP_{\text{CIS}_2,P_{n-3}}(x,y)$ for blue and $yP_{\text{CIS}_2,P_{n-3}}(x,y)$ for red. The second case results in two colored vertices followed by two empty vertices, leaving any legal position on P_{n-4}. In this case, the two leftmost vertices can be any combination of one blue and one red, two blue, or two red, giving us the term $(x^2+y^2+2xy)P_{\text{CIS}_2,P_{n-4}}(x,y)$.

This gives the recursion

$$\begin{aligned}P_{\text{CIS}_2,P_n}(x,y) = P_{\text{CIS}_2,P_{n-1}}(x,y) + (x+y)P_{\text{CIS}_2,P_{n-3}}(x,y)\\ + (x^2+y^2+2xy)P_{\text{CIS}_2,P_{n-4}}(x,y).\end{aligned}$$

Setting $x = y = 1$, we get the recursion for the total number of positions as desired. □

Using the shorthand $\mathbf{U} = BB|BR|RB|RR|B|R$, the regular expression is

$$E^*(\mathbf{U}EEE^*)^*(\mathbf{U}|\mathbf{U}E|\epsilon).$$

Thus we get the following proposition.

Proposition 5. *The generating function for the polynomial profile of* CIS_2 *played on paths is*

$$GF_{\text{CIS}_2,P_n}(x,y,t) = \frac{1+xt+yt+(xt+yt)^2+et(xt+yt+(xt+yt)^2)}{1-et-e^2t^2(xt+yt+(xt+yt)^2)}.$$

[2]OEIS sequence A138495

The total number of positions is generated by

$$GF_{\text{Cis}_2,P_n}(1,1,t) = \frac{1+2t+6t^2+4t^3}{1-t-2t^3-4t^4}.$$

4. Col, Snort, and Cis

4.1. Cis on Cycles

We will generalize the results of [2] for Cis on paths to cycles. We will find the polynomial profiles using the polynomial profiles on paths, as well as the generating function for the polynomial profile.

Proposition 6. *The polynomial profile when playing* Cis *on* C_n *for* $n \geq 4$ *is given by*

$$P_{\text{Cis},C_n}(x,y) = P_{\text{Cis},P_{n-1}}(x,y) + (x+y)P_{\text{Cis},P_{n-2}}(x,y).$$

The number of positions is $2^{n-2} + (-1)^n$.

Proof. We fix one vertex in the cycle and label it v_0. We then label the rest of the vertices as $v_1, \ldots, v_{n-1}$ in a clockwise rotation.

If v_0 is colored, then the adjacent vertices (v_1 and v_{n-1}) must be empty. This leaves any possible position on P_{n-3} which gives $xP_{\text{Cis},P_{n-3}}(x,y)$ if it is a blue vertex and $yP_{\text{Cis},P_{n-3}}(x,y)$ if it is a red vertex.

If v_0 is uncolored, this leaves any possible position on P_{n-1}, thus contributing the term $P_{\text{Cis},P_{n-1}}(x,y)$.

This gives the recursion

$$P_{\text{Cis},C_n}(x,y) = (x+y)P_{\text{Cis},P_{n-3}}(x,y) + P_{\text{Cis},P_{n-1}}(x,y).$$

We then set $x = y = 1$ to get the number of positions recursively as

$$P_{\text{Cis},C_n}(1) = 2P_{\text{Cis},P_{n-3}}(1) + P_{\text{Cis},P_{n-1}}(1).$$

In [2], the number of positions for Cis on paths was found to be $P_{\text{Cis},P_n}(1) = \frac{2^n-(-1)^n}{3}$. Substituting this into the recursion, we get

$$\begin{aligned} P_{\text{Cis},C_n}(1) &= 2P_{\text{Cis},P_{n-3}}(1) + P_{\text{Cis},P_{n-1}}(1) \\ &= 2\left(\frac{2^{n-3}-(-1)^{n-3}}{3}\right) + \left(\frac{2^{n-1}-(-1)^{n-1}}{3}\right) \\ &= 2^{n-2} + (-1)^n. \end{aligned}$$

□

Proposition 7. *The generating function for the polynomial profile of* Cis *played on cycles is*

$$GF_{\text{Cis},C_n}(1,x,y,t) = \frac{1+x^2t^2+y^2t^2}{1-t-xt^2-yt^2},$$

and thus the generating function for the total number of positions is

$$GF_{\text{CIS},C_n}(1,1,1,t) = \frac{1+2t^2}{1-t-2t^2}.$$

Proof. For CIS on a cycle, we will fix a vertex v and will consider a position on C_n as a position on P_{n+1}, where the first and last vertex are simultaneously either empty, blue, or red. Recall that no adjacent vertices can both be colored. Therefore, the positions on the equivalent path P_{n+1} are exactly those that satisfy one of the following patterns:

1. Starts with an empty vertex;
   - followed by zero or more empty vertices;
   - followed by repeated patterns taken from
     - a blue vertex followed by an empty vertex
     - a red vertex followed by an empty vertex

     followed by zero or more empty vertices; or
2. Starts with a blue vertex;
   - followed by at least one empty vertex;
   - followed by repeated patterns taken from
     - a blue vertex followed by an empty vertex
     - a red vertex followed by an empty vertex

     followed by zero or more empty vertices;
   - ends with a blue vertex; or
3. Starts with a red vertex;
   - followed by at least one empty vertex;
   - followed by repeated patterns taken from
     - a blue vertex followed by an empty vertex
     - a red vertex followed by an empty vertex

     followed by zero or more empty vertices;
   - ends with a red vertex.

The regular expression for CIS then is

$$[EE^*(BEE^*|REE^*)^*]|[BEE^*(BEE^*|REE^*)^*(B)]|[REE^*(BEE^*|REE^*)^*(R)].$$

The generating function for the three cases count the first and last vertex separately, while they are the same for us on the cycle. To adjust for this, we divide the generating function for the first case by et, the second by xt, and the third by yt. This gives us the following generating function for the cycle C_n:

$$\begin{aligned}GF_{\text{CIS},C_n}(e,x,y,t) &= \left(\frac{1}{1-et}\right)\left(\frac{1-et}{1-et-xet^2-yet^2}\right)\\ &\quad+\left(\frac{xet^2}{1-et}\right)\left(\frac{1-et}{1-et-xet^2-yet^2}\right)\\ &\quad+\left(\frac{yet^2}{1-et}\right)\left(\frac{1-et}{1-et-xet^2-yet^2}\right).\end{aligned}$$

Setting $e = 1$ and simplifying gives the desired generating function. □

4.2. Recursion for Col and Snort on Cycles

We will give the recursions for the total number of positions of Col and Snort on cycles. The recursions for the polynomial profile can be found similarly. As the latter do not simplify nicely, we will only give the former.

Proposition 8. *For $n \geq 4$, the number of positions*[3] *when playing* Col *on C_n is given by*

$$P_{\text{Col},C_n}(1) = P_{\text{Col},P_{n-1}}(1) + 3P_{\text{Col},P_{n-3}}(1) + 2P_{\text{Col},P_{n-4}}(1) + P_{\text{Col},C_{n-2}}(1),$$

with initial terms $P_{\text{Col},C_2}(1) = 7$ and $P_{\text{Col},C_3}(1) = 13$.

Proof. We fix one vertex in the cycle and label it v_0. We then label the rest of the vertices as $v_1, \ldots, v_{n-1}$ in a clockwise rotation.

If v_0 is uncolored, this leaves any possible position on P_{n-1}, which gives us $P_{\text{Col},P_{n-1}}(1)$ possible positions.

If v_0 is colored, then there are three cases to consider:

1. both v_1 and v_{n-1} are empty;
2. both v_1 and v_{n-1} are colored, necessarily the opposite color to v_0; and
3. either v_1 or v_{n-1} is colored, necessarily the opposite color to v_0, with the other being empty.

The first case leaves any possible position on P_{n-3}, whether v_0 is blue or red, thus there are $2P_{\text{Col},P_{n-3}}(1)$ possible positions for this case.

The second case can be reduced to a cycle on $n-2$ vertices where the fixed vertex cannot be empty. This gives us $P_{\text{Col},C_{n-2}}(1) - P_{\text{Col},P_{n-3}}(1)$ possible positions.

The third case reduces to a colored path on $n-2$ vertices. Let $f_B(n)$ denote the polynomial enumerating the positions when playing Col on P_n with the first

[3]This sequence seems to be the OEIS sequence A051927. We have confirmed this for $n \leq 12$.

vertex colored blue, and similarly for $f_R(n)$ and $f_E(n)$. Using this notation, the number of positions in this case is

$$\begin{aligned} f_E(n-3)(1) + f_B(n-3)(1) + f_E(n-3)(1) + f_R(n-3)(1) \\ &= P_{\text{Col},P_{n-3}}(1) - f_R(n-3)(1) + P_{\text{Col},P_{n-3}}(1) - f_B(n-3)(1) \\ &= 2P_{\text{Col},P_{n-3}}(1) - (f_R(n-3)(1) + f_B(n-3)(1)) \\ &= 2P_{\text{Col},P_{n-3}}(1) - (P_{\text{Col},P_{n-3}}(1) - P_{\text{Col},P_{n-4}}(1)) \\ &= P_{\text{Col},P_{n-3}}(1) + P_{\text{Col},P_{n-4}}(1). \end{aligned}$$

We count this case twice, once for v_0 being empty and v_{n-1} being colored and once for the opposite; this gives us $2P_{\text{Col},P_{n-3}}(1) + 2P_{\text{Col},P_{n-4}}(1)$.

All together we get

$$P_{\text{Col},C_n}(1) = P_{\text{Col},P_{n-1}}(1) + 3P_{\text{Col},P_{n-3}}(1) + 2P_{\text{Col},P_{n-4}}(1) + P_{\text{Col},C_{n-2}}(1). \quad \square$$

The proof for SNORT on cycles is similar to the proof for COL on cycles and is thus omitted.

Proposition 9. *For $n \geq 4$, the number of positions*[4] *when playing* SNORT *on C_n is given by*

$$P_{\text{Snort},C_n}(1) = P_{\text{Snort},P_{n-1}}(1) + 3P_{\text{Snort},P_{n-3}}(1) + 2P_{\text{Snort},P_{n-4}}(1) + P_{\text{Snort},C_{n-2}}(1),$$

with initial terms $P_{\text{Snort},C_2}(1) = 7$ and $P_{\text{Snort},C_3}(1) = 15$.

4.3. COL, SNORT, and CIS on Complete Bipartite Graphs

When playing COL, SNORT, or CIS on the star $K_{1,n}$, the state of the central vertex will completely determine the possible positions.

For COL and SNORT, if the central vertex is colored blue, the outer vertices can each independently be red or empty and respectively, blue or empty. Similarly if it is colored red. If the central vertex is uncolored, the outer vertices can independently be empty or either color.

For CIS, if the central vertex is colored, the outer vertices have to be empty, while if it is uncolored, then the outer vertices can independently be empty or either color.

This gives us the following result.

Proposition 10. *When playing on a star $K_{1,n}$, $n \geq 1$, the total number of positions is given by*

$$\begin{aligned} P_{\text{Col},K_{1,n}}(1) &= 2^{n+1} + 3^n \\ P_{\text{Snort},K_{1,n}}(1) &= 2^{n+1} + 3^n \\ P_{\text{Cis},K_{1,n}}(1) &= 2 + 3^n. \end{aligned}$$

[4] This sequence seems to be OEIS sequence A124696. We have confirmed this for $n \leq 13$.

That the number of positions for COL and SNORT on $K_{1,n}$ is the same is not a surprise. It is known that this is the case for any bipartite board.

Theorem 1 ([2]). *If B is a bipartite graph, then $P_{\text{COL},B}(x) = P_{\text{SNORT},B}(x)$. In particular, $P_{\text{COL},B}(1) = P_{\text{SNORT},B}(1)$, i.e., the number of positions is the same for* COL *and* SNORT *when playing on a bipartite graph.*

We have computed the number of positions on other complete bipartite graphs for COL and SNORT and these can be found in Table 3.

m/n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	3	9	27	81	243	729	2187	6561	19683	59049	177147	531441	1594323
1	3	7	17	43	113	307	857	2443	7073	20707	61097	181243	539633	
2	9	17	35	77	179	437	1115	2957	8099	22757	65195	189437		
3	27	43	77	151	317	703	1637	3991	10157	26863	73397			
4	81	113	179	317	611	1253	2699	6077	14291					
5	243	307	437	703	1253	2407	4877	10303						
6	729	857	1115	1637	2699	4877	9395							
7	2187	2443	2957	3991	6077	10303								
8	6561	7073	8099	10157	14291									
9	19683	20707	22757	26863										
10	59049	61097	65195	73397										
11	177147	181243	189437											
12	531441	539633												
13	1594323													

Table 3: The number of positions when playing COL or SNORT on $K_{m,n}$.

Based on this data, we pose the following conjecture.

Conjecture 1. The number of positions when playing COL or SNORT on the complete bipartite graph $K_{m,n}$ is recursively given by

$$P_{\text{COL},K_{m,n}}(1) = 5P_{\text{COL},K_{m,n-1}}(1) - 6P_{\text{COL},K_{m,n-2}}(1) + c_m$$

with initial terms as per Table 3, and c_m is given by the OEIS sequence A260217 (the first few terms are $c_2 = 4$, $c_3 = 24$, $c_4 = 100$, $c_5 = 360$, and $c_6 = 1204$).

For CIS, similar to the situation of the star, as soon as a single vertex is colored, only vertices in the same part of $K_{m,n}$ are able to be colored. Ensuring that we do not count the empty position twice, we get the following result.

Proposition 11. *The number of positions when playing* CIS *on the complete bipartite graph $K_{m,n}$, $m, n \geq 1$, is given by*

$$P_{\text{CIS},K_{m,n}}(1) = 3^m + 3^n - 1.$$

5. Future Work

As was the case in [2] for COL, ENCOL(k) appears the most complicated of the generalizations of COL, SNORT, and CIS that we considered. We are still looking into finding a recursion, as well as a generating function, for EnCol(k) played on paths.

For Col and Snort played on cycles we have found recursions for the polynomial profiles and the number of positions, but have not yet found the generating functions. For complete bipartite graphs, in addition to trying to prove Conjecture 1, we also would like to find generating functions.

Of course, this work could also be extended by considering other distance games or looking at the generalizations when played on boards other than paths.

Finally, we are interested in the ratio of positions in purely alternating play, which can also be found from the polynomial profile, to the total number of positions.

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