INVERTIBLE ELEMENTS OF THE DICOT MISÈRE UNIVERSE

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Abstract

We present a necessary and sufficient condition for a dicot misère game to be invertible.

1. Introduction

Many two-person games played on boards, such as DOMINEERING, AMAZONS, and SNORT, at some point decompose into regions and players are only allowed to play in one region on each move. If $A, B, \ldots, K$ are the regions this is represented as $A + B + \ldots + K$. Let 0 represent the game where neither player has any moves. The '+' is called a disjunctive sum and it induces an equivalence relation, a partial order, and an algebra.

The analysis of the disjunctive sum $A + B + \ldots + K$ would be simplified if $A + B$ were equivalent to 0 because then $A + B$ could be ignored and dropped from the sum. Under the *normal play* winning convention, i.e., last player to move wins, if B were the same position as A but with the players' roles reversed then no matter where the first player played the second player would have the same move in the other region. Colloquially, this is called 'turning the board around'. After each pair of moves, the same situation holds and, eventually, all moves are exhausted, i.e., at 0, and it is the first player's move. Essentially nothing has changed from progressing from $A + B$ to 0 and it is appropriate then to write $B = -A$, have $A + B = A + (-A) = 0$, and call B the inverse of A.

Under the *misère* winning convention, last player to move loses and there are no games equivalent to 0 [9]; consequently, there are no inverses. Dicot[1] games have the property that either both players have a move or the game is over. Misère dicot games (i.e., dicot games under misère rules) have more structure [5, 8, 15]. Allen asks, regarding the dicot structure, when it is true that $G - G = 0$ (for example, $* + * = 0$) [12]. McKay, Milley and Nowakowski show that this is true if $G' - G'$ is a next player win for every follower G' of G [8].

In Theorem 12, the main result of this paper, we prove the converse implication establishing that a dicot canonical form, G, is invertible if and only if there is no subposition G' of G, such that $G' - G'$ is a previous player win. There are two problems that need to be overcome.

(1) For all games G and H, it is necessary to find a reasonable way of deciding if G is greater than H. This is given in Theorem 5.

(2) If $-H$ is the result of reversing the roles in H, then we cannot guarantee that $H + (-H)$ is equivalent to 0. Fortunately, something less powerful is sufficient and that is given in Lemma 1.

This document is self-contained; see [1, 2, 4, 16] for more information. The necessary game theory background is given in Section 2. Readers fluent in combinatorial game theory may wish to proceed to Section 3.

2. General Games Background

Combinatorial game theory studies perfect information games in which there are no chance devices (for example, dice) and two players take turns moving alternately. Using standard notation, where the two players are usually called Left and Right[2], a position is written in the form $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, where $G^{\mathcal{L}} = \{G^{L_1}, G^{L_2}, \ldots\}$ is the set of left options from G, and G^L is a particular left option (and the same for $G^{\mathcal{R}}$ and G^R). In normal play the last player able to move is the winner; by contrast, in misère play the last player is the loser. Here we are concerned with short dicot games under misère play (short games are games with finitely many distinct subpositions and no infinite run).

Given a game G, any position that can be reached from G, alternating play is not necessary, is called a *follower* of G (G itself is a follower of G). It is assumed that both Left and Right are *optimal players*; that means that, whenever one can force a win, the player does so. With that in mind, the four possible *outcomes* of a position are $\mathscr{L}$, $\mathscr{P}$, $\mathscr{N}$, and $\mathscr{R}$, where $\mathscr{L}$eft wins, regardless of moving first or second; $\mathscr{R}$ight

[1] Originally called all-small games because, under normal play, their values are infinitesimal. This is not true in misère or scoring games and the term 'dicotic' is in [16] but authors have preferred the shorter 'dicot'.

[2] For the purpose of using distinguishing pronouns, Left is taken to be female and Right taken to be male.

wins, regardless of moving first or second; $\mathscr{N}$ext player wins regardless of whether this is Left or Right; $\mathscr{P}$revious player wins regardless of whether this is Left or Right. Conventionally, these are ordered as $\mathscr{L} > \mathscr{P}$, $\mathscr{L} > \mathscr{N}$, $\mathscr{P} > \mathscr{R}$, $\mathscr{N} > \mathscr{R}$, and $\mathscr{P}$ and $\mathscr{N}$ are incomparable. The outcome function $o(G)$ will be used to denote the outcome of G. The *outcome classes* $\mathcal{L}, \mathcal{N}, \mathcal{R}, \mathcal{P}$ are the sets of all games with the indicated outcome ($G \in \mathcal{L}$ when $o(G) = \mathscr{L}$).

Often, games decompose into components during play. For those situations, the disjunctive sum is formalized:

$$G + H = \{G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \,|\, G^{\mathcal{R}} + H, G + H^{\mathcal{R}}\}.$$

Given a winning convention and associated outcomes, the relations *inequality* and *equivalence* of games are defined by

$$G \succcurlyeq H \text{ if and only if } o(G + X) \geqslant o(H + X) \text{ for all games } X;$$

$$G \equiv H \text{ if and only if } o(G + X) = o(H + X) \text{ for all games } X.$$

The first means that replacing H by G can never hurt Left, no matter what the context is; the second means that G acts like H in any context. In this paper, we use the symbol $=$ for both games and outcomes, trusting context to distinguish between them. Also, we will use the same symbols for different game conventions.

Normal play is a very special case. Combinatorial games played under the normal play convention, together with the disjunctive sum, induce a group structure [1, 2, 4, 16]. The inverse of G is its *conjugate*, obtained recursively by $\sim G = \{\sim G^{\mathcal{R}} \,|\, \sim G^{\mathcal{L}}\}$. To check if $G \succcurlyeq H$, that is $G + (\sim H) \succcurlyeq 0$, one only needs to play $G + (\sim H)$ and check if Left wins going second. Also, in game practice, a component $G + (\sim G)$ can be removed from the analysis as it behaves like an empty region of the board. These facts show the importance of invertibility. Under the normal play convention, all games are invertible.

The game $\sim G$ is obtained from G by reversing the roles of Left and Right – ‘turning the board around’ or ‘switching colors’. In normal play, $G + (\sim G) = 0$, which is decidedly not true in misère play. Despite this, from now on, we will write $-G$ instead of $\sim G$.

There are certain subsets of games that occur often that have similar abstract definitions and have attracted much attention.

Definition 1. A *universe* is a class of positions satisfying the following properties:

1. If $G \in \mathcal{U}$ and G' is an option of G, then $G' \in \mathcal{U}$;
2. If $G, H \in \mathcal{U}$, then $G + H \in \mathcal{U}$;
3. If $G \in \mathcal{U}$, then $-G \in \mathcal{U}$.

Universes under the normal play convention have group structure. On the other hand, universes under the misère play convention are almost never groups, only monoids. Even worse, under misère play, it is even possible to have universes where $G + H = 0$ but $H \neq \sim G$ [10, 14].

Among many other reasons, the fact that misère structures lose the group structure makes general misère analysis very difficult; see [11] for a survey. A breakthrough in the study of misère games occurred when Plambeck and Siegel [13, 14] suggested weakened equality and inequality relations in order to compare games only within a particular universe. For example, with this idea, it is possible to say that two dicot positions are equivalent 'modulo dicots', even if they are different in the full misère structure. The *restricted* relations are defined below.

Definition 2 ([14]). For a universe $\mathcal{U}$ and games G, H, the terms *equivalence* and *inequality, modulo* $\mathcal{U}$, are defined by

$$G =_{\mathcal{U}} H \text{ if and only if } o(G + X) = o(H + X) \text{ for all games } X \in \mathcal{U},$$

$$G \succcurlyeq_{\mathcal{U}} H \text{ if and only if } o(G + X) \geqslant o(H + X) \text{ for all games } X \in \mathcal{U}.$$

Here, we will need some standard results on the order relation of games. The proofs are not especially difficult and can be found, for example, in [6].

Theorem 1. *For any universe $\mathcal{U}$ and any games $G, H, J \in \mathcal{U}$, if $G \succcurlyeq_{\mathcal{U}} H$, then $G + J \succcurlyeq_{\mathcal{U}} H + J$.*

Theorem 2. *Let $G, H \in \mathcal{U}$ and let $J \in \mathcal{U}$ be invertible. Then $G + J \succcurlyeq_{\mathcal{U}} H + J$ if and only if $G \succcurlyeq_{\mathcal{U}} H$.*

Theorem 3. *For any universe $\mathcal{U}$ and any games $G, H, J \in \mathcal{U}$, if $G \succ_{\mathcal{U}} 0$ and $H \succcurlyeq_{\mathcal{U}} 0$, then $G + H \succ_{\mathcal{U}} 0$.*

Theorem 4. *Let $G \in \mathcal{U}$. If $|G^{\mathcal{L}}| \geqslant 1$, then for any $A \in \mathcal{U}$ we have $\{G^{\mathcal{L}} \cup \{A\} \mid G^{\mathcal{R}}\} \succcurlyeq_{\mathcal{U}} G$.*

3. Misère Play: Dicot Forms

Recently, some advances have been made with regard to $\mathcal{D}^-$, the dicot universe under the misère play convention. The most useful of these are presented below. Theorem 5 shows how to check for equivalence and inequality between two games, G and H, only looking at their followers.

Theorem 5 ([6]). *For games G and H, $G \succcurlyeq_{\mathcal{D}^-} H$ if and only if*

1. $o(G) \geqslant o(H)$;
2. (a) *For all G^R, there is H^R such that $G^R \succcurlyeq_{\mathcal{D}^-} H^R$ or there is G^{RL} such that $G^{RL} \succcurlyeq_{\mathcal{D}^-} H$;*
   (b) *For all H^L, there is G^L such that $G^L \succcurlyeq_{\mathcal{D}^-} H^L$ or there is H^{LR} such that $G \succcurlyeq_{\mathcal{D}^-} H^{LR}$.*

As usual, from now on, due to the fact that we are only concerned with the dicot misère universe, we will use $=$ and $\succcurlyeq$ instead of $=_{\mathcal{D}^-}$ and $\succcurlyeq_{\mathcal{D}^-}$, respectively.

Corollary 1. *Let G be a dicotic form. Then, $G \succcurlyeq 0$ if and only if $o(G) \geqslant \mathcal{N}$ and for all G^R there is a $G^{RL} \succcurlyeq 0$.*

Corollary 2. *Let G be a dicotic form. Then, $G = 0$ if and only if $o(G) = \mathcal{N}$, for all G^R there is a $G^{RL} \succcurlyeq 0$, and for all G^L there is a $G^{LR} \preccurlyeq 0$.*

Corollary 3. *Modulo $\mathcal{D}^-$, we have $* + * = \{* \mid *\} = 0$.*

The rest of this section concerns reductions and canonical forms for the dicot misère universe.

Theorem 6 (Domination, [5]). *Let $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$ be a dicot form. If $A, B \in G^{\mathcal{L}}$ and $A \preccurlyeq B$, then $G = \{G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\}$.*

Definition 3. For a game G in any universe $\mathcal{U}$, suppose there are followers $A \in G^{\mathcal{L}}$ and $B \in A^{\mathcal{R}}$ with $B \preccurlyeq_{\mathcal{U}} G$. Then the Left option A is *reversible*, and sometimes, to be specific, A is said to be *reversible through* its right option B. In addition, B is called a *reversing* option for A and, if $B^{\mathcal{L}}$ is non-empty, then $B^{\mathcal{L}}$ is a *replacement set* for A. In this case, A is said to be *non-atomic-reversible*. If the reversing option is left-atomic, that is, if $B^{\mathcal{L}} = \emptyset$, then A is said to be *atomic-reversible*.

Theorem 7 (Non-atomic reversibility, [5]). *Let G be a dicot form and suppose that A is a left option of G that is reversible through B. If $B^{\mathcal{L}}$ is non-empty, then $G = \big\{(G^{\mathcal{L}} \setminus \{A\}) \cup B^{\mathcal{L}} \mid G^{\mathcal{R}}\big\}$.*

Theorem 8 (Atomic reversibility, [5]). *Let G be a dicot form and suppose that $A \in G^{\mathcal{L}}$ is reversible through $B = 0$.*

1. *If, in G, there is a Left winning move $C \in G^{\mathcal{L}} \setminus \{A\}$, then $G = \big\{G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\big\}$;*
2. *If A is the only winning Left move in G, then $G = \big\{*, G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\big\}$.*

Theorem 9 (Substitution Theorem, [5]). *If $G = \{A \mid C\}$ where A and C are atomic-reversible options , then $G = 0$.*

A game G is said to be in *canonical form* if none of the previous theorems can be applied to G or to followers to obtain an equivalent game in the dicot misère universe with different sets of options. In [5], there is a proof for uniqueness and simplicity (game tree of least depth) of dicot misère canonical forms. We now take a closer look at the conjugate property. Recall that $-G$ is obtained by reversing the roles of Left and Right.

Theorem 10 ([7])**.** *For all dicot forms G, H, if $G + H = 0$, then $H = -G$.*

In a misère universe, it is helpful to have the notion of the *adjoint* of a game G, denoted by G°. The adjoint, whose definition is given below, is a game such that $G + G^\circ \in \mathcal{P}$. The game G° can be thought of as the misère analogue of the inverse of G.

Definition 4 ([16])**.** $$G^\circ = \begin{cases} * & \text{if } G^{\mathcal{L}} = \emptyset \text{ and } G^{\mathcal{R}} = \emptyset; \\ \{(G^{\mathcal{R}})^\circ \,|\, 0\} & \text{if } G^{\mathcal{L}} = \emptyset \text{ and } G^{\mathcal{R}} \neq \emptyset; \\ \{0 \,|\, (G^{\mathcal{L}})^\circ\} & \text{if } G^{\mathcal{L}} \neq \emptyset \text{ and } G^{\mathcal{R}} = \emptyset; \\ \{(G^{\mathcal{R}})^\circ \,|\, (G^{\mathcal{L}})^\circ\} & \text{if } G^{\mathcal{L}} \neq \emptyset \text{ and } G^{\mathcal{R}} \neq \emptyset. \end{cases}$$

We end this section with some facts about *symmetric forms*.

Definition 5. A *symmetric form* G is a form $\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$ such that $G^{\mathcal{R}} = -G^{\mathcal{L}}$.

Theorem 11. *Let G be a symmetric form. Then,*

1. $G \in \mathcal{N} \cup \mathcal{P}$;
2. $G = 0$ *or* $G \parallel 0$.

Proof. The proof uses the 'strategy-stealing argument'. Suppose that $G \in \mathcal{L}$. Left wins G playing first with some option G^L. Hence, by symmetry, Right wins G playing first with $-G^L$. This contradicts $G \in \mathcal{L}$, therefore G cannot be in $\mathcal{L}$. A similar argument can be used to prove that G cannot be in $\mathcal{R}$.

Suppose now that $G \succ 0$. By Corollary 1, Left wins G playing first and for all G^R, there is some $G^{RL} \succcurlyeq 0$. But, by symmetry, this means that Left wins G playing first and for all $-G^L$, there is some $-G^{LR} \succcurlyeq 0$. This implies that Right wins G playing first and for all G^L, there is some $G^{LR} \preccurlyeq 0$. Hence, by Corollary 2, $G = 0$. This contradicts $G \succ 0$. Thus G cannot be strictly larger than 0. A similar argument can be used to prove that G cannot be strictly less than 0. □

Corollary 4. *Let G be a game form. Then, $G - G \in \mathcal{N} \cup \mathcal{P}$ and $G - G = 0$ or $G - G \parallel 0$.*

Proof. By definition of conjugate and disjunctive sum,

$$\begin{aligned} &G - G &= \\ &\{G^{\mathcal{L}} \,|\, G^{\mathcal{R}}\} + \{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\} &= \\ &\{G^{\mathcal{L}} - G, G - G^{\mathcal{R}} \,|\, G^{\mathcal{R}} - G, G - G^{\mathcal{L}}\} &= \\ &\{G^{\mathcal{L}} - G, G - G^{\mathcal{R}} \mid -(G - G^{\mathcal{R}}), -(G^{\mathcal{L}} - G)\} & \end{aligned}$$

Therefore, $G-G$ is symmetric, and, by Theorem 11, the result follows. □

4. Invertible Elements in the Dicot Misère Universe

Considering the classic game NIM [3], a game form of a finite heap of size n is designated by $*n$ and formally $*n=\{0,*,\dots,*(n-1)\,|\,0,*,\dots,*(n-1)\}$. It is easy to check that, in the dicot misère universe, $*2=\{0,*\,|\,0,*\}$ is not invertible. By the conjugate property, if it was invertible, we would have $*2+*2=0$. However, $*2+*2\in\mathcal{P}$ (against a move by the first player, the second player wins by moving to $*$) and $0\in\mathcal{N}$. So, a natural question arises: 'Is it true that a non-invertible element G always satisfies the property $G-G\in\mathcal{P}$?'. The answer is no. To see this consider $G=\{0\,|\,*2\}$. This game is in canonical form, $G-G\in\mathcal{N}$, and $G-G\neq 0$. The next question that one could ask is 'Is it true that a non-invertible element G always has $*2$ as a follower?'. The answer to this question is again no. Consider $H=\{0,*\,|\,\{*\,|\,0,*\},\{0\,|\,0,*\}\}$ (in canonical form). Then, $*2$ is not a follower of $G=\{0\,|H\}$, $G-G\in\mathcal{N}$, and $G-G\neq 0$. These questions touch the essence of the problem, but the characterization of the invertible elements of the dicot misère universe is more sophisticated. It is presented in Theorem 12, our main result.

Theorem 12 (Characterization of invertible elements of the dicot misère universe)**.** *Let G be a dicot in canonical form. Then, G is invertible if and only if there is no G', a follower of G, such that $G'-G'\in\mathcal{P}$.*

4.1. Structure of the Proof

Consider G in canonical form. By Corollary 2, $G-G=0$ if and only if (i) $o(G-G)=\mathcal{N}$, (ii) for all $(G-G)^R$ there is a $(G-G)^{RL}\succcurlyeq 0$, and (iii) for all $(G-G)^L$ there is a $(G-G)^{LR}\preccurlyeq 0$. The difficult part is to prove that if G is invertible, then there is no G', a subposition of G, such that $G'-G'\in\mathcal{P}$.

The proof is by contradiction. If G were the simplest invertible canonical form with an option G^{L_1} that has a follower $G^{L_1'}$ such that $G^{L_1'}-G^{L_1'}\in\mathcal{P}$, then it is necessary to have some $(G^{L_1}-G)^R\preccurlyeq 0$ by Corollary 2. Moreover, we must have $G^{L_1}-G^{L_2}\prec 0$ for all Left options G^{L_2} as G^{L_1} cannot be invertible. Then Corollary 2 again assures us that we have $(G-G^{L_2})^L\succcurlyeq 0$. Repeating this process, the contradiction we reach is the existence of an infinite sequence of follower differences, meaningless in the context of short games (infinite descent is not possible in a short game):

$$\begin{gathered}G^{L_1}-G^{L_2}\prec 0\\ G^{L_3}-G^{L_2}\succ 0\\ G^{L_3}-G^{L_4}\prec 0\\ \cdots\end{gathered}$$

A crucial fact needed in the creation of the preceding infinite sequence pertains to Right's response to $G^{L_3} - G$. Specifically, why must there be a G^{L_4} with $G^{L_3} - G^{L_4} \prec 0$, and not, say $G^{L_3} - G^{L_1} \prec 0$? If so, then $G^{L_3} - G^{L_2} \succ 0$ and $G^{L_1} - G^{L_2} \prec 0$ (used as $-G^{L_1} + G^{L_2} \succ 0$) yield $G^{L_3} - G^{L_2} + G^{L_2} - G^{L_1} \succ 0$. Under the normal play ending condition, this would be equivalent to $G^{L_3} - G^{L_1} \succ 0$, contradicting our initial assumption $G^{L_3} - G^{L_1} \prec 0$.

However, we are playing under the misère play ending condition and thus no such simplification is possible. (In general, if H is not invertible, then $G+H-H+W \succ 0$ is not equivalent to $G+W \succ 0$ in a misère universe.) Fortunately for us, it is possible to prove a weaker result (Lemma 1) that makes the proof work inside the monoid: *If $G \succ 0$, then $G + H - H \not\prec 0$.*

4.2. Characterization of the Invertible Elements of $\mathcal{D}^-$

Lemma 1. *Let G and H be two dicots. If $G \succ 0$, then $G + H - H \not\prec 0$.*

Proof. By Corollary 4, $H - H \in \mathcal{P}$ or $H - H \in \mathcal{N}$.

If $H - H = 0$, then $G \succ 0$ implies $G + H - H \succ 0$, and, of course, we have $G + H - H \not\prec 0$.

If $H - H \neq 0$ and $H - H \in \mathcal{P}$, then Left, playing first, has a winning move in $H - H + *$ (she removes the star). Moreover, since $G \succ 0$, Left also has a winning move in $G + H - H + *$ playing first. However, if Left plays first in $0 + *$, she loses. Hence, $G + H - H \not\prec 0$.

If $H - H \neq 0$ and $H - H \in \mathcal{N}$, then we let

$$X = \{0 \,|\, \{\mathcal{F}^\circ(H - H) \,|\, 0\}\},$$

where $\mathcal{F}^\circ(H-H)$ is the set of the adjoints of all followers of $H-H$. By Corollary 2, if $H-H \neq 0$, then there must be some $(H-H)^L$ such that there is no $(H-H)^{LR} \preccurlyeq 0$. We claim that $(H - H)^L + X$ is a Left winning move in $H - H + X$.

In fact, if Right answers in X, Left replies with $(H - H)^L + \big((H - H)^L\big)^\circ$ and wins. On the other hand, if Right answers with $(H - H)^{LR} + X$, we have two possibilities: (1) if $(H - H)^{LR} \in \mathcal{L} \cup \mathcal{P}$, Left replies with $(H - H)^{LR} + 0$ and wins; (2) if $(H - H)^{LR} \in \mathcal{N} \cup \mathcal{R}$, then $(H - H)^{LR} \not\preccurlyeq 0$ implies that there exists some $(H - H)^{LRL}$ such that there is no $(H - H)^{LRLR} \preccurlyeq 0$. Left replies with $(H - H)^{LRL} + X$ and the process is repeated, but it cannot go on indefinitely (again, we only consider short games). So, at some point, Right's move has to fall into one of the previous cases, and Left wins. Thus Left can win $H - H + X$ playing first. Since $G \succ 0$, this then implies that Left can win $G + H - H + X$ playing first. However, Left loses $0 + X$ playing first. Hence, $G + H - H \not\prec 0$. □

Proof. (Theorem 12)
($\Leftarrow$) Suppose that there is no G', a follower of G, such that $G' - G' \in \mathcal{P}$. Because

G is a follower of itself, we have that $G - G \in \mathcal{N}$. Now if Left moves in $G - G$ to $G^L - G$, Right can reply with $G^L - G^L$. By induction G^L must be invertible and thus, by the conjugate property, $G^L - G^L = 0$. By Corollary 2 we now have that $G - G = 0$ which tells us that G is invertible.

($\Rightarrow$) For contradiction, suppose that there is a game G which is invertible and has a follower G' such that $G' - G' \in \mathcal{P}$. Moreover, we will assume that G is the simplest possible game satisfying such hypotheses. By the conjugate property, it follows that $G - G = 0$. Thus, by Corollary 2, for all G^R there is a $G^{RL} \succcurlyeq 0$ and for all G^L there is a $G^{LR} \preccurlyeq 0$. Note that the existence of G' assures us that $G \neq \{\ |\ \}$ and there are in fact moves in $G - G$.

Thus, without loss of generality, we are able to consider a Left move G^{L_1} such that $G^{L_1} = G'$ or G^{L_1} contains G'. To get the argument started, we consider the Left option $G^{L_1} - G$. By Corollary 2, Right must have an answer which is less than or equal to zero.

The game $G^{L_1} - G^{L_1}$ cannot be 0 because G is the simplest invertible element satisfying the hypotheses of the theorem. Hence, G^{L_1} is not invertible. Further $G^{L_1} - G^{L_1}$ cannot be less than zero because its game tree is symmetric. Therefore, $G^{L_1} - G^{L_1}$ cannot be Right's response to $G^{L_1} - G$.

On the other hand, we cannot have $G^{L_1R} - G \preccurlyeq 0$. If so, the fact that G is invertible allows us to conclude that $G^{L_1R} \preccurlyeq G$ and, due to Theorem 8 and the fact that G is in canonical form, G^{L_1} must be the atomic reversible option $*$, making G^{L_1} invertible.

Right's reply must be to some $G^{L_1} - G^{L_2} \preccurlyeq 0$. Moreover, we must have $G^{L_1} - G^{L_2} \prec 0$ because, if $G^{L_1} - G^{L_2} = 0$, then G^{L_1} would be invertible. Note too that G^{L_2} is also not invertible, otherwise we would have $G^{L_1} \prec G^{L_2}$, but recall that G is given to us in canonical form.

Next consider $G - G^{L_2}$, a right option of $G - G$. Using arguments similar to those given above, there exists $G^{L_3} - G^{L_2} \succ 0$ (G^{L_3} not invertible).

Against $G^{L_3} - G$, a left option of $G - G$, we cannot have $G^{L_3} - G^{L_1} \prec 0$. If this were the case, then we would have $G^{L_3} - G^{L_1} + G^{L_1} - G^{L_2} \prec 0$, and this contradicts Lemma 1. In fact, against $G^{L_k} - G$, a left option of $G - G$, we cannot have $G^{L_k} - G^{L_{k-i}} \prec 0$. If so, we would have

$$G^{L_k} - G^{L_{k-i}} + G^{L_{k-i}} - G^{L_{k-i+1}} + \ldots - G^{L_{k-1}} \prec 0,$$

contradicting Lemma 1.

Hence, there exists G^{L_4} such that $G^{L_3} - G^{L_4} \prec 0$ (G^{L_4} not invertible).

This process necessarily results in the following infinite sequence:

$$G^{L_1} - G^{L_2} \prec 0,\, G^{L_3} - G^{L_2} \succ 0,\, G^{L_3} - G^{L_4} \prec 0,\, G^{L_5} - G^{L_4} \succ 0, \ldots$$

But, due to the fact that we are considering short games, such an infinite sequence cannot exist. So, there is no invertible dicot G in canonical form with a follower G'

such that $G' - G' \in \mathcal{P}$. $\square$

Observation 13. The result works only for canonical forms. For example, $\{0, *, *2 \,|\, 0\}$ is invertible and $*2 + *2 \in \mathcal{P}$. However, $\{0, *, *2 \,|\, 0\}$ is not in canonical form; its canonical form is $\{0, * \,|\, 0\}$.

Corollary 5. *Let G be a dicot in canonical form. If $*2$ is a follower of G, then G is not invertible.*

Proof. This is an immediate consequence of Theorem 12 and the fact that $*2 + *2$ is a $\mathcal{P}$–position. $\square$

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