



**TWO IDENTITIES FOR COMPLEX NUMBERS WITH
APPLICATIONS TO SPECIAL SEQUENCES AND FUNCTIONS**

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Abstract

We present two identities for n complex numbers.

1. Let $P(z) = \alpha z^m + \beta z^{m-1} + \dots$ be a polynomial of degree m with $m \leq n$, and let a_1, \dots, a_n be pairwise distinct complex numbers. Then,

$$\sum_{k=1}^n P(a_k) \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{a_k - a_j} = \begin{cases} 0, & \text{if } m < n - 1, \\ \alpha, & \text{if } m = n - 1, \\ \alpha(a_1 + \dots + a_n) + \beta, & \text{if } m = n. \end{cases}$$

2. For all pairwise distinct complex numbers z_1, \dots, z_n ($n \geq 2$),

$$1 + \sum_{k=1}^n z_k - \prod_{k=1}^n z_k = \sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^n \left(1 + \frac{z_j(z_j - 1)(z_k - 1)}{z_k - z_j} \right).$$

Applications lead to formulas involving special sequences and functions like, for example,

$$F_n = n - \sum_{k=1}^{n-1} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} \left(1 + \frac{2(2 \cos \frac{j\pi}{n} + i) \cos \frac{j\pi}{n} \cos \frac{k\pi}{n}}{\cos \frac{j\pi}{n} - \cos \frac{k\pi}{n}} \right),$$

where F_n denotes the n -th Fibonacci number.

1. Introduction and Statement of the Main Results

The work on this paper has been inspired by an interesting article published by J.L. Díaz-Barrero [1] in 2003. He applied methods from complex analysis to obtain identities involving the classical Fibonacci and Lucas numbers. One of his results states that for $n \geq 0$,

$$\frac{L_n L_{n+1}}{(L_n - L_{n+2})(L_{n+1} - L_{n+2})} + \frac{L_{n+1} L_{n+2}}{(L_{n+1} - L_n)(L_{n+2} - L_n)} + \frac{L_{n+2} L_n}{(L_{n+2} - L_{n+1})(L_n - L_{n+1})} = 1, \tag{1.1}$$

where L_k denotes the k -th Lucas number. The following companion to (1.1) is valid,

$$\frac{(L_n L_{n+1})^{-1}}{(L_n - L_{n+2})(L_{n+1} - L_{n+2})} + \frac{(L_{n+1} L_{n+2})^{-1}}{(L_{n+1} - L_n)(L_{n+2} - L_n)} + \frac{(L_{n+2} L_n)^{-1}}{(L_{n+2} - L_{n+1})(L_n - L_{n+1})} = 0. \tag{1.2}$$

However, formulas (1.1) and (1.2) do not characterize the Lucas numbers. Indeed, both are, respectively, special cases of

$$\frac{ab}{(a - c)(b - c)} + \frac{bc}{(b - a)(c - a)} + \frac{ca}{(c - b)(a - b)} = 1$$

and

$$\frac{(ab)^{-1}}{(a - c)(b - c)} + \frac{(bc)^{-1}}{(b - a)(c - a)} + \frac{(ca)^{-1}}{(c - b)(a - b)} = 0 \tag{1.3}$$

which hold for all pairwise distinct complex numbers a, b, c . In (1.3), additionally, $abc \neq 0$.

The aim of this paper is to present related identities with n complex numbers. Our first theorem offers an extension of (1.3).

Theorem 1. *Let $P(z) = \alpha z^m + \beta z^{m-1} + \dots$ be a polynomial of degree m with $m \leq n$, and let a_1, \dots, a_n be pairwise distinct complex numbers. Then,*

$$\sum_{k=1}^n P(a_k) \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{a_k - a_j} = \begin{cases} 0, & \text{if } m < n - 1, \\ \alpha, & \text{if } m = n - 1, \\ \alpha(a_1 + \dots + a_n) + \beta, & \text{if } m = n. \end{cases} \tag{1.4}$$

Remark 1. Let a_1, \dots, a_n be pairwise distinct complex numbers. From Theorem 1 with $P(z) = z$, we obtain for $n \geq 3$,

$$\sum_{k=1}^n a_k \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{a_k - a_j} = 0. \tag{1.5}$$

If we set $n = 3$ and multiply by $(a_1 a_2 a_3)^{-1}$, then (1.5) leads to (1.3).

Remark 2. Let a_1, \dots, a_n be pairwise distinct complex numbers which are different from 0. We apply (1.4) with $P(z) = 1$ and $n + 1$ instead of n . Then, we set $a_{n+1} = 0$ and multiply both sides by $(-1)^n a_1 \cdots a_n$. This yields for $n \geq 2$,

$$\sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^n \frac{a_j}{a_j - a_k} = 1. \tag{1.6}$$

Remark 3. An application of (1.6) leads to the following counterpart of (1.5). Let

$$\prod_{j=1}^n a_j = Q \neq 0,$$

where a_1, \dots, a_n are pairwise distinct complex numbers. Then,

$$\sum_{k=1}^n \frac{1}{a_k} \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{a_j - a_k} = \frac{1}{Q}. \tag{1.7}$$

Our second theorem presents an identity which involves the sum and the product of n complex numbers.

Theorem 2. Let z_1, \dots, z_n ($n \geq 2$) be pairwise distinct complex numbers. Then,

$$1 + \sum_{k=1}^n z_k - \prod_{k=1}^n z_k = \sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^n \left(1 + \frac{z_j(z_j - 1)(z_k - 1)}{z_k - z_j} \right). \tag{1.8}$$

Remark 4. An application of the arithmetic mean - geometric mean inequality and Theorem 2 shows that the inequality

$$\sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^n \left(1 + \frac{z_j(z_j - 1)(z_k - 1)}{z_k - z_j} \right) > 1 + nQ^{1/n} - Q$$

is valid for all positive real numbers z_1, \dots, z_n ($n \geq 2$) which are pairwise distinct and satisfy $\prod_{k=1}^n z_k = Q$. Let $\epsilon > 0$. If we use (1.8) with

$$\begin{aligned} z_k &= (1 + k\epsilon)Q^{1/n} && \text{for } k = 1, \dots, [n/2], \\ z_k &= \frac{Q^{1/n}}{1 + (k - [n/2])\epsilon} && \text{for } k = [n/2] + 1, \dots, 2[n/2], \\ z_n &= Q^{1/n} && \text{if } n \text{ is odd,} \end{aligned}$$

and let $\epsilon \rightarrow 0$, then we conclude that the constant lower bound $1 + nQ^{1/n} - Q$ is sharp.

The proofs of the two theorems are given in the next section. In Section 3, we apply Theorem 1, Theorem 2 and Remark 3 to obtain various identities involving Lucas and Fibonacci numbers, harmonic numbers, the central binomial coefficient, Chebyshev polynomials of the second kind and an n -th root of unity.

2. Proofs

Proof of Theorem 1. We consider two cases.

Case 1. $m \leq n - 1$. Using the partial fraction decomposition yields

$$\frac{P(z)}{(z - a_1) \cdots (z - a_n)} = \frac{A_1}{z - a_1} + \cdots + \frac{A_n}{z - a_n},$$

where

$$A_k = P(a_k) \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{a_k - a_j}.$$

This leads to

$$P(z) = \alpha z^m + \beta z^{m-1} + \cdots = \prod_{j=1}^n (z - a_j) \sum_{j=1}^n \frac{A_j}{z - a_j} = c_{n-1} z^{n-1} + \cdots \quad (2.1)$$

with

$$c_{n-1} = \sum_{k=1}^n A_k.$$

Comparing the coefficients gives that if $m < n - 1$, then (2.1) leads to $c_{n-1} = 0$. If $m = n - 1$, then we obtain from (2.1) that $c_{n-1} = \alpha$.

Case 2. $m = n$. We have $P(z) = \alpha z^n + P_1(z)$, where P_1 is a polynomial of degree $\leq n - 1$. It follows that

$$\sum_{k=1}^n P(a_k) \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{a_k - a_j} = \alpha S_1 + S_2 \quad (2.2)$$

with

$$S_1 = \sum_{k=1}^n a_k^n \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{a_k - a_j} \quad \text{and} \quad S_2 = \sum_{k=1}^n P_1(a_k) \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{a_k - a_j}.$$

From

$$\frac{z^{n-1}}{(z - a_1) \cdots (z - a_n)} = \frac{A_1^*}{z - a_1} + \cdots + \frac{A_n^*}{z - a_n}$$

with

$$A_k^* = a_k^{n-1} \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{a_k - a_j}$$

we obtain

$$z^n = \prod_{j=1}^n (z - a_j) \sum_{k=1}^n A_k^* + \prod_{j=1}^n (z - a_j) \sum_{k=1}^n \frac{a_k A_k^*}{z - a_k}.$$

Since

$$\prod_{j=1}^n (z - a_j) = z^n - \sum_{k=1}^n a_k \cdot z^{n-1} + \dots,$$

we get

$$z^n = \left(z^n - \sum_{k=1}^n a_k \cdot z^{n-1} + \dots \right) \sum_{k=1}^n A_k^* + \sum_{k=1}^n a_k A_k^* \cdot z^{n-1} + \dots.$$

Next, we compare the coefficients. This gives

$$\sum_{k=1}^n A_k^* = 1 \quad \text{and} \quad \sum_{k=1}^n a_k \sum_{k=1}^n A_k^* = \sum_{k=1}^n a_k A_k^*.$$

It follows that

$$\sum_{k=1}^n a_k = \sum_{k=1}^n a_k A_k^* = S_1. \tag{2.3}$$

Since $\deg P_1 \leq n - 1$, we obtain (as in Case 1 with P_1 instead of P) $\beta = S_2$. From (2.2) and (2.3) we conclude that

$$\sum_{k=1}^n P(a_k) \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{a_k - a_j} = \alpha \sum_{k=1}^n a_k + \beta.$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. We use induction on n . Let

$$F_n(z_1, \dots, z_n) = 1 + \sum_{k=1}^n z_k - \prod_{k=1}^n z_k - \sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^n \left(1 + \frac{z_j(z_j - 1)(z_k - 1)}{z_k - z_j} \right).$$

We have for $z_1 \neq z_2$,

$$\begin{aligned} F_2(z_1, z_2) &= 1 + z_1 + z_2 - z_1z_2 - \left(1 + \frac{z_2(z_2 - 1)(z_1 - 1)}{z_1 - z_2}\right) \\ &\quad - \left(1 + \frac{z_1(z_1 - 1)(z_2 - 1)}{z_2 - z_1}\right) \\ &= 0. \end{aligned}$$

Now, we assume that $F_n(w_1, \dots, w_n) = 0$ for all pairwise distinct complex numbers w_1, \dots, w_n . Let z_1, \dots, z_{n+1} be complex numbers which are pairwise distinct. We consider two cases.

Case 1. There exists a number z_r ($1 \leq r \leq n + 1$) such that $z_r = 0$ or $z_r = 1$. Since

$$F_{n+1}(z_1, \dots, z_{r-1}, 0, z_{r+1}, \dots, z_{n+1}) = F_n(z_1, \dots, z_{r-1}, z_{r+1}, \dots, z_{n+1})$$

and

$$F_{n+1}(z_1, \dots, z_{r-1}, 1, z_{r+1}, \dots, z_{n+1}) = F_n(z_1, \dots, z_{r-1}, z_{r+1}, \dots, z_{n+1}),$$

we conclude from the induction hypothesis that

$$F_{n+1}(z_1, \dots, z_r, \dots, z_{n+1}) = 0.$$

Case 2. Let $z_k \neq 0$ and $z_k \neq 1$ for $k = 1, \dots, n + 1$. We obtain for $z \in \mathbb{C} \setminus \{z_1, \dots, z_n\}$,

$$\begin{aligned} F_{n+1}(z_1, \dots, z_n, z) &= 1 + \sum_{k=1}^n z_k + z - \left(\prod_{k=1}^n z_k\right) z \\ &\quad - \sum_{k=1}^n \left[\left(1 + \frac{z(z - 1)(z_k - 1)}{z_k - z}\right) \prod_{\substack{j=1 \\ j \neq k}}^n \left(1 + \frac{z_j(z_j - 1)(z_k - 1)}{z_k - z_j}\right) \right] \\ &\quad - \prod_{j=1}^n \left(1 + \frac{z_j(z_j - 1)(z - 1)}{z - z_j}\right). \end{aligned}$$

Let

$$G(z) = F_{n+1}(z_1, \dots, z_n, z) \prod_{j=1}^n (z - z_j).$$

Here, G is a polynomial of degree at most $n + 1$. Since

$$G(0) = F_n(z_1, \dots, z_n) \prod_{j=1}^n (-z_j) \quad \text{and} \quad G(1) = F_n(z_1, \dots, z_n) \prod_{j=1}^n (1 - z_j),$$

we conclude from the induction hypothesis that $G(0) = G(1) = 0$. Next, we prove that G has additional zeros. Let $r \in \{1, \dots, n\}$. Then,

$$\begin{aligned} \lim_{z \rightarrow z_r} (z - z_r) \left(1 + \sum_{k=1}^n z_k + z - \left(\prod_{k=1}^n z_k \right) z \right) &= 0, \\ \lim_{z \rightarrow z_r} (z - z_r) \sum_{k=1}^n \left[\left(1 + \frac{z(z-1)(z_k-1)}{z_k-z} \right) \prod_{\substack{j=1 \\ j \neq k}}^n \left(1 + \frac{z_j(z_j-1)(z_k-1)}{z_k-z_j} \right) \right] \\ &= -z_r(z_r-1)^2 \prod_{\substack{j=1 \\ j \neq r}}^n \left(1 + \frac{z_j(z_j-1)(z_r-1)}{z_r-z_j} \right) \end{aligned}$$

and

$$\begin{aligned} \lim_{z \rightarrow z_r} (z - z_r) \prod_{j=1}^n \left(1 + \frac{z_j(z_j-1)(z-1)}{z-z_j} \right) \\ = z_r(z_r-1)^2 \prod_{\substack{j=1 \\ j \neq r}}^n \left(1 + \frac{z_j(z_j-1)(z_r-1)}{z_r-z_j} \right). \end{aligned}$$

This leads to

$$\lim_{z \rightarrow z_r} (z - z_r) F_{n+1}(z_1, \dots, z_n, z) = 0.$$

Thus,

$$G(z_1) = \dots = G(z_n) = 0.$$

It follows that G has $n + 2$ distinct zeros. This can only happen if $G(z) \equiv 0$. Hence,

$$F_{n+1}(z_1, \dots, z_{n+1}) = G(z_{n+1}) \prod_{j=1}^n \frac{1}{z_{n+1} - z_j} = 0.$$

This completes the proof of (1.8). □

3. Applications

We apply both theorems and Remark 3 to provide various identities involving special sequences and functions.

I. Let L_n and F_n be the n -th Lucas and Fibonacci numbers, respectively.

(i) Using the representation

$$L_{n+2} = 1 + \sum_{k=0}^n L_k$$

we conclude from (1.8) with $z_k = L_{k-1}$ and $n + 1$ instead of n that

$$L_{n+2} = \prod_{k=0}^n L_k + \sum_{k=0}^n \prod_{\substack{j=0 \\ j \neq k}}^n \left(1 + \frac{L_j(L_j - 1)(L_k - 1)}{L_k - L_j} \right).$$

(ii) We have the product formula

$$F_n = \prod_{k=1}^{n-1} \left(1 - 2i \cos \frac{k\pi}{n} \right),$$

see Rudolph-Lilith [2]. Applying (1.8) gives

$$F_n = n - \sum_{k=1}^{n-1} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} \left(1 + \frac{2(2 \cos \frac{j\pi}{n} + i) \cos \frac{j\pi}{n} \cos \frac{k\pi}{n}}{\cos \frac{j\pi}{n} - \cos \frac{k\pi}{n}} \right)$$

and (1.7) yields

$$\frac{1}{F_n} = \left(\frac{i}{2} \right)^{n-2} \sum_{k=1}^{n-1} \frac{1}{1 - 2i \cos \frac{k\pi}{n}} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} \frac{1}{\cos \frac{j\pi}{n} - \cos \frac{k\pi}{n}}.$$

II. Let $H_n = 1 + 1/2 + \dots + 1/n$ be the n -th harmonic number.

(i) From (1.8) with $z_k = 1 + 1/k$ we conclude that

$$H_n = \sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^r \left(1 + \frac{j+1}{j(j-k)} \right).$$

(ii) Let B_n be the Bernoulli polynomial of degree $n \geq 2$. Then,

$$B_n(z) = z^n - \frac{n}{2} z^{n-1} + \dots.$$

We use (1.4) with $P = B_n$ and $a_k = 1/k$. This leads to

$$H_n = \frac{n}{2} + \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} k^{n-1} B_n\left(\frac{1}{k}\right).$$

(iii) We set $z_k = (-1)^{k+1}/k$ and apply (1.8) with $2n$ instead of n . Since

$$H_{2n} - H_n = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k},$$

we get

$$1 + H_{2n} - H_n = \frac{(-1)^n}{(2n)!} + \sum_{k=1}^{2n} \prod_{\substack{j=1 \\ j \neq k}}^{2n} \left(1 + \frac{(1 + (-1)^j j) (1 + (-1)^k k)}{j(j - (-1)^{j+k} k)} \right).$$

III. We have

$$\prod_{k=1}^{n-1} \left(1 - \frac{n^2}{k^2}\right) = \frac{(-1)^{n-1}}{2} \binom{2n}{n} \quad \text{and} \quad \sum_{k=1}^{n-1} \left(1 - \frac{n^2}{k^2}\right) = n - 1 - n^2 H_{n-1}^{(2)},$$

where $H_n^{(2)} = 1 + 1/2^2 + \dots + 1/n^2$ denotes the n -th harmonic number of second order. Then, (1.8) with $z_k = 1 - n^2/k^2$ yields the following representation for the central binomial coefficient,

$$\frac{(-1)^n}{2} \binom{2n}{n} = n^2 H_{n-1}^{(2)} - n + \sum_{k=1}^{n-1} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} \left(1 + \frac{n^2(n^2 - j^2)}{j^2(j^2 - k^2)}\right).$$

IV. Let U_n be the Chebyshev polynomial of the second kind of degree n . Applying (1.8) and

$$U_n(z) = 2^n \prod_{k=1}^n \left(z - \cos \frac{k\pi}{n+1}\right)$$

leads to

$$\frac{1}{2^n} U_n(z) = 1 + nz - \sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^n \left(1 + \frac{(z - \cos \frac{j\pi}{n+1})(z - 1 - \cos \frac{j\pi}{n+1})(z - 1 - \cos \frac{k\pi}{n+1})}{\cos \frac{j\pi}{n+1} - \cos \frac{k\pi}{n+1}}\right).$$

Provided that $U_n(z) \neq 0$, we conclude from (1.7) that

$$\frac{2^n}{U_n(z)} = \sum_{k=1}^n \frac{1}{z - \cos \frac{k\pi}{n+1}} \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{\cos \frac{k\pi}{n+1} - \cos \frac{j\pi}{n+1}}.$$

In particular, since $U_n(1) = n + 1$, we obtain

$$(n + 1) \left(1 - \frac{1}{2^n}\right) = \sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^n \left(1 + \frac{(1 - \cos \frac{j\pi}{n+1}) \cos \frac{j\pi}{n+1} \cos \frac{k\pi}{n+1}}{\cos \frac{j\pi}{n+1} - \cos \frac{k\pi}{n+1}}\right)$$

and

$$\frac{2^n}{n + 1} = \sum_{k=1}^n \frac{1}{1 - \cos \frac{k\pi}{n+1}} \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{\cos \frac{k\pi}{n+1} - \cos \frac{j\pi}{n+1}}.$$

V. Let $\omega_n = \exp(2\pi i/n)$ be an n -th root of unity. An application of

$$1 - z^n = \prod_{k=0}^{n-1} (1 - z\omega_n^k)$$

gives

$$z^n + n = \sum_{k=0}^{n-1} \prod_{\substack{j=0 \\ j \neq k}}^{n-1} \left(1 + \frac{z\omega_n^{j+k}(1 - z\omega_n^j)}{\omega_n^j - \omega_n^k}\right).$$

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