ON THE ALMKVIST–MEURMAN THEOREM FOR BERNOULLI POLYNOMIALS

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Abstract

Almkvist and Meurman showed that if h and k are integers, then so is

$$k^n\big(B_n(h/k) - B_n\big),$$

where $B_n(u)$ is the Bernoulli polynomial. We give here a new and simpler proof of the Almkvist–Meurman theorem using generating functions. We describe some properties of these numbers and prove a common generalization of the Almkvist–Meurman theorem and a result of Gy on Bernoulli–Stirling numbers. We then give a simple generating function proof of an analogue of the Almkvist–Meurman theorem for Euler polynomials, due to Fox.

1. The Almkvist–Meurman Theorem

1.1. Introduction

Let $B_n(u)$ denote the nth Bernoulli polynomial, defined by

$$\sum_{n=0}^{\infty} B_n(u)\frac{x^n}{n!} = \frac{xe^{ux}}{e^x - 1}. \tag{1}$$

Then $B_n(0)$ is the nth Bernoulli number B_n, defined by

$$\sum_{n=0}^{\infty} B_n\frac{x^n}{n!} = \frac{x}{e^x - 1}.$$

These generating functions may be viewed as formal power series or as convergent series for $|x| < 2\pi$.

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For $k \neq 0$, let

$$M_n(h,k) = k^n\big(B_n(h/k) - B_n\big) = \sum_{i=0}^{n-1} \binom{n}{i} B_i h^i k^{n-i}. \tag{2}$$

Almkvist and Meurman [1] showed in 1991 that if h and k are integers then $M_n(h,k)$ is an integer. Other proofs were given by Sury [9], Bartz and Rutkowski [2], and Clarke and Slavutskii [4].

We give here a simple new proof, using generating functions, of the Almkvist–Meurman theorem, and we discuss some of the properties of these integers. We also prove a common generalization of the Almkvist–Meurman theorem and a theorem of Gy [6] on "Bernoulli-Stirling numbers," and we give a simple generating function proof of an analogue of the Almkvist–Meurman theorem for Euler polynomials due to Fox [5].

1.2. Vandiver's Theorem

A closely related result, from which Almkvist and Meurman's result follows easily, was proved by Vandiver much earlier. (Previous authors on the Almkvist–Meurman theorem seem to be unaware of Vandiver's work.) Vandiver stated his result and gave a brief indication of the proof in 1937 [11] and gave a complete proof using a different approach in 1941 [12, Theorem III]. Carlitz [3, Theorem 2] gave another proof of Vandiver's theorem. We state Vandiver's theorem here and explain how the Almkvist–Meurman theorem follows from it.

Theorem (Vandiver)**.** *Let h and k be integers with $k \neq 0$. If n is even and positive then*

$$k^n B_n(h/k) = G_n - \sum_p \frac{1}{p} \tag{3}$$

where G_n is an integer and the sum is over all primes p such that $p - 1 \mid n$ but $p \nmid k$. If n is odd then $k^n B_n(h/k)$ is an integer unless $n = 1$ and k is odd, and in this case $kB_1(h/k) = G_1 + 1/2$, where G_1 is an integer.

Since $B_n(0) = B_n$, the case $h = 0, k = 1$ of Vandiver's theorem is the well-known von Staudt–Clausen theorem [3], which describes the fractional part of the Bernoulli numbers. Vandiver's theorem implies that, in all cases, the fractional part of $k^n B_n(h/k)$ is independent of h and thus is equal to the fractional part of $k^n B_n(0) = k^n B_n$. Therefore $k^n\big(B_n(h/k) - B_n\big)$ is an integer. Conversely, Vandiver's theorem may be derived easily from the Almkvist–Meurman theorem together with the von Staudt–Clausen theorem. The Almkvist–Meurman theorem implies that the fractional part of $k^n B_n(h/k)$ is the same as the fractional part of $k^n B_n$, and the fractional part of $k^n B_n$ is easily determined by the von Staudt-Clausen theorem. Vandiver's theorem was rediscovered by Bartz and Rutkowski [2], who derived the Almkvist–Meurman theorem from it.

1.3. A Generating Function Proof of the Almkvist–Meurman Theorem

A *Hurwitz* series is a power series $\sum_{n=0}^{\infty} r_n x^n/n!$ for which each r_n is an integer; i.e., it is the exponential generating function for a sequence of integers. It is well known that Hurwitz series are closed under addition and multiplication, and that if $f(x)$ is a Hurwitz series with constant term 0 then $f(x)^k/k!$ is a Hurwitz series. In particular, the Stirling numbers of the second kind $S(n,k)$, defined by

$$\frac{(e^x-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!},$$

are integers.

Since all of the generating function we will be concerned with are exponential, by the "coefficients" of a generating function we mean its coefficients as an exponential generating function.

It follows from Equation (1) and Equation (2) that

$$\sum_{n=0}^{\infty} M_n(h,k)\frac{x^n}{n!} = \sum_{n=0}^{\infty} k^n\big(B_n(h/k)-B_n\big)\frac{x^n}{n!} = kx\frac{e^{hx}-1}{e^{kx}-1}. \tag{4}$$

We present here a new proof of the Almkvist–Meurman theorem using the generating function of Equation (4) and basic facts about Hurwitz series.

Theorem 1 (Almkvist–Meurman)**.** *For all integers h and k, with $k \neq 0$, $M_n(h,k)$ is an integer.*

Proof. From Equation (4) we have

$$\sum_{n=0}^{\infty} M_n(h,k)\frac{x^n}{n!} = kx\frac{e^{hx}-1}{e^{kx}-1} = kx\frac{e^{x}-1}{e^{kx}-1}\cdot\frac{e^{hx}-1}{e^{x}-1}.$$

If $h \geq 0$ then $(e^{hx}-1)/(e^x-1)$ is a Hurwitz series, since

$$\frac{e^{hx}-1}{e^x-1} = 1 + e^x + \cdots + e^{(h-1)}.$$

If $h < 0$ then

$$\frac{e^{hx}-1}{e^x-1} = -e^{hx}\frac{e^{-hx}-1}{e^x-1},$$

so $(e^{hx}-1)/(e^x-1)$ is also a Hurwitz series in this case. Thus it is sufficient to show that $kx(e^x-1)/(e^{kx}-1)$ is a Hurwitz series.

We have

$$\begin{aligned}
kx\frac{e^x-1}{e^{kx}-1} &= \frac{e^x-1}{e^{kx}-1}\log\bigl(1+(e^{kx}-1)\bigr)\\
&= \frac{e^x-1}{e^{kx}-1}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{(e^{kx}-1)^j}{j}\\
&= \sum_{j=1}^{\infty}(-1)^{j-1}\frac{(e^x-1)^j}{j}\left(\frac{e^{kx}-1}{e^x-1}\right)^{j-1}\\
&= \sum_{j=1}^{\infty}(-1)^{j-1}(j-1)!\,\frac{(e^x-1)^j}{j!}\left(\frac{e^{kx}-1}{e^x-1}\right)^{j-1}. \qquad (5)
\end{aligned}$$

Since $(e^x-1)^j/j!$ and $(e^{kx}-1)/(e^x-1)$ are Hurwitz series, so is the sum in Equation (5). □

1.4. The Almkvist–Meurman Numbers

In this section we discuss the Almkvist–Meurman numbers $M_n(h,k)$.

Proposition 2. *The numbers $M_n(h,k)$ have the following properties:*

(a) *For $h>0$ we have*

$$M_n(h,1) = n(0^{n-1}+1^{n-1}+\cdots+(h-1)^{n-1}).$$

(b) *As a polynomial in h and k, $M_n(h,k)$ is homogeneous of degree n.*

(c) *We have*

$$M_n(k-h,k) = (-1)^n M_n(h,k) \text{ for } n\neq 1, \qquad (6)$$

$$M_n(h+k,k) = M_n(h,k) + nkh^{n-1} \text{ for all } n. \qquad (7)$$

Proof.
(a) By Equation (4) we have $\sum_{n=0}^{\infty} M_n(h,1)x^n/n! = x(1+e^x+\cdots+e^{(h-1)x})$.
(b) This follows immediately from Equation (2).
(c) By the identity $B_n(1-u) = (-1)^n B_n(u)$ for Bernoulli polynomials, we have

$$k^n\left(B_n\Bigl(\frac{k-h}{k}\Bigr)-B_n\right) = (-1)^n k^n\left(B_n\Bigl(\frac{h}{k}\Bigr)-(-1)^n B_n\right).$$

Then Equation (6) follows since $B_n=0$ if n is odd and not equal to 1. Equation (7) follows similarly from the identity $B_n(u+1) = B_n(u) + nu^{n-1}$. □

The first few values of $M_n(h,k)$ as polynomials in h and k are

$$M_0(h,k) = 0,$$
$$M_1(h,k) = h,$$
$$M_2(h,k) = h^2 - hk = -h(k-h),$$
$$M_3(h,k) = h^3 - \tfrac{3}{2}h^2k + \tfrac{1}{2}hk^2 = \tfrac{1}{2}h(k-h)(k-2h),$$
$$M_4(h,k) = h^4 - 2h^3k + h^2k^2 = h^2(k-h)^2,$$
$$M_5(h,k) = h^5 - \tfrac{5}{2}h^4k + \tfrac{5}{3}h^3k^2 - \tfrac{1}{6}hk^4 = -\tfrac{1}{6}h(k-h)(k-2h)(k^2+3hk-3h^2),$$
$$M_6(h,k) = h^6 - 3h^5k + \tfrac{5}{2}h^4k^2 - \tfrac{1}{2}h^2k^4 = \tfrac{1}{2}h^2(k-h)^2(k^2+2hk-2h^2).$$

Proposition 3. *As a polynomial in h and k, $M_n(h,k)$ has the following divisibility properties:*

(a) *$M_n(h,k)$ is divisible by h for all n.*

(b) *$M_n(h,k) - h^n$ is divisible by hk for $n \geq 1$.*

(c) *$M_n(h,k)$ is divisible by $k-h$ for $n > 1$.*

(d) *$M_n(h,k)$ is divisible by $k-2h$ for n odd and greater than 1.*

(e) *$M_n(h,k)$ is divisible by $h^2(k-h)^2$ for n even and greater than 2.*

Proof. The proofs of (a), (b), (c), and (d) are straightforward. For example, (a) is equivalent to $M_n(0,k) = 0$, which is clear either from the generating function or from Equation (2) and $B_n(0) = B_n$.

For (e), it is enough to show that for n even and greater than 2, $B_n(u) - B_n$ is divisible by $u^2(1-u)^2$. Suppose that n is even and greater than 2. Since $B_n(u) = B_n + nB_{n-1}u + \cdots + u^n$ and $B_{n-1} = 0$, it follows that $B_n(u) - B_n$ is divisible by u^2. Since $B_n(u) = B_n(1-u)$, $B_n(u) - B_n$ is also divisible by $(1-u)^2$. □

We note that by (b), as a polynomial in h and k, $M_n(h,0) = h^n$ for $n > 0$. Empirically, it seems that for n even, $(-1)^{n/2}M_n(a, a+b)$ is a polynomial in a and b with positive coefficients, but the coefficients are not all integers.

Let $A_n(k) = M_n(1,k)$, so

$$\sum_{n=0}^{\infty} A_n(k)\frac{x^n}{n!} = kx\frac{e^x - 1}{e^{kx} - 1} = \frac{kx}{1 + e^x + \cdots + e^{(k-1)x}}. \tag{8}$$

Some values of $A_n(k)$ are given in Table 1.4. The rows in this table are sequences A083007 through A083014 in the On-Line Encyclopedia of Integer Sequences [8]. In particular, the numbers $A_n(2)$ are the well-known Genocchi numbers (A036968; see also A001469 and A110501) with exponential generating function $2x/(e^x+1)$. The numbers $(-1)^{n+1}A_{2n+1}(3)$ are Glaisher's G-numbers (A002111).

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
2	1	-1	0	1	0	-3	0	17	0	-155
3	1	-2	1	4	-5	-26	49	328	-809	-6710
4	1	-3	3	9	-25	-99	427	2193	-12465	-79515
5	1	-4	6	16	-74	-264	1946	9056	-88434	-512024
6	1	-5	10	25	-170	-575	6370	28225	-415826	-2294975

Table 1: Values of $A_n(k)$

It is apparent from the table that $(-1)^{\lceil n/2 \rceil} A_n(k)$ is positive for $n > 1$. We will prove this in Theorem 6 below. To do this we use properties of Bernoulli polynomials. These properties are known (see Nörlund [7, pp. 22–23]), but we include proofs for completeness.

Lemma 4.

(a) *For $m \geq 1$, $(-1)^m B_{2m}(u)$ is increasing on $[0, 1/2]$.*

(b) *For $m \geq 1$, $(-1)^m B_{2m-1}(u)$ is positive and $(-1)^{m+1} B''_{2m+1}(u)$ is negative on $(0, 1/2)$.*

Proof. From the definition of the Bernoulli polynomials in Equation (1) it follows that

$$B'_n(u) = nB_{n-1}(u). \tag{9}$$

We will also use the well-known facts that $B_n = 0$ for n odd and greater than 1 and $B_n(1/2) = 0$ for all odd n.

We proceed by induction on m. We have $-B_1(u) = 1/2 - u$, so $-B_1(u)$ is positive on $(0, 1/2)$. Now suppose that $m \geq 1$ and that $(-1)^m B_{2m-1}(u)$ is positive on $(0, 1/2)$. It follows from Equation (9) that $(-1)^m B_{2m}(u)$ is increasing on $[0, 1/2]$.

Setting $p(u) = (-1)^{m+1} B_{2m+1}(u)$, we have $p(0) = p(1/2) = 0$ and

$$p''(u) = (-1)^{m+1}(2m+1)(2m)B_{2m-1}(u),$$

which is negative on $(0, 1/2)$. So $p(u)$ is strictly concave on $[0, 1/2]$, and therefore takes on its minimum values only at the endpoints of this interval. Thus $p(u)$ is positive on $(0, 1/2)$. □

Lemma 5. *Let $\tilde{B}_n(u) = B_n(u) - B_n$.*

(a) *If n is odd and greater than 1 then $(-1)^{\lceil n/2 \rceil} \tilde{B}_n(u)$ is positive for $0 < u < 1/2$.*

(b) *If n is even then $(-1)^{\lceil n/2 \rceil} \tilde{B}_n(u)$ is positive for $0 < u < 1$.*

(c) *If n is divisible by 4 then $\tilde{B}_n(u)$ is nonnegative for all u.*

Proof. Part (a) follows immediately from (b) of Lemma 4, since for n odd and greater than 1, $\tilde{B}_n(u) = B(u)$. Part (b) follows from (a) of Lemma 4 together with the facts that $\tilde{B}_n(0) = 0$ and $B_n(1-u) = (-1)^n B_n(u)$. Part (c) follows from part (b) and the formula $B_n(u+1) = B_n(u) + nu^{n-1}$. □

Lemma 5 implies the following positivity results for $M_n(h,k)$.

Theorem 6. *Let k be a positive integer.*

(a) *If n is odd and greater than 1, and $0 < h < k/2$, then $(-1)^{\lceil n/2 \rceil} M_n(h,k)$ is positive.*

(b) *If n is even and $0 < h < k$ then $(-1)^{\lceil n/2 \rceil} M_n(h,k)$ is positive.*

(c) *If n is divisible by 4 then $M_n(h,k)$ is nonnegative for all h.*

2. A Generalization of Gy's Theorem

Gy [6, Theorems 6.1 and 6.2] has studied the numbers with exponential generating function

$$\frac{(e^x-1)^k}{k!}\frac{kx}{e^{kx}-1}$$

and has shown that they are integers. Comparison with the Almkvist–Meurman theorem suggests that we look at the more general exponential generating function

$$\frac{(e^{hx}-1)^j}{j!}\frac{kx}{e^{kx}-1}, \tag{10}$$

which reduces to Gy's generating function for $h = 1, j = k$ and to the Almkvist–Meurman generating function for $j = 1$. The coefficient of $x^n/n!$ in (10) is

$$\sum_{i=0}^{n-j} \binom{n}{i} h^{n-i} S(n-i,j) k^i B_i. \tag{11}$$

It is not true that these numbers are always integers. However, in Theorem 7 below we give a sufficient condition for (10) to be a Hurwitz series, and in Theorem 8 we show that this condition is necessary. In particular, Theorem 7 implies that (10) is a Hurwitz series for $j = 1$ and for $j = k$, so it is a common generalization of Gy's theorem and the Almkvist–Meurman theorem. However, our proof of Theorem 7 is more complicated than the proof of the Almkvist–Meurman theorem given in Section 1.3, and requires the von Staudt–Clausen theorem.

Theorem 7. *Let j be a positive integer, and let h and k integers. Suppose that every prime divisor of j also divides h or k. Then*

$$\frac{(e^{hx}-1)^j}{j!}\frac{kx}{e^{kx}-1} \tag{12}$$

is a Hurwitz series.

Proof. Our proof is based on Gy's proof for the case $h = 1, j = k$.

Let p be a prime. A rational number is called *p-integral* if its denominator is not divisible by p. We call an exponential generating function a *p-Hurwitz series* if its coefficients are p-integral. Every Hurwitz series is p-Hurwitz, and p-Hurwitz series are closed under multiplication. It is clear that a power series is a Hurwitz series if and only if it is a p-Hurwitz series for every prime p.

Let $B(x) = x/(e^x - 1)$ be the Bernoulli number generating function. We will use two consequences of the von Staudt–Clausen theorem: (i) the denominator of B_n is square-free, and (ii) for any prime p, B_n is p-integral unless $p - 1 \mid n$. The first consequence implies that for any integer k, $pB(kx)$ is p-Hurwitz and that if p divides k then $B(kx)$ is p-Hurwitz. The second, together with Fermat's theorem, implies that if neither h nor k is divisible by p then $(k^n - h^n)B_n$ is p-integral for all n, and thus $B(kx) - B(hx)$ is p-Hurwitz.

To prove the theorem it is sufficient to show that for every prime p, (12) is p-Hurwitz. We consider three cases (with some overlap): (a) $p \mid h$, (b) $p \mid k$, and (c) $p \nmid k$ and $p \nmid h$.

(a) Suppose that $p \mid h$. Then since $(e^x - 1)^j/j!$ is a Hurwitz series with no constant term, every coefficient of $(e^{hx} - 1)^j/j!$ is divisible by p. Thus we may write (12) as

$$\frac{(e^{hx}-1)^j}{j!\,p}\cdot pB(kx)$$

and both factors are p-Hurwitz.

(b) Suppose that $p \mid k$. Then $B(kx)$ is p-Hurwitz, so (12) is p-Hurwitz.

(c) Suppose that $p \nmid h$ and $p \nmid k$. Then $p \nmid j$. We write (12) as

$$\frac{(e^{hx}-1)^j}{j!}\big(B(kx)-B(hx)\big)+\frac{(e^{hx}-1)^j}{j!}B(hx). \tag{13}$$

Since $B(kx) - B(hx)$ is p-Hurwitz, so is the first term in (13). The second term in (13) may be written

$$\frac{hx}{j}\frac{(e^{hx}-1)^{j-1}}{(j-1)!}. \tag{14}$$

Since $p \nmid j$, (14) is also p-Hurwitz. □

The condition in Theorem 7 that every prime divisor of j also divides h or k is necessary, as shown by the following result.

Theorem 8. *Let j be a positive integer, and let h and k be integers. Let p be a prime divisor of j that divides neither h nor k. Then the coefficient of $x^{j+p-1}/(j+p-1)!$ in (12) is not p-integral.*

Proof. From (11), the coefficient of $x^{j+p-1}/(j+p-1)!$ is

$$\sum_{i=0}^{p-1} \binom{j+p-1}{i} h^{j+p-1-i} S(j+p-1-i, j) k^i B_i. \tag{15}$$

By the von Staudt–Clausen theorem, B_i is p-integral for $i < p-1$, but not for $i = p-1$. Thus the terms of (15) with $i < p-1$ are all p-integral, so it is sufficient to show that the $i = p-1$ term is not p-integral. Since $S(j, j) = 1$, the $i = p-1$ term is $\binom{j+p-1}{j} h^j k^{p-1} B_{p-1}$. Neither h nor k is divisible by p, so it is enough to show that $\binom{j+p-1}{j}$ is not divisible by p. To do this we apply Lucas's congruence for binomial coefficients: since p divides j we may set $j = pr$, and then $\binom{j+p-1}{j} = \binom{pr+p-1}{pr} \equiv \binom{r}{r}\binom{p-1}{0} \equiv 1 \pmod{p}$. □

3. Fox's Theorem on Euler Polynomials

There is an analogue of the Almkvist–Meurman theorem for Euler polynomials due to Fox [5]. The Euler polynomials $E_n(u)$ are defined by

$$\sum_{n=0}^{\infty} E_n(u) \frac{x^n}{n!} = \frac{2e^{ux}}{e^x + 1}. \tag{16}$$

Fox showed that if r and s are integers then $s^n\big(E_n(r/s) - (-1)^{rs} E_n(0)\big)$ is an integer. Another proof of Fox's theorem was given by Sury [10]. We give here a simple proof of a slight strengthening of Fox's theorem using generating functions.

Theorem 9. *Let r and s be integers, with $s \neq 0$. If s is even then $s^n E_n(r/s)$ is an integer, and if s is odd then $\frac{1}{2} s^n \big(E_n(r/s) - (-1)^r E_n(0)\big)$ is an integer.*

Proof. We first note that if $f(x)$ is a Hurwitz series with constant term 1 then $1/f(x)$ is also a Hurwitz series.

By Equation (16) we have the generating function

$$\sum_{n=0}^{\infty} s^n E_n(r/s) \frac{x^n}{n!} = \frac{2e^{rx}}{e^{sx} + 1}. \tag{17}$$

If s is even then $2/(e^{sx} + 1)$ has integer coefficients since its reciprocal is

$$\tfrac{1}{2}(e^{sx} + 1) = 1 + \sum_{n=1}^{\infty} \tfrac{1}{2} s^n \frac{x^n}{n!},$$

which has integer coefficients and constant term 1. Thus the series in Equation (17) is a Hurwitz series, and in particular, $s^n E_n(0)$ is an integer.

Now suppose that s is odd. We have

$$\begin{aligned}\sum_{n=0}^{\infty} \tfrac{1}{2} s^n \big(E_n(r/s) - (-1)^r E_n(0)\big) \frac{x^n}{n!} &= \frac{e^{rx} - (-1)^r}{e^{sx}+1} \\ &= \frac{e^{rx} - (-1)^r}{e^{x}+1} \cdot \frac{e^{x}+1}{e^{sx}+1} \\ &= \frac{e^{(r-1)x} - e^{(r-2)x} + \cdots + (-1)^{r-1}}{e^{(s-1)x} - e^{(s-2)x} + \cdots + 1}.\end{aligned}$$

The denominator has constant term 1, so the quotient is a Hurwitz series. □

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