#A15 INTEGERS 23 (2023)

LINE SIDON SETS

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Received: 8/22/22, Accepted: 1/27/23, Published: 2/10/23

Abstract

In this paper we define the notion of a line Sidon set, expanding the idea of Sidon sets in $\mathbb{R}$ to sets of lines in the real plane, where the operation under consideration is composition. We prove that any set of n lines in the plane contains a line Sidon subset of size $n^{\frac{1}{3}+\frac{1}{24}}$, where $n^{\frac{1}{3}}$ represents the trivial lower bound given by a probabilistic argument.

1. Introduction

A finite set[1] $A \subseteq \mathbb{R}$ is called an *additive Sidon set* if it contains no solutions $a, b, c, d \in A$ to the equation $a + b = c + d$ with $\{a, b\} \neq \{c, d\}$. Analogously, a set which contains no solutions to the equation $ab = cd$, with $\{a, b\} \neq \{c, d\}$ is called a *multiplicative Sidon set*. Sidon sets are highly studied objects in combinatorial number theory, with much research being focused on finding the size of the largest additive Sidon subsets of $[n] := \{1, 2, ..., n\}$, which is known to be[2] $\Theta(n^{\frac{1}{2}})$. See [3] for a thorough review of additive Sidon sets.

The case of the first n integers turns out to be a minimizer (up to multiplicative constants) for finding large additive Sidon subsets, which is shown in the following theorem of Komlós, Sulyok, and Szemerédi [2].

Theorem 1 (Komlós, Sulyok, Szemerédi)**.** *For all finite sets $A \subseteq \mathbb{Z}$ there is a subset $B \subseteq A$ which is additive Sidon. The size of B satisfies $|B| \gg |A|^{\frac{1}{2}}$.*

Theorem 1 has since been extended to apply to sets of real numbers [5], and

DOI: 10.5281/zenodo.7625119

[1]Throughout this note, all sets are finite.

[2]We will write $X \ll Y$ to mean that there exists an absolute constant c such that $X \leq cY$. The expression $Y \gg X$ means that $X \ll Y$, and $X = \Theta(Y)$ means that we have both $X \ll Y$ and $Y \ll X$.

furthermore can be altered to apply in the multiplicative Sidon case, by considering the set $\log A$.

In this note, we extend the notion of Sidon sets to lines in $\mathbb{R}^2$. Let L be a set of non-vertical and non-horizontal lines[3] in $\mathbb{R}^2$. We consider two lines $l_1 : y = ax + b$ and $l_2 : y = cx + d$ in L. We can compose them as linear functions, as

$$(l_1 \circ l_2)(x) = a(cx + d) + b = acx + ad + b.$$

For each line $y = mx + c$, we can find the inverse line $y = \frac{1}{m}x - \frac{c}{m}$.

A set of lines L is called *line Sidon* if it contains no non-trivial solutions $l_1, l_2, l_3, l_4 \in L$ to the equation

$$l_1^{-1} \circ l_2 = l_3^{-1} \circ l_4 \tag{1}$$

where a solution is called trivial if $\{l_1, l_4\} = \{l_2, l_3\}$. Our main result is an analog of Theorem 1 for sets of lines.

Theorem 2. *Let L be a set of non-vertical and non-horizontal lines in $\mathbb{R}^2$. Then there exists a subset $S \subseteq L$ which is line Sidon, and such that*

$$|S| \gg |L|^{\frac{1}{3}+\frac{1}{24}}.$$

Equation (1) defines the *energy* E_L of a set of lines, in analogy to the commonly used additive and multiplicative energy of sets of real numbers. This notion originated with Elekes; see for instance [1]. Formally, we define E_L as the number of quadruples $(l_1, l_2, l_3, l_4) \in L^4$ which solve Equation (1). We have the trivial bounds

$$|L|^2 \le E_L \le |L|^3.$$

A simple application of the probabilistic method yields the following lemma, see [7] for a sketch of the proof.

Lemma 1. *For every set of lines L in $\mathbb{R}^2$, there exists a subset $S \subseteq L$ which is line Sidon, and with*

$$|S| \gg \frac{|L|^{\frac{4}{3}}}{E_L^{\frac{1}{3}}}.$$

Thus, if the energy is small, we find a large line Sidon subset. Lemma 1 gives the trivial lower bound of $|L|^{\frac{1}{3}}$ for the size of S in Theorem 2, when the energy E_L is as large as possible; the main feature of Theorem 2 is that the exponent is greater than $\frac{1}{3}$.

In order to prove Theorem 2, we make use of a simple corollary of the following theorem of Petridis et al. [4].

[3]From here on we assume all lines are non-vertical and non-horizontal.

Theorem 3 (Petridis, Roche-Newton, Rudnev, Warren)**.** *If L is a set of lines in $\mathbb{R}^2$ with no more than m parallel lines, and no more than M concurrent lines, then we must have*

$$E_L \ll m^{\frac{1}{2}}|L|^{\frac{5}{2}} + M|L|^2.$$

Corollary 1. *Suppose L is a set of lines in $\mathbb{R}^2$ with $E_L \gg |L|^{3-\delta}$. Then, one of the following two cases must occur:*

1. *There exists a subset $S \subseteq L$, with all lines in S being parallel, and*

$$|S| \gg |L|^{1-2\delta}.$$

2. *There exists a subset $S \subseteq L$, with all lines in S being concurrent, and*

$$|S| \gg |L|^{1-\delta}.$$

2. Proof of Theorem 2

In this section we prove Theorem 2.

2.1. Case 1 - Small Energy

Proof. When the energy E_L is relatively small, we use Lemma 1 to find the subset S. That is, suppose that $E_L \ll |L|^{3-\delta}$, for some parameter $\delta > 0$ to be chosen later. Upon applying Lemma 1, we find a subset $S \subseteq L$ which is line Sidon, with

$$|S| \gg |L|^{\frac{1}{3}+\frac{\delta}{3}}.$$

Therefore, we instead suppose that $E_L \gg |L|^{3-\delta}$. We will apply Corollary 1 to L, and split into two cases depending on whether the subset $S \subseteq L$ contains parallel or concurrent lines.

2.2. Case 2a - Parallel Lines

We begin with the case of parallel lines, in which we find a set $S \subseteq L$ of size $|S| \gg |L|^{1-2\delta}$, and each line in S has the form $y = mx + c$, for some fixed non-zero $m \in \mathbb{R}$, and c from some set $C \subseteq \mathbb{R}$. There is a clear bijection between S and C, mapping each line to its intercept.

We take two lines $l_1, l_2 \in S$, corresponding to $c_1, c_2 \in C$ respectively. Then $l_1^{-1} \circ l_2$ is the line

$$y = x + \frac{c_2 - c_1}{m}.$$

Therefore if the two lines $l_1^{-1} \circ l_2$ and $l_3^{-1} \circ l_4$ are equal, we must have

$$x + \frac{c_2 - c_1}{m} = x + \frac{c_4 - c_3}{m}$$

for $c_1, c_2, c_3, c_4 \in C$. This implies that (c_1, c_2, c_3, c_4) is a solution to the additive equation $c_2 + c_3 = c_4 + c_1$.

Now we can apply Theorem 1 to the set C, in order to find $C' \subseteq C$ which is additive Sidon, and such that $|C'| \gg |C|^{\frac{1}{2}}$. We claim that since there are no non-trivial additive solutions in C', there cannot exist any non-trivial line energy solutions in the subset $S' \subseteq L$ given by the lines $y = mx + c$ for $c \in C'$. Indeed, suppose we have a non-trivial line energy solution. Then, as above, we must find $c_1, c_2, c_3, c_4 \in C'$ with $c_2 + c_3 = c_4 + c_1$. As C' is an additive Sidon set, this must imply $\{c_2, c_3\} = \{c_1, c_4\}$. But then we have $\{l_2, l_3\} = \{l_1, l_4\}$, contradicting the non-triviality of the line energy solution. We have therefore found a line Sidon set $S' \subseteq L$, which is of size

$$|S'| = |C'| \gg |C|^{\frac{1}{2}} \gg |L|^{\frac{1}{2} - \delta}.$$

2.3. Case 2b - Concurrent Lines

Something similar to the above happens in the concurrent lines case. In this case, Corollary 1 yields a set of lines $S \subseteq L$ of size $|S| \gg |L|^{1-\delta}$, such that each line in S has the form $y = c(x - t) + s$, for some fixed center of concurrency (t, s) and $c \in C$ for some subset $C \subset \mathbb{R}$ corresponding to the slopes of the lines in S. Again, there is a clear bijection between S and C.

Let l_1 be the line $y = c_1 x + s - c_1 t$, and l_2 be $y = c_2 x + s - c_2 t$. Then $l_1^{-1} \circ l_2$ is the line

$$y = \frac{c_2}{c_1} x + \frac{c_1 t - c_2 t}{c_1}.$$

Therefore, if we were to have a solution $l_1^{-1} \circ l_2 = l_3^{-1} \circ l_4$, then we must have equality of the corresponding slopes, implying that $c_2 c_3 = c_1 c_4$.

We apply Theorem 1 to C to find a subset $C' \subseteq C$, which is multiplicatively Sidon, and $|C'| \gg |C|^{\frac{1}{2}}$. In the same way as above, since there are no non-trivial solutions to $c_2 c_3 = c_1 c_4$ in C', there cannot exist any non-trivial solutions to (1) in the set of lines $S' \subseteq L$, corresponding to the slopes from C'. Therefore, we have found a line Sidon set S', which has size

$$|S'| = |C'| \gg |C|^{\frac{1}{2}} \gg |L|^{\frac{1}{2} - \frac{\delta}{2}}.$$

2.4. Choosing δ

Now we have three lower bounds for the size of a line Sidon subset $S \subseteq L$, corresponding to the three cases above:

- `Case 1:` $|S| \gg |L|^{\frac{1}{3}+\frac{\delta}{3}}$
- `Case 2a:` $|S| \gg |L|^{\frac{1}{2}-\delta}$
- `Case 2b:` $|S| \gg |L|^{\frac{1}{2}-\frac{\delta}{2}}$.

Since Case 2b is always better than Case 2a, we will choose a δ which optimizes between Case 1 and 2a. Therefore,

$$|L|^{\frac{1}{2}-\delta} = |L|^{\frac{1}{3}+\frac{\delta}{3}}$$

and hence we make the choice $\delta = \frac{1}{8}$. We then conclude that $|S| \gg |L|^{\frac{1}{3}+\frac{1}{24}}$ as needed. □

Acknowledgements. This work was completed while the first author was doing an internship supported by FFG Project 895224 - JKU Young Scientists. The second author was supported by Austrian Science Fund FWF Project P 34180. We thank the anonymous referee for helpful comments.

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