THE RECIPROCALS OF TAILS OF THE ALTERNATING RIEMANN ZETA FUNCTION

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Received: 9/13/22, Accepted: 2/7/23, Published: 3/3/23

Abstract

In this paper, we give the integer parts of reciprocals of tails of the alternating Riemann zeta function $\zeta^*(s)$ at $s = 1, 2, 3, 4$ by using several new inequalities.

1. Introduction

In this paper, we study the reciprocals of tails of the alternating Riemann zeta function. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad (\Re(s) > 1),$$

which can be continued meromorphically to the whole complex plane with a simple pole at $s = 1$ whose residue is 1. The alternating Riemann zeta function $\zeta^*(s)$ is defined by

$$\zeta^*(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s}, \quad (\Re(s) > 0),$$

which can be continued analytically to the whole complex plane. The following easily derivable relationships are useful (see, e.g., [8, section 2.3]):

$$\zeta(s) = \begin{cases} \frac{1}{1-2^{-s}} \sum\limits_{k=1}^{\infty} \frac{1}{(2k-1)^s} & (\Re(s) > 1), \\ \frac{1}{1-2^{1-s}} \zeta^*(s) & (\Re(s) > 0; s \neq 1). \end{cases}$$

DOI: 10.5281/zenodo.7692104

The Riemann zeta function has a number of graceful features. Recently, some authors have begun to investigate the tails of the Riemann zeta function and the tails of the alternating Riemann zeta function which are defined by

$$\zeta_n(s) = \sum_{k=n}^{\infty} \frac{1}{k^s} \quad (\Re(s) > 1)$$

and

$$\zeta_n^*(s) = \sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{k^s} \quad (\Re(s) > 0),$$

respectively. Here and below, n is a positive integer. For example, in 2016 Lin [5] studied the integer parts of reciprocals of tails of the Riemann zeta function at the integer points $s \geq 2$ and proved that

$$\left\lfloor \zeta_n(2)^{-1} \right\rfloor = n - 1,$$

and

$$\left\lfloor \zeta_n(3)^{-1} \right\rfloor = 2n(n-1),$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Soon afterwards, Lin and Li [6] considered the computational formula for the case $s = 4$, and they obtained that

$$\left\lfloor \zeta_n(4)^{-1} \right\rfloor = \begin{cases} 24m^3 - 18m^2 + \left\lfloor \frac{3(5m-1)}{2} \right\rfloor & \text{if } n = 2m; \\ \\ 24m^3 - 54m^2 + \left\lfloor \frac{3(58m-17)}{4} \right\rfloor & \text{if } n = 2m-1. \end{cases}$$

It is to be noted that Xu [9] also proved two computation formulas for $s = 4, 5$, and Hwang and Song [1] obtained a complicated formula for the case $s = 6$, which depends on the residue of n modulo 48. In 2018, Kim and Song [3] studied the integer parts of the inverses of tails of the alternating Riemann zeta function and they obtained that for $s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, we have

$$\left\lfloor \zeta_n^*(s)^{-1} \right\rfloor = \left\lfloor (-1)^{n+1} 2 \left(n - \frac{1}{2}\right)^s \right\rfloor. \tag{1}$$

Later, Hwang and Song [2] proved that for $s = \frac{1}{p}$ with any integer $p \geq 5$ or for $s = \frac{2}{p}$ with any odd integer $p \geq 5$, there exists an integer $N > 0$ such that Equation (1) still holds for every integer $n \geq N$.

We want to mention that some similar questions on Fibonacci numbers have already been considered. For example, in 2008, Ohtsuka and Nakamura [7] studied the properties of infinite sums of reciprocal Fibonacci numbers, and they proved that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \geq 2 \text{ is even;} \\ F_{n-2} - 1 & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

and

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1 & \text{if } n \geq 2 \text{ is even;} \\ F_{n-1}F_n & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

Here, the Fibonacci sequence $\{F_k\}$ is defined by the recursive formula $F_{k+1} = F_k + F_{k-1}$ with $k \geq 1$ and the initial values $F_0 = 0$ and $F_1 = 1$. Then in 2013, Kuhapatanakul [4] considered the infinite sums of reciprocals of generalized Fibonacci numbers defined by the recursive formula $V_{k+1} = aV_k + bV_{k-1}$ and the initial values $V_0 = c$ and $V_1 = 1$. And it was shown in [4] that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{V_k} \right)^{-1} \right\rfloor = (-1)^n (V_n + V_{n-1}) - 1,$$

for integers a, b with $1 \leq b \leq a$ and $c = 0$. A direct corollary of the above formula is the following result for the Fibonacci numbers:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{F_k} \right)^{-1} \right\rfloor = (-1)^n F_{n+1} - 1.$$

Inspired by the results mentioned above, we try to find the closed formulas of the integer parts of reciprocals of tails of the alternating Riemann zeta function at integer points. By using several new inequalities, we obtain four interesting computational formulas for $\lfloor \zeta_n^*(s)^{-1} \rfloor$ with $s = 1, 2, 3, 4$. That is, we shall prove the following four theorems.

Theorem 1. *For any positive integer n, we have*

$$\lfloor \zeta_n^*(1)^{-1} \rfloor = \begin{cases} -2n & \text{if } n \geq 2 \text{ is even;} \\ 2n - 1 & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

Theorem 2. *For any positive integer n, we have*

$$\lfloor \zeta_n^*(2)^{-1} \rfloor = \begin{cases} -(2n^2 - 2n + 1) - 1 & \text{if } n \geq 2 \text{ is even;} \\ 2n^2 - 2n + 1 & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

Theorem 3. *For any positive integer n, we have*

$$\lfloor \zeta_n^*(3)^{-1} \rfloor = \begin{cases} -\left(2n^3 - 3n^2 + \frac{9}{2}n - \frac{5}{2}\right) - \frac{3}{2} & \text{if } n \geq 22 \text{ is even;} \\ 2n^3 - 3n^2 + \frac{9}{2}n - \frac{5}{2} & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

Theorem 4. *For any positive integer n, we have*

$$\left\lfloor \zeta_n^*(4)^{-1} \right\rfloor = \begin{cases} -(2n^4-4n^3+8n^2-6n-8)-1 & \text{if } n \geq 10 \text{ is even;} \\ 2n^4-4n^3+8n^2-6n-8 & \text{if } n \geq 11 \text{ is odd.} \end{cases}$$

The following is the structure of this article. Section 2 establishes various inequalities required for the proofs of our theorems. Section 3 begins with a unified result (Theorem 5) for all numbers $s \geq 1$. Then we give the proofs of Theorems 1-4.

2. Several Inequalities

For a fixed integer s with $1 \leq s \leq 4$, we want to prove there exist functions $f_s(k), g_s(k)$ with $\lim_{k\to\infty} f_s(k) = \lim_{k\to\infty} g_s(k) = \infty$ and integers $k_{s,even}, k_{s,odd} \geq 1$ such that

$$\frac{1}{f_s(k)+1} - \frac{1}{f_s(k+1)+1} < -\frac{1}{(2k)^s} + \frac{1}{(2k+1)^s} < \frac{1}{f_s(k)} - \frac{1}{f_s(k+1)}$$

holds for any integer $k \geq k_{s,even}$, and

$$\frac{1}{g_s(k)+1} - \frac{1}{g_s(k+1)+1} < \frac{1}{(2k-1)^s} - \frac{1}{(2k)^s} < \frac{1}{g_s(k)} - \frac{1}{g_s(k+1)}$$

holds for any integer $k \geq k_{s,odd}$.

Lemma 1. *Let $f_1(k) = -4k$ and $g_1(k) = 4k-3$. For any positive integer k, we have*

$$\frac{1}{f_1(k)+1} - \frac{1}{f_1(k+1)+1} < -\frac{1}{2k} + \frac{1}{2k+1} < \frac{1}{f_1(k)} - \frac{1}{f_1(k+1)} \tag{2}$$

and

$$\frac{1}{g_1(k)+1} - \frac{1}{g_1(k+1)+1} < \frac{1}{2k-1} - \frac{1}{2k} < \frac{1}{g_1(k)} - \frac{1}{g_1(k+1)}. \tag{3}$$

Proof. As for any positive integer k it holds

$$-\frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{4k} - \frac{1}{4k+4} = \frac{-1}{4k(k+1)(2k+1)} < 0,$$

the right-hand side of (2) is proved. For the left-hand side of (2), we use

$$-\frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{4k-1} - \frac{1}{4k+3} = \frac{3}{2k(2k+1)(4k-1)(4k+3)} > 0$$

which holds for any positive integer k.

Similarly, let us prove Inequality (3). The right-hand side follows from

$$\frac{1}{2k-1}-\frac{1}{2k}-\frac{1}{4k-3}+\frac{1}{4k+1}=\frac{-3}{2k(2k-1)(4k-3)(4k+1)}<0,$$

and the left-hand side follows from

$$\frac{1}{2k-1}-\frac{1}{2k}-\frac{1}{4k-2}+\frac{1}{4k+2}=\frac{1}{2k(2k-1)(2k+1)}>0.$$

This completes the proof of this lemma. □

Lemma 2. *Let $f_2(k)=-2(2k-\frac{1}{2})^2-\frac{3}{2}$ and $g_2(k)=2(2k-\frac{3}{2})^2+\frac{1}{2}$. For any positive integer k, we have*

$$\frac{1}{f_2(k)+1}-\frac{1}{f_2(k+1)+1}<-\frac{1}{(2k)^2}+\frac{1}{(2k+1)^2}<\frac{1}{f_2(k)}-\frac{1}{f_2(k+1)} \tag{4}$$

and

$$\frac{1}{g_2(k)+1}-\frac{1}{g_2(k+1)+1}<\frac{1}{(2k-1)^2}-\frac{1}{(2k)^2}<\frac{1}{g_2(k)}-\frac{1}{g_2(k+1)}. \tag{5}$$

Proof. Since

$$-\frac{1}{(2k)^2}+\frac{1}{(2k+1)^2}=-\frac{4k+1}{16k^4+16k^3+4k^2}$$

and

$$\frac{1}{f_2(k)}-\frac{1}{f_2(k+1)}=-\frac{16k+4}{64k^4+64k^3+16k^2+12},$$

the right-hand side of (4) is equivalent to

$$(16k+4)\left(16k^4+16k^3+4k^2\right)<(4k+1)\left(64k^4+64k^3+16k^2+12\right).$$

The above inequality is just $12>0$, which holds for any positive integer k. Using

$$\frac{1}{f_2(k)+1}-\frac{1}{f_2(k+1)+1}=-\frac{16k+4}{64k^4+64k^3-8k+5},$$

we find the left-hand side of (4) is equivalent to

$$(16k+4)\left(16k^4+16k^3+4k^2\right)>(4k+1)\left(64k^4+64k^3-8k+5\right).$$

Through proper simplification, the above inequality is just

$$16k^2+8k-5>0,$$

which holds for any positive integer k. So this completes the proof of Inequality (4).

Similarly, for Inequality (5), we have

$$\frac{1}{(2k-1)^2}-\frac{1}{(2k)^2}=\frac{4k-1}{16k^4-16k^3+4k^2}.$$

Since

$$\frac{1}{g_2(k)}-\frac{1}{g_2(k+1)}=\frac{16k-4}{64k^4-64k^3+8k+5},$$

through proper simplification, the right-hand side of (5) is just

$$16k^2-8k-5>0,$$

which holds for any positive integer k. For the left-hand side, we have

$$\frac{1}{g_2(k)+1}-\frac{1}{g_2(k+1)+1}=\frac{16k-4}{64k^4-64k^3+16k^2+12}.$$

Then the left-hand side of (5) is just $12>0$, which holds for any positive integer k. So this completes the proof of Inequality (5). □

Lemma 3. *Let* $f_3(k)=-16k^3+12k^2-9k+1$*. For any positive integer* $k\geq 11$*, we have*

$$\frac{1}{f_3(k)+1}-\frac{1}{f_3(k+1)+1}<-\frac{1}{(2k)^3}+\frac{1}{(2k+1)^3}<\frac{1}{f_3(k)}-\frac{1}{f_3(k+1)}. \tag{6}$$

Proof. We have

$$-\frac{1}{(2k)^3}+\frac{1}{(2k+1)^3}=-\frac{12k^2+6k+1}{64k^6+96k^5+48k^4+8k^3}.$$

Since

$$\frac{1}{f_3(k)}-\frac{1}{f_3(k+1)}=-\frac{48k^2+24k+13}{256k^6+384k^5+240k^4+104k^3+117k^2+75k-12},$$

the right-hand side of (6) is equivalent to

$$\begin{aligned}&\left(12k^2+6k+1\right)\left(256k^6+384k^5+240k^4+104k^3+117k^2+75k-12\right)\\&>\left(48k^2+24k+13\right)\left(64k^6+96k^5+48k^4+8k^3\right).\end{aligned}$$

The above inequality is just

$$288k^5+1452k^4+1602k^3+423k^2+3k-12>0,$$

which holds for all integers $k\geq 1$. Since

$$\frac{1}{f_3(k)+1}-\frac{1}{f_3(k+1)+1}=-\frac{48k^2+24k+13}{256k^6+384k^5+240k^4+72k^3+93k^2+33k-22},$$

the left-hand side of (6) is equivalent to

$$\begin{aligned}&\left(12k^2+6k+1\right)\left(256k^6+384k^5+240k^4+72k^3+93k^2+33k-22\right)\\&<\left(48k^2+24k+13\right)\left(64k^6+96k^5+48k^4+8k^3\right).\end{aligned}$$

The above inequality is just

$$96k^5-972k^4-922k^3-27k^2+99k+22>0,$$

which holds for all integers $k\geq 11$. So this completes the proof. $\square$

Lemma 4. *Let $g_3(k)=16k^3-36k^2+33k-12$. For any positive integer $k\geq 4$, we have*

$$\frac{1}{g_3(k)+1}-\frac{1}{g_3(k+1)+1}<\frac{1}{(2k-1)^3}-\frac{1}{(2k)^3}<\frac{1}{g_3(k)}-\frac{1}{g_3(k+1)}. \tag{7}$$

Proof. We have

$$\frac{1}{(2k-1)^3}-\frac{1}{(2k)^3}=\frac{12k^2-6k+1}{64k^6-96k^5+48k^4-8k^3}.$$

Since

$$\frac{1}{g_3(k)}-\frac{1}{g_3(k+1)}=\frac{48k^2-24k+13}{256k^6-384k^5+240k^4-104k^3+117k^2-75k-12},$$

the right-hand side of (7) is equivalent to

$$\begin{aligned}&\left(12k^2-6k+1\right)\left(256k^6-384k^5+240k^4-104k^3+117k^2-75k-12\right)\\&<\left(48k^2-24k+13\right)\left(64k^6-96k^5+48k^4-8k^3\right).\end{aligned}$$

The above inequality is just

$$288k^5-1452k^4+1602k^3-423k^2+3k+12>0,$$

which holds for all integers $k\geq 4$. Since

$$\frac{1}{g_3(k)+1}-\frac{1}{g_3(k+1)+1}=\frac{48k^2-24k+13}{256k^6-384k^5+240k^4-72k^3+93k^2-33k-22},$$

the left-hand side of (7) is equivalent to

$$\begin{aligned}&\left(12k^2-6k+1\right)\left(256k^6-384k^5+240k^4-72k^3+93k^2-33k-22\right)\\&>\left(48k^2-24k+13\right)\left(64k^6-96k^5+48k^4-8k^3\right).\end{aligned}$$

The above inequality is just

$$96k^5+972k^4-922k^3+27k^2+99k-22>0,$$

which holds for all integers $k\geq 1$. This proves Inequality (7). $\square$

Lemma 5. *Let* $f_4(k) = -32k^4 + 32k^3 - 32k^2 + 12k + 7$. *For any positive integer* $k \geq 5$, *we have*

$$\frac{1}{f_4(k)+1} - \frac{1}{f_4(k+1)+1} < -\frac{1}{(2k)^4} + \frac{1}{(2k+1)^4} < \frac{1}{f_4(k)} - \frac{1}{f_4(k+1)}. \tag{8}$$

Proof. We have

$$-\frac{1}{(2k)^4} + \frac{1}{(2k+1)^4} = -\frac{32k^3 + 24k^2 + 8k + 1}{256k^8 + 512k^7 + 384k^6 + 128k^5 + 16k^4}.$$

Since

$$\frac{1}{f_4(k)} - \frac{1}{f_4(k+1)} =$$
$$-\frac{128k^3 + 96k^2 + 96k + 20}{1024k^8 + 2048k^7 + 2048k^6 + 1280k^5 + 448k^4 + 64k^3 - 1488k^2 - 744k - 91},$$

the right-hand side of (8) is equivalent to

$$2048k^7 + 3584k^6 - 45312k^5 - 58880k^4 - 32608k^3 - 9624k^2 - 1472k - 91 > 0,$$

which holds for all integers $k \geq 5$. Since

$$\frac{1}{f_4(k)+1} - \frac{1}{f_4(k+1)+1} =$$
$$-\frac{128k^3 + 96k^2 + 96k + 20}{1024k^8 + 2048k^7 + 2048k^6 + 1280k^5 + 384k^4 - 1648k^2 - 816k - 96},$$

the left-hand side of (8) is equivalent to

$$52480k^5 + 65600k^4 + 35840k^3 + 10480k^2 + 1584k + 96 > 0,$$

which holds for all integers $k \geq 1$. This completes the proof. □

Lemma 6. *Let* $g_4(k) = 32k^4 - 96k^3 + 128k^2 - 84k + 12$. *For any positive integer* $k \geq 6$, *we have*

$$\frac{1}{g_4(k)+1} - \frac{1}{g_4(k+1)+1} < \frac{1}{(2k-1)^4} - \frac{1}{(2k)^4} < \frac{1}{g_4(k)} - \frac{1}{g_4(k+1)}. \tag{9}$$

Proof. We have

$$\frac{1}{(2k-1)^4} - \frac{1}{(2k)^4} = \frac{32k^3 - 24k^2 + 8k - 1}{256k^8 - 512k^7 + 384k^6 - 128k^5 + 16k^4}.$$

Since

$$\frac{1}{g_4(k)} - \frac{1}{g_4(k+1)} =$$
$$\frac{128k^3 - 96k^2 + 96k - 20}{1024k^8 - 2048k^7 + 2048k^6 - 1280k^5 + 384k^4 - 1648k^2 + 816k - 96},$$

the denominator of the above fraction on the right-hand side is greater than zero if $k \geq 2$. Then for any integer $k \geq 2$, the right-hand side of (9) is equivalent to

$$52480k^5 - 65600k^4 + 35840k^3 - 10480k^2 + 1584k - 96 > 0,$$

which holds for all integers $k \geq 2$. Since

$$\frac{1}{g_4(k)+1} - \frac{1}{g_4(k+1)+1} = \frac{128k^3 - 96k^2 + 96k - 20}{1024k^8 - 2048k^7 + 2048k^6 - 1280k^5 + 448k^4 - 64k^3 - 1488k^2 + 744k - 91},$$

the denominator of the above fraction on the right-hand side is greater than zero if $k \geq 2$. Then for any integer $k \geq 2$, the left-hand side of (9) is equivalent to

$$2048k^7 - 3584k^6 - 45312k^5 + 58880k^4 - 32608k^3 + 9624k^2 - 1472k + 91 > 0,$$

which holds for all integers $k \geq 6$. This completes the proof. □

3. Proofs of the Theorems

For proving Theorems 1-4, we give a unified idea of their proofs.

Theorem 5. *Assume that for any positive integer $s \geq 1$, there exist functions $f_s(x), g_s(x) \in \mathbb{Q}[x]$, such that*

(i) $\lim_{k\to\infty} f_s(k) = \lim_{k\to\infty} g_s(k) = \infty$;

(ii) *For some positive integer $k_{s,even}$,*

$$\frac{1}{f_s(k)+1} - \frac{1}{f_s(k+1)+1} < -\frac{1}{(2k)^s} + \frac{1}{(2k+1)^s} < \frac{1}{f_s(k)} - \frac{1}{f_s(k+1)} \quad (10)$$

holds for any integer $k \geq k_{s,even}$;

(iii) *For some positive integer $k_{s,odd}$,*

$$\frac{1}{g_s(k)+1} - \frac{1}{g_s(k+1)+1} < \frac{1}{(2k-1)^s} - \frac{1}{(2k)^s} < \frac{1}{g_s(k)} - \frac{1}{g_s(k+1)} \quad (11)$$

holds for any integer $k \geq k_{s,odd}$.

Then

$$\frac{1}{f_s\left(\frac{n}{2}\right)+1} < \zeta_n^*(s) < \frac{1}{f_s\left(\frac{n}{2}\right)}$$

holds for any positive even integer n with $n \geq 2k_{s,even}$, and

$$\frac{1}{g_s\left(\frac{n+1}{2}\right)+1} < \zeta_n^*(s) < \frac{1}{g_s\left(\frac{n+1}{2}\right)}$$

holds for any positive odd integer n with $n \geq 2k_{s,odd} - 1$.

Proof. Let n be a positive even integer. We have

$$\zeta_n^*(s) = \sum_{k=\frac{n}{2}}^{\infty} \left(-\frac{1}{(2k)^s} + \frac{1}{(2k+1)^s} \right).$$

Using assumption (ii), we get

$$\begin{aligned}\sum_{k=\frac{n}{2}}^{\infty} \left(\frac{1}{f_s(k)+1} - \frac{1}{f_s(k+1)+1} \right) &< \sum_{k=\frac{n}{2}}^{\infty} \left(-\frac{1}{(2k)^s} + \frac{1}{(2k+1)^s} \right) \\ &< \sum_{k=\frac{n}{2}}^{\infty} \left(\frac{1}{f_s(k)} - \frac{1}{f_s(k+1)} \right),\end{aligned}$$

which holds for any positive even integer n with $n \geq 2k_{s,even}$. From the formulas above and assumption (i), we obtain

$$\frac{1}{f_s\left(\frac{n}{2}\right)+1} < \zeta_n^*(s) < \frac{1}{f_s\left(\frac{n}{2}\right)},$$

which holds for any positive even integer n with $n \geq 2k_{s,even}$.

Similarly, let n be a positive odd integer. We have

$$\zeta_n^*(s) = \sum_{k=\frac{n+1}{2}}^{\infty} \left(\frac{1}{(2k-1)^s} - \frac{1}{(2k)^s} \right).$$

Using assumption (iii), we obtain

$$\begin{aligned}\sum_{k=\frac{n+1}{2}}^{\infty} \left(\frac{1}{g_s(k)+1} - \frac{1}{g_s(k+1)+1} \right) &< \sum_{k=\frac{n+1}{2}}^{\infty} \left(\frac{1}{(2k-1)^s} - \frac{1}{(2k)^s} \right) \\ &< \sum_{k=\frac{n+1}{2}}^{\infty} \left(\frac{1}{g_s(k)} - \frac{1}{g_s(k+1)} \right),\end{aligned}$$

which holds for any positive odd integer n with $n \geq 2k_{s,odd} - 1$. From the formulas above and assumption (i), we find

$$\frac{1}{g_s\left(\frac{n+1}{2}\right)+1} < \zeta_n^*(s) < \frac{1}{g_s\left(\frac{n+1}{2}\right)},$$

which holds for any positive even integer n with $n \geq 2k_{s,odd} - 1$. □

3.1. Proof of Theorem 1

For $s = 1$, we take $k_{1,even} = k_{1,odd} = 1$, $f_1(k) = -4k$ and $g_1(k) = 4k - 3$. Then $f_1(k)$ and $g_1(k)$ satisfy the three conditions of Theorem 5 by Lemma 1. Hence, using Theorem 5, we obtain that

$$-\frac{1}{2n-1} < \zeta_n^*(1) < -\frac{1}{2n}$$

holds for any positive even integer n, and

$$\frac{1}{2n} < \zeta_n^*(1) < \frac{1}{2n-1}$$

holds for any positive odd integer n. This completes the proof of Theorem 1. □

3.2. Proof of Theorem 2

For $s = 2$, we take $k_{2,even} = k_{2,odd} = 1$, $f_2(k) = -2(2k - \frac{1}{2})^2 - \frac{3}{2}$ and $g_2(k) = 2(2k - \frac{3}{2})^2 + \frac{1}{2}$. Then using Lemma 2 and Theorem 5, we obtain

$$\frac{1}{f_2\left(\frac{n}{2}\right) + 1} < \zeta_n^*(2) < \frac{1}{f_2\left(\frac{n}{2}\right)}$$

or equivalently

$$-\frac{1}{2n^2 - 2n + 1} < \zeta_n^*(2) < -\frac{1}{2n^2 - 2n + 2}$$

holds for any positive even integer n, and

$$\frac{1}{g_2\left(\frac{n+1}{2}\right) + 1} < \zeta_n^*(2) < \frac{1}{g_2\left(\frac{n+1}{2}\right)}$$

or equivalently

$$\frac{1}{2n^2 - 2n + 2} < \zeta_n^*(2) < \frac{1}{2n^2 - 2n + 1}$$

holds for any positive odd integer n. So this proves Theorem 2. □

3.3. Proof of Theorem 3

For $s = 3$, we take $k_{3,even} = 11$, $k_{3,odd} = 4$, $f_3(k) = -16k^3 + 12k^2 - 9k + 1$ and $g_3(k) = 16k^3 - 36k^2 + 33k - 12$. Then from Lemma 3, Lemma 4 and Theorem 5, we get

$$\frac{1}{f_3\left(\frac{n}{2}\right) + 1} < \zeta_n^*(3) < \frac{1}{f_3\left(\frac{n}{2}\right)},$$

or equivalently

$$-\frac{1}{2n^3 - 3n^2 + \frac{9}{2}n - 2} < \zeta_n^*(3) < -\frac{1}{2n^3 - 3n^2 + \frac{9}{2}n - 1}$$

holds for any positive even integer $n \geq 22$. So we have the inequality

$$-\left(2n^3 - 3n^2 + \frac{9}{2}n - 1\right) < \zeta_n^*(3)^{-1} < -\left(2n^3 - 3n^2 + \frac{9}{2}n - 2\right).$$

Since $-\left(2n^3-3n^2+\frac{9}{2}n-1\right)$ and $-\left(2n^3-3n^2+\frac{9}{2}n-2\right)$ are two consecutive integers, it follows that for any positive even integer $n \geq 22$,

$$\left\lfloor \zeta_n^*(3)^{-1} \right\rfloor = -\left(2n^3-3n^2+\frac{9}{2}n-1\right).$$

Similarly, for any positive odd integer $n \geq 7$, we get

$$\frac{1}{g_3\left(\frac{n+1}{2}\right)+1} < \zeta_n^*(3) < \frac{1}{g_3\left(\frac{n+1}{2}\right)},$$

or equivalently

$$\frac{1}{2n^3-3n^2+\frac{9}{2}n-\frac{3}{2}} < \zeta_n^*(3) < \frac{1}{2n^3-3n^2+\frac{9}{2}n-\frac{5}{2}}.$$

Then we have the inequality

$$2n^3-3n^2+\frac{9}{2}n-\frac{5}{2} < \zeta_n^*(3)^{-1} < 2n^3-3n^2+\frac{9}{2}n-\frac{3}{2}.$$

Since $2n^3-3n^2+\frac{9}{2}n-\frac{3}{2}$ and $2n^3-3n^2+\frac{9}{2}n-\frac{5}{2}$ are two consecutive positive integers, it follows that for any positive odd integer $n \geq 7$,

$$\left\lfloor \zeta_n^*(3)^{-1} \right\rfloor = 2n^3-3n^2+\frac{9}{2}n-\frac{5}{2}.$$

This completes the proof of Theorem 3. □

3.4. Proof of Theorem 4

For $s=4$, we take $k_{4,even}=5$, $k_{4,odd}=6$, $f_4(k)=-32k^4+32k^3-32k^2+12k+7$ and $g_4(k)=32k^4-96k^3+128k^2-84k+12$. Hence by Lemma 5, Lemma 6 and Theorem 5, we have

$$\frac{1}{f_4\left(\frac{n}{2}\right)+1} < \zeta_n^*(4) < \frac{1}{f_4\left(\frac{n}{2}\right)},$$

or equivalently

$$-\frac{1}{2n^4-4n^3+8n^2-6n-8} < \zeta_n^*(4) < -\frac{1}{2n^4-4n^3+8n^2-6n-7}$$

holds for any positive even integer $n \geq 10$. Therefore, we find

$$\left\lfloor \zeta_n^*(4)^{-1} \right\rfloor = -\left(2n^4-4n^3+8n^2-6n-7\right)$$

holds for any positive even integer $n \geq 10$. Similarly, for any positive odd integer $n \geq 11$, we have

$$\frac{1}{g_4\left(\frac{n+1}{2}\right)+1} < \zeta_n^*(4) < \frac{1}{g_4\left(\frac{n+1}{2}\right)},$$

or equivalently

$$\frac{1}{2n^4 - 4n^3 + 8n^2 - 6n - 7} < \zeta_n^*(4) < \frac{1}{2n^4 - 4n^3 + 8n^2 - 6n - 8}.$$

Then it follows that for any positive odd integer $n \geq 11$,

$$\left\lfloor \zeta_n^*(4)^{-1} \right\rfloor = 2n^4 - 4n^3 + 8n^2 - 6n - 8.$$

This completes the proof of Theorem 4. □

References

[1] W. Hwang and K. Song, A reciprocal sum related to the Riemann zeta function at $s = 6$, arxiv: 1709.07994.

[2] W. Hwang and K. Song, On the integer part of the reciprocal of the Riemann zeta function tail at certain rational numbers in the critical strip, *J. Inequal. Appl.* **270** (2019), 12 pp.

[3] D. Kim and K. Song, The inverses of tails of the Riemann zeta function, *J. Inequal. Appl.* **157** (2018), 13 pp.

[4] K. Kuhapatanakul, On the sums of reciprocal generalized Fibonacci numbers, *J. Integer Seq.* **16** (2013), Article 13.7.1, 8 pp.

[5] X. Lin, Some identities related to Riemann zeta-function, *J. Inequal. Appl.* **32** (2016), 6 pp.

[6] X. Lin and X. Li, A reciprocal sum related to the Riemann ζ-function, *J. Math. Inequal.* **11** (2017), 209-215.

[7] H. Ohtsuka and S. Nakamura, On the sum of reciprocal Fibonacci numbers, *Fibonacci Quart.* **46/47** (2008/09), 153-159.

[8] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[9] H. Xu, Some computational formulas related the Riemann zeta-function tails, *J. Inequal. Appl.* **132** (2016), 7 pp.