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FINDING A WIDELY DIGITALLY DELICATE PRIME

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Abstract

This paper describes the first construction of an explicitly-known widely digitally delicate prime.

1. Introduction

In a recent paper, Filaseta and Southwick [2] prove the remarkable theorem that a positive proportion of primes become composite when any digit is changed, *including leading zeros*. Such primes are known as **widely digitally delicate primes**. This fact is all the more surprising when one considers that no such prime is explicitly known. Using an existing table of digitally delicate primes at the Online Encyclopedia of Integer Sequences [3], they show that there is no widely digitally delicate prime below 10^9.

It is the aim of the current paper to remove that element of surprise. We do so by giving the first explicit example of a widely digitally delicate prime.

For a comprehensive history of related problems, see [2]. We will define a few terms for clarity.

1.1. Terminology

Definition 1. A *covering system* is a collection of residue classes such that every integer is in at least one congruence class in the collection.

Definition 2. A *digitally delicate prime* is a prime such that changing any one of its digits gives a composite. The digits under consideration do not include the leading zeros.

Definition 3. A *widely digitally delicate prime* is a digitally delicate prime that also becomes composite when any one of its leading zeros is changed.

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2. Finding the Covering Systems

Filaseta and Southwick use covering systems to construct a congruence class such that any digitally delicate prime in that class is a widely digitally delicate prime. They then show that a positive proportion of primes in this congruence class are digitally delicate.

Unfortunately, they do not explicitly construct the congruence class. They rely on the fact that ten specific unfactored composite numbers have at least two prime factors. In particular, any prime factor of the cyclotomic polynomial $\Phi_n(10)$ is a prime where 10 has order n. Any composite factor of $\Phi_n(10)$ (that is not a power) is divisible by at least two distinct primes p_1, p_2 where 10 has order n. The unknown primes can be used for two pieces of the covering systems.

They note that it is possible to construct an explicit congruence class, but that the resulting numbers to be tested for primality would be around $20,000$ digits.

By modifying their original covering systems, we only need to test numbers of slightly more than $4,000$ digits. We do not reproduce the congruence in [2], but recommend a careful reading of that paper to those who want to see the exact nature of the modifications described below.

The first step in their technique constructs a congruence class a mod M such that for each digit $d = 1, \ldots, 9$, the quantity $a + d \times 10^k$ is guaranteed to be composite, for all k. The recipe given for the digits 1, 2, 4, 5, 6, 7, 8, and 9 is very easy to follow — all primes are given explicitly. The modifications required are to make sure that adding 3×10^k produces a composite.

The first observation is that the numbers of the form $\Phi_n(10)$ are factors of repunits. The practice of factoring numbers is very popular on the Internet, as are repunits. Kamada [4] runs a website that collects many of these factors.

We use these factorizations to replace $p_{242,1}$, $p_{242,2}$, $p_{275,1}$, $p_{275,2}$, $p_{363,1}$, $p_{363,2}$, $p_{396,1}$, $p_{396,2}$, $p_{484,1}$, $p_{484,2}$, $p_{605,1}$, $p_{605,2}$, $p_{726,1}$, $p_{792,1}$, $p_{1188,1}$, $p_{2420,1}$, $p_{4356,1}$, $p_{5808,1}$, and $p_{5808,2}$ with explicit prime values.

Kamada's website does not give us values for $p_{1210,1}$, $p_{1210,2}$, $p_{2904,1}$, and $p_{2904,2}$, so we have to improvise. Instead of $p_{1210,1}$, we use a third new factor of $\Phi_{605}(10)$, which we call $p_{605,3}$. Because 605 is a divisor of 1210, a congruence for k modulo 605 also holds modulo 1210. Instead of $p_{1210,2}$, we use the remaining composite factor of $\Phi_{605}(10)$ after dividing the composite C_{605} from [2] by the three new primes. Although knowing a prime factor would give a shorter overall congruence, a composite factor is equally valid.

Instead of $p_{2904,1}$, we use $p_{363,3}$, and instead of $p_{2904,2}$, we use $p_{726,3}$.

Filaseta and Juillerat [1] have produced alternate covering systems, but the congruence class would be too large for the techniques in this paper.

3. Implementation

All of the code used in this computation is at github.com/31and8191/delicate.

In order to construct the congruence class for the prime, we put all of the congruence classes from [2] into a PARI/GP [6] script and then use the Chinese Remainder Theorem. This part of the computation takes a fraction of a second. We add the restriction $p \equiv 1 \bmod 2$ so that we only search over odd numbers.

The next step is a C program using the PARI library. It proceeds through the congruence class, testing each number with PARI's `ispseudoprime()` function. Once a number passes, the program tests each variation with a changed digit by using the same function. It is possible to inadvertently reject a digitally delicate number if a variation is a pseudoprime, but that is a small risk. Moreover, we are not concerned with false negatives.

That code ran for about 8 hours on 50 Xeon E5-2699 processors, each running at 2.3 GHz, before producing a positive answer. Note that this answer is a widely digitally delicate **probable** prime.

That brings us to the last step. I used the PARI/GP `isprime()` function to prove the primality of the number. That proof took 10 hours and 45 minutes on one of the Xeon processors described above. The prime contains 4030 digits; it is given its own section below.

4. Verification

In order to verify that the prime is widely digitally delicate, I used PARI/GP to verify that each of the prime factors of the congruence modulus is a factor of $10^{439084800} - 1$. Then I verified that $p + d \times 10^j$ for $1 \leq j \leq 439084800$ and $1 \leq d \leq 9$ has a non-trivial GCD with the congruence modulus.

An ECPP primality certificate is at the GitHub repository.

5. Other Computation

I used the exact congruences provided in [5] and similar code to provide examples of widely digitally delicate primes for bases 4, 5, 6, 9 and 11. Southwick had given examples for bases 2 and 3. Bases 7 and 8 should be equally easy.

I computed a list of digitally delicate primes up to 10^{11}; none of them are widely digitally delicate.

6. The Prime

2858945704919870011781537243745879385155011253521887655208864363343953253
908162183231168020433985595885849898174484619772705429763745991194664461
1001637271234290796863053715952950110064335655619433332495514961468987867
776550562050563528909231406272849064430203150357126420677812739927895202
546618565727359110480207958673019425654563382405130810590043829832715380
016952742364731312668598733740964023055663765267200830877802707668653471
933777180767950036688773110101833483505861901417331345780047986628791794
326507501874968194285942890205589193464254902430887558565879364674586977
542869103259950737623441567819481362009791661429525863338817583418337750
636854201374491211035260749630474538478350982057067390182173675337406552
432949290894282153403327225579837893941254410712722039794085468380534381
131239387917463610086595526879843146781456368648816955679603677489665210
301585644557371116185244779145233755095968806972088867036525332000563482
661120311917367253346938362615371782983097701381566032284628055925636748
898126820650464827690220782884190798343116909690410841541683928928146059
651185336459496576809728038333262955417851244474541192580656810039197263
444963081193148783858884674684291290222330733896006088695759175030396064
034822228704608841906934859287094672381558955431945725037617470641744551
971184997856292928496591534921211236168099241705439755139254869750171630
873039529038801552200374073595158674067785012041849784559706868066251558
959007352028060909364833937786587386985075068208343448492898929874743163
115366206843020913389386172497510663145775351914248509567074373964353926
550855085313243203493891126882173079296972161009951526577264968933125572
307190814735375219620927527926344910610632478049454126284831311027582454
324305015354771565731123702418064737323403806151773310815070975731441875
702723456262422122995757282908371118151810622179472346353894805386163739
494513263542196086802617326314422658695401574953618504325207626570369827
335044357601565464406200866061462414874388427226428378855463876957838538
058667545547011005526175955126923212015500245199711223649048445695487924
834121120688806897224963006252750771969496500665634692331145254779980486
343967092211316107065827196341907200951131788071359401349146624378022752
052545844144930311500503905988265128637482488955981286529793467809095581
440352363832112873230199489841731268453354502588069324055161268443331216
962521399803300022233553354419767585266713647261688468914028856910813765
524543565884834951623937277749823243669586887491792446589934309152360145
402223702059870976018312422587314837330651273101910480339409326664560463
527738695382754181288243632972126722859624445527255292587157441921530826
113552602921728700347440467945156253246324775061989113520577721694133490
784399906305246134471659941228999430202113778814079195332045833820656458

901527522411086824910166211024837473058912892928937245474268883280684242480487874698030206629697898794434472627736433485220385368119684099619061236512191519875021429108929407668260565433218275587502476105227425889862121206865754501735824432651679406459045203787776692980263412366742788331072209322658634582697941520974990757805582353440188334311782884631555114175534159974173334421080734338168929817480812898846370530054327828885261092395358871603433955916853259029118665402592591212769522753087657181383889092152274888705253938497543051178220510833709854540377531040026335538883439006407548027704742980284105860308607796896327127214198076159411035112242982961440264096944328564609012135336998306466140086042893352680052097192600330626485072400319291412252578437918736963043218871598100051612110090345840283216887928981948760162910616377942967779894553056956511250045645042017691094927754292707440718212298955678804377942875527266376928166048118383050605417536635038030584921442558997485565729401000730292149168921659416381049130296671584907248290245314396708303820511731834810811383331428645024458864506105438802952519781329205899113620752868037776637274722119629626665263194401602076490693769774059953268451781114501333003

References

[1] M. Filaseta and J. Juillerat, Consecutive primes which are widely digitally delicate, *Integers* **21A** (2021) #A12, 37pp.

[2] M. Filaseta and J. Southwick, Primes that become composite after changing an arbitrary digit, *Math. Comp.* **90** (2021)(328), 979–993, URL https://doi.org/10.1090/mcom/3593.

[3] O. Foundation, Inc., The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A050249, 2021.

[4] M. Kamada, Factorization of 11...11(repunit), https://stdkmd.net/nrr/repunit/, accessed 2021-04-24.

[5] J. T. Southwick, *Two Inquiries Related to the Digits of Prime Numbers*, ProQuest LLC, Ann Arbor, MI, 2020, URL http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:27828596, Thesis (Ph.D.)–University of South Carolina.

[6] The PARI Group, Univ. Bordeaux, *PARI/GP Version 2.11.0*, 2018, available at http://pari.math.u-bordeaux.fr/.