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EXTREMAL SEQUENCES FOR THE UNIT-WEIGHTED GAO CONSTANT OF $\mathbb{Z}_n$

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Abstract

For $A \subseteq \mathbb{Z}_n$, the A-weighted Gao constant $E_A(n)$ is defined to be the smallest natural number k such that any sequence of k elements in $\mathbb{Z}_n$ has a subsequence of length n whose A-weighted sum is zero. Sequences of length $E_A(n) - 1$ in $\mathbb{Z}_n$ which do not have any A-weighted zero-sum subsequence of length n are called E-extremal sequences for A. Such a sequence which has $n - 1$ zeroes is said to be of the standard type. In this paper we take A to be $U(n)$ which is the set of units in $\mathbb{Z}_n$. When n is odd, we characterize all such sequences and show that they are of the standard type. When n is even, we give examples of such sequences which are not of the standard type. We also characterize the E-extremal sequences for $U(n)$ when $n = 2^r p$ where p is an odd prime.

1. Introduction

We denote the number of elements in a finite set S by $|S|$. Let R be a ring with unity.

Definition 1.1. Given an R-module M, $A \subseteq R$ and a sequence $S = (x_1, x_2, \ldots, x_k)$

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in M, a subsequence T of S is called an *A-weighted zero-sum subsequence* if the set $I = \{i : x_i \in T\}$ is non-empty and for each $i \in I$, there exists $a_i \in A$ such that $\sum_{i\in I} a_i x_i = 0$.

Definition 1.2. Given a finite R-module M and $A \subseteq R$, the *A-weighted Davenport constant* $D_A(M)$ is the least positive integer k such that any sequence in M of length k has an A-weighted zero-sum subsequence.

Definition 1.3. Given a finite R-module M and $A \subseteq R$, the *A-weighted Gao constant* $E_A(M)$ is the least positive integer k such that any sequence in M of length k has an A-weighted zero-sum subsequence of length $|M|$.

The constant $D_A(M)$ was defined in Section 1 of [5]. We denote the ring $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}_n$ and the group of units in $\mathbb{Z}_n$ by $U(n)$. When $M = R = \mathbb{Z}_n$, we denote the constants $D_A(\mathbb{Z}_n)$ and $E_A(\mathbb{Z}_n)$ by $D_A(n)$ and $E_A(n)$, respectively. From Theorem 1.2 of [8], we have $E_A(n) = D_A(n) + n - 1$.

Definition 1.4. Let $A \subseteq \mathbb{Z}_n$. A sequence in $\mathbb{Z}_n$ of length $E_A(n) - 1$ which does not have any A-weighted zero-sum subsequence of length n, is called an *E-extremal sequence for A*. A sequence in $\mathbb{Z}_n$ of length $D_A(n) - 1$ which does not have any A-weighted zero-sum subsequence is called a *D-extremal sequence for A*.

Clearly, an E-extremal sequence for A can have at most $n - 1$ zeroes.

Definition 1.5. Let A be a subgroup of $U(n)$. Suppose $S = (x_1, \dots, x_k)$ and $T = (y_1, \dots, y_k)$ are sequences in $\mathbb{Z}_n$. We say that S and T are *A-equivalent* if there is a unit $c \in U(n)$, a permutation $\sigma \in S_k$ and we can find $a_1, \dots, a_k \in A$ such that when $1 \leq i \leq k$, we have $c\, y_{\sigma(i)} = a_i x_i$.

Remark 1.6. Suppose A is a subgroup of $U(n)$ and S is an E-extremal sequence for A. If a sequence T in $\mathbb{Z}_n$ is A-equivalent to S, then T is also an E-extremal sequence for A. A similar statement also holds for D-extremal sequences for A.

2. Some Known Results

For $a, b \in \mathbb{Z}$ we use the notation $[a, b]$ to denote the set $\{k \in \mathbb{Z} : a \leq k \leq b\}$. When $n = p_1^{r_1} \dots p_s^{r_s}$ where the p_i's are distinct primes, we let $\omega(n) = s$ and $\Omega(n) = r_1 + \cdots + r_s$. Let m be a divisor of n. We refer to the ring homomorphism $f_{n,m} : \mathbb{Z}_n \to \mathbb{Z}_m$ given by $a + n\mathbb{Z} \mapsto a + m\mathbb{Z}$ as the natural map.

The ring $\mathbb{Z}_n$ is a unique factorization ring if and only if n is a prime power (definitions and related results are given in [2]). If p is a prime divisor of n, we say that an element m of $\mathbb{Z}_n$ is coprime to p if $f_{n,p}(m) \neq 0$.

Definition 2.1. An E-extremal sequence for A which has $n - 1$ zeroes is said to be of the *standard type*.

Observation 2.2. Let $A \subseteq \mathbb{Z}_n \setminus \{0\}$, $k = D_A(n) - 1$, $S = (x_1, \ldots, x_k, \overbrace{0, \ldots, 0}^{n-1 \text{ times}})$ and $T = (x_1, \ldots, x_k)$. As $E_A(n) = D_A(n) + n - 1$, we see that S is an E-extremal sequence for A if and only if T is a D-extremal sequence for A. Thus, the set of all E-extremal sequences for A which are of the standard type is in bijection with the set of all D-extremal sequences for A.

From Theorem 1.3 of [3] or Theorem 1 of [4], we have that $E_{U(n)}(n) = n + \Omega(n)$ for any n. So if S is an E-extremal sequence for $U(n)$, we see that S has length $n - 1 + \Omega(n)$.

Let $n = p_1 \ldots p_k$, where the p_i's are primes and for each $i \in [1, k]$ let $b_i \in \mathbb{Z}_n$ be such that $f_{n,p_i}(b_i) \neq 0$. In [1] it was shown that the following sequence is a D-extremal sequence for $U(n)$:

$$(\, b_1,\, p_1 b_2,\, p_1 p_2 b_3,\, \ldots,\, p_1 p_2 \cdots p_{k-1} b_k \,).$$

Now using this fact and Observation 2.2 we get the next result.

Lemma 2.3. *Let $n = p_1 \ldots p_k$ where the p_i's are primes, and for each $i \in [1, k]$ let $b_i \in \mathbb{Z}_n$ be such that $f_{n,p_i}(b_i) \neq 0$. Then the following sequence is an E-extremal sequence for $U(n)$:*

$$(\, b_1,\, p_1 b_2,\, p_1 p_2 b_3,\, \ldots,\, p_1 p_2 \cdots p_{k-1} b_k,\, \overbrace{0,\, \ldots,\, 0}^{n-1 \text{ times}} \,). \tag{1}$$

When $n = p^r$ where p is an odd prime, in Theorem 3 of [1] it is shown that a sequence in $\mathbb{Z}_n$ is an E-extremal sequence for $U(n)$ if and only if it is $U(n)$-equivalent to the sequence

$$(\, 1,\, p,\, p^2,\, \ldots,\, p^{r-1},\, \overbrace{0,\, \ldots,\, 0}^{n-1 \text{ times}} \,).$$

Let $r \geq 2$ and $n = 2^r$. Suppose S is a sequence of length $n + r - 1$ in which 2^{r-1} occurs an odd number of times, 2^i occurs exactly once for each $i \in [0, r-2]$ and the remaining terms are zero. In Theorem 10 of [7] it is shown that a sequence in $\mathbb{Z}_n$ is an E-extremal sequence for $U(n)$ if and only if it is $U(n)$-equivalent to S. For example, a sequence in $\mathbb{Z}_8$ is an E-extremal sequence for $U(8)$ if and only if it is $U(8)$-equivalent to one of the following sequences:

$$(\, 1, 2, 4, 0, 0, 0, 0, 0, 0, 0 \,),\ (\, 1, 2, 4, 4, 4, 0, 0, 0, 0, 0 \,),$$
$$(\, 1, 2, 4, 4, 4, 4, 4, 0, 0, 0 \,),\ (\, 1, 2, 4, 4, 4, 4, 4, 4, 4, 0 \,).$$

Observe that a sequence of the type (1) is of the standard type. In this article, we have proved the following results:

- For any odd n, a sequence in $\mathbb{Z}_n$ is an E-extremal sequence for $U(n)$ if and only if it is $U(n)$-equivalent to a unique sequence which is of the type (1).
- When n is even and squarefree, we give examples of E-extremal sequences for $U(n)$ which are of the standard type but not of the type (1).
- For any even n, we give two types of E-extremal sequences for $U(n)$, which we denote by (2) and (3), which are not of the type (1).
- When $n = 2^r p$ where p is an odd prime, we show that a sequence in $\mathbb{Z}_n$ is an E-extremal sequence for $U(n)$ if and only if it is a permutation of a sequence which is of the type (1), (2) or (3).

3. When n Is Odd

The following result is Lemma 2.1 (ii) from [3], which we restate here.

Lemma 3.1. *Let $A = U(p^r)$ where p is an odd prime. If a sequence S over $\mathbb{Z}_{p^r}$ has at least two terms coprime to p, then S is an A-weighted zero-sum sequence.*

Lemma 3.2. *Let $S = (x_1, \ldots, x_k)$ be a sequence in $\mathbb{Z}_n$, d be a proper divisor of n which divides every element of S and $n' = n/d$. For each $i \in [1, k]$ let $x'_i = f_{n,n'}(x_i/d)$ and $S' = (x'_1, \ldots, x'_k)$. Suppose $A \subseteq \mathbb{Z}_n$, $A' \subseteq f_{n,n'}(A)$ and S' is an A'-weighted zero-sum sequence. Then S is an A-weighted zero-sum sequence.*

Proof. Let $S = (x_1, \ldots, x_k)$ and $S' = (x'_1, \ldots, x'_k)$ where $f_{n,n'}(x_i/d) = x'_i$ for each $i \in [1, k]$. Suppose S' is an A'-weighted zero-sum sequence. For each $i \in [1, k]$ there exists $a'_i \in A'$ such that $a'_1 x'_1 + \ldots + a'_k x'_k = 0$. As $A' \subseteq f_{n,n'}(A)$, for each $i \in [1, k]$ there exist $a_i \in A$ such that $f_{n,n'}(a_i) = a'_i$. Let $x = a_1 x_1 + \ldots + a_k x_k$. Then x is divisible by d and $f_{n,n'}(x/d) = a'_1 x'_1 + \ldots + a'_k x'_k = 0$. Thus n' divides x/d and so $n = n'd$ divides x. Hence, S is an A-weighted zero-sum sequence. □

Theorem 3.3. *Let n be odd. Then a sequence in $\mathbb{Z}_n$ is an E-extremal sequence for $U(n)$ if and only if it is a permutation of a sequence which is of the type* (1).

Proof. From Lemma 2.3, a sequence which is a permutation of a sequence of the type (1) is an E-extremal sequence for $U(n)$.

Let $S = (x_1, \ldots, x_l)$ be an E-extremal sequence for $U(n)$. Suppose for each prime divisor p of n, at least two terms of S are coprime to p. As $2\,\omega(n) < n$, we can find a subsequence T of S of length n such that for each prime divisor p of n, at least two terms of T are coprime to p. As n is odd, by Lemma 3.1 we get the contradiction that T is a $U(n)$-weighted zero-sum subsequence of S of length n. Thus, there is a prime divisor p of n such that at most one term of S is coprime to p.

Suppose p divides all the terms of S. Let $n' = n/p$ and $S' = (x'_1, \ldots, x'_l)$ denote the sequence in $\mathbb{Z}_{n'}$ where $x'_i = f_{n,n'}(x_i/p)$ for each $i \in [1, l]$. We have $E_{U(n')}(n') = n' + \Omega(n')$ and the length of S' is $n - 1 + \Omega(n) = (p-1)n' + n' + \Omega(n')$. So it follows that S' contains p disjoint subsequences $S'_1, \ldots, S'_p$ which are $U(n')$-weighted zero-sum sequences of length n'. So we get a $U(n')$-weighted zero-sum subsequence of S' of length n. Thus, by Lemma 3.2 we get the contradiction that S has a $U(n)$-weighted zero-sum subsequence of length n.

Thus, there is exactly one term of S which is coprime to p which we may assume to be x_1. By repeating the arguments in the above two paragraphs for the sequence $(x'_2, \ldots, x'_l)$, we see that there is a prime divisor p' of n' such that exactly one term of this sequence is coprime to p'. We may assume that that term is x'_2. Let us denote p by p_1 and p' by p_2.

Thus, we have found a prime divisor p_2 of n such that x_2/p_1 is coprime to p_2 and p_1p_2 divides x_j for $j > 2$. By continuing this process, for each $i \in [2, \Omega(n)]$ we get prime divisors p_i of n such that $x_i/(p_1 \ldots p_{i-1})$ is coprime to p_i and $p_1p_2 \ldots p_i$ divides x_j for $j > i$. Thus, it follows that $x_j = 0$ for all $j > \Omega(n)$. Hence, we see that S is a permutation of a sequence which is of the type (1). □

4. When n Is Even

From Theorem 3.3 we see that when n is odd, any E-extremal sequence for $U(n)$ is of the standard type. However, we will now show that there are E-extremal sequences for $U(n)$ which are not of the standard type when n is even.

When n is even and $\omega(n) \leq 2$, it can be shown that there is only one type of D-extremal sequence for $U(n)$. When n is even with $\omega(n) \geq 3$, there are other types of D-extremal sequences for $U(n)$. So we will get E-extremal sequences for $U(n)$ which are of the standard type but not of the type (1).

Lemma 4.1. *Let $n = 2\,p_1p_2 \ldots p_r$ be squarefree, where the p_i's are odd primes. Then the sequence $S = (a, \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_r)$ is a D-extremal sequence for $U(n)$ where $\hat{p}_i = \dfrac{n}{2p_i}$ for each $i \in [1, r]$, $a = 1$ if r is even and $a = 2$ if r is odd.*

Proof. Let n and S be as in the statement of the lemma. Suppose T is a $U(n)$-weighted zero-sum subsequence of S. The first term of T cannot be $\hat{p}_i$ for any $i \in [1, r-1]$, as all the other terms of T are divisible by p_i. Suppose the first term of T is a. If there exists $i \in [1, r]$ such that T does not contain $\hat{p}_i$, we get the contradiction that a is divisible by p_i. So the only remaining possibility for T is the sequence S. However, by our choice of a we see that S has an odd number of odd terms. So any $U(n)$-weighted sum of the terms of S cannot be even. Thus T cannot be S. □

Lemma 4.2. *Let $n = 2p_1 \dots p_k$, where the p_i's are primes and $k \geq 1$. Suppose*

$$S = (\, b_1,\, p_1 b_2,\, p_1 p_2 b_3,\, \dots,\, p_1 \cdots p_{k-1} b_k,\, \overbrace{n/2,\, \dots,\, n/2}^{m \ times},\, \overbrace{0,\, \dots,\, 0}^{n-m \ times}\,) \qquad (2)$$

where m is odd, and for each $i \in [1, k]$ we have that b_i is coprime with p_i. Then S is an E-extremal sequence for $U(n)$.

Proof. As S has length $n + k = E_{U(n)}(n) - 1$, it is enough to show that S does not have any $U(n)$-weighted zero-sum subsequence T of length n. Suppose such a subsequence T exists. Then the first term of T cannot be any of the first k terms of S. Suppose T consists of the last n terms of S.

Let T' be the sequence whose terms are obtained by dividing the terms of T by $n/2$ and then taking their images under $f_{n,2}$. The sequence T' in $\mathbb{Z}_2$ has an odd number of non-zero terms which are all equal to one. So we see that T' cannot be a zero-sum sequence. As the map $f_{n,2}$ sends elements of $U(n)$ to 1, it follows that T cannot be a $U(n)$-weighted zero-sum sequence. □

Remark 4.3. A sequence of the type (2) in which $m = 1$ is also of the type (1).

We now give an example of an E-extremal sequence for $U(n)$ when n is even, all of whose terms are non-zero.

Lemma 4.4. *Suppose $n = 2^{r+1} p_1 \dots p_k$, where the p_i's are odd primes and $r \geq 1$. Let S be the sequence which is of the type*

$$\left(a_0\,,\, 2a_1\,,\, \dots,\, 2^{r-1} a_{r-1}\,,\, c\,,\, 2^r b_1\,,\, 2^r p_1 b_2\,,\, \dots,\, 2^r p_1 \cdots p_{k-1} b_k\,,\, \frac{n}{2}^{[n-1]} \right) \qquad (3)$$

where $\frac{n}{2}^{[n-1]}$ denotes $n-1$ consecutive terms which are $n/2$, the a_i's and b_j's are odd, c is divisible by 2^{r+1} and b_j is coprime to p_j for each $j \in [1, k]$. Then S is an E-extremal sequence for $U(n)$.

Proof. Let T be a $U(n)$-weighted zero-sum subsequence of S having length n. It is easy to see that the first term of T cannot be any of the first r terms. Suppose the first term of T is c. We see that except c, all the terms of T are an odd multiple of 2^r and there are an odd number of such terms, as the length of T is even. So if T is a $U(n)$-weighted zero-sum sequence, we get the contradiction that c is not divisible by 2^{r+1}. It follows that the first term of T cannot be c.

It is easy to see that the first term of T cannot be any of the next k terms of S. For example, suppose the first term of T is $2^r p_1 b_2$. As all the other terms of T are divisible by $p_1 p_2$, it follows that p_2 divides $2^r b_2$. As p_2 is odd, this contradicts the fact that b_2 is coprime to p_2. As there are only $n-1$ terms remaining, S has no $U(n)$-weighted zero-sum subsequence of length n. □

5. When $n = 2p$ Where p Is an Odd Prime

We begin by re-stating Observation 2.2 of [3].

Observation 5.1. For a prime p, let $v_p(n)$ denote the highest power of p which divides n. Let S be a sequence in $\mathbb{Z}_n$ and for each prime divisor p of n, let $S^{(p)}$ denote the image of S under f_{n,p^s} where $s = v_p(n)$. Then S is a $U(n)$-weighted zero-sum sequence if and only if for every prime divisor p of n, the sequence $S^{(p)}$ in $\mathbb{Z}_{p^s}$ is a $U(p^s)$-weighted zero-sum sequence.

Lemma 5.2. *Let p be an odd prime. Suppose W is a sequence in $\mathbb{Z}_{2p}$ such that $W^{(2)}$ has an even number of ones and $W^{(p)}$ does not have exactly one unit. Then W is a $U(2p)$-weighted zero-sum sequence.*

Proof. By Lemma 3.1 if $W^{(p)}$ has at least two units, then it is a $U(p)$-weighted zero-sum sequence. If $W^{(p)}$ has no unit, then all its terms are zero and so it is a $U(p)$-weighted zero-sum sequence. Also $W^{(2)}$ is a $U(2)$-weighted zero-sum sequence as it has an even number of ones. Thus, by Observation 5.1 we see that W is a $U(2p)$-weighted zero-sum sequence. □

Theorem 5.3. *Let p be an odd prime and k any natural number. Suppose S is a sequence in $\mathbb{Z}_{2p}$ having length $k + 1$ such that S does not have any $U(2p)$-weighted zero-sum subsequence of length k. Then S is a permutation of a sequence which is of one of the following types:*

- $(a, 2b, \overbrace{0, \ldots, 0}^{k-1 \text{ times}})$ *where a is odd and b is coprime to p;*
- $(u, c, \overbrace{p, \ldots, p}^{k-1 \text{ times}})$ *where u is a unit and c is even;*
- $(b, \overbrace{p, \ldots, p}^{m \text{ times}}, \overbrace{0, \ldots, 0}^{k-m \text{ times}})$ *where b is coprime to p and m is odd.*

Proof. Let p and S be as in the statement of the theorem.

`Case 5.3.1.` Either all the terms of $S^{(2)}$ are zero or all the terms of $S^{(2)}$ are one.

Suppose all the terms of $S^{(2)}$ are zero. If $S^{(p)}$ has at most one non-zero term, then we get the contradiction that S has a subsequence of length k whose all terms are zero. So $S^{(p)}$ must have at least two non-zero terms. Let T denote a subsequence of S of length k such that $T^{(p)}$ has at least two non-zero terms. Then by Lemma 5.2 we get the contradiction that T is a $U(2p)$-weighted zero-sum subsequence of S of length k. So all terms of $S^{(2)}$ cannot be zero. By a similar argument, we see that all terms of $S^{(2)}$ cannot be one.

`Case 5.3.2.` $S^{(2)}$ has exactly one term which is one.

Let T denote the subsequence of S of length k such that all the terms of $T^{(2)}$ are zero. By Lemma 5.2 we see that $T^{(p)}$ must have exactly one unit. Thus, S is a permutation of the sequence $(a, 2b, 0, \ldots, 0)$ of length $k+1$ where a is odd and b is coprime to p.

Case 5.3.3. $S^{(2)}$ has exactly one term which is zero.

Let T denote the subsequence of S of length k such that all the terms of $T^{(2)}$ are one. By Lemma 5.2 we see that $T^{(p)}$ must have exactly one unit. Thus, S is a permutation of the sequence $(u, c, p, \ldots, p)$ of length $k+1$ where u is a unit and c is even.

Case 5.3.4. $S^{(2)}$ has at least two terms which are zero and at least two terms which are one.

Suppose $S^{(p)}$ does not have exactly one unit. By using Lemma 5.2 we will get a $U(2p)$-weighted zero-sum subsequence of S of length k. So $S^{(p)}$ has exactly one unit. Let T be the subsequence of S of length k such that $T^{(p)}$ is the zero sequence. If $T^{(2)}$ has an even number of ones, then by Lemma 5.2 we get the contradiction that T is a $U(2p)$-weighted zero-sum subsequence of S having length k. Thus, $T^{(2)}$ has an odd number of ones. Hence S is a permutation of the sequence $(b, p, \ldots, p, 0, \ldots, 0)$ of length $k+1$ where b is coprime to p and p occurs an odd number of times. □

Corollary 5.4. *Let p be an odd prime. Then S is an E-extremal sequence for $U(2p)$ if and only if S is a permutation of a sequence which is of one of the types* (1)*,* (2) *or* (3)*.*

Proof. From Lemmas 2.3, 4.2 and 4.4 we see that when n is even, if S is a sequence in $\mathbb{Z}_n$ which is a permutation of a sequence which is of one of the types (1), (2) or (3), then S is an E-extremal sequence for $U(n)$.

Let p be an odd prime. Suppose S is an E-extremal sequence for $U(2p)$. As $E_{U(n)}(n) = n + \Omega(n)$ for any n, we see that S has length $2p+1$. By Theorem 5.3, we see that one of the following three cases occur.

- S is a permutation of a sequence which is of the type $(a, 2b, \overbrace{0, \ldots, 0}^{2p-1 \text{ times}})$ where a is odd and b is coprime to p. In this case S is a permutation of a sequence which is of the type (1).

- S is a permutation of a sequence which is of the type $(u, c, \overbrace{p, \ldots, p}^{2p-1 \text{ times}})$ where u is a unit and c is even. In this case S is a permutation of a sequence which is of the type (3).

- S is a permutation of a sequence which is of the type $(b, \overbrace{p, \ldots, p}^{m \text{ times}}, \overbrace{0, \ldots, 0}^{2p-m \text{ times}})$

where b is coprime to p and m is odd. In this case S is a permutation of a sequence which is of the type (2). □

6. When $n = 2^r p$ Where p Is an Odd Prime and r Is Greater than 1

The next result is Lemma 1 (ii) of [4].

Lemma 6.1. *Let S be a sequence in $\mathbb{Z}_{2^r}$ which has a non-zero even number of units. Then S is a $U(2^r)$-weighted zero-sum sequence.*

Lemma 6.2. *Let p be an odd prime. Suppose W is a sequence in $\mathbb{Z}_{2^r p}$ such that $W^{(2)}$ has a non-zero even number of units and $W^{(p)}$ does not have exactly one unit. Then W is a $U(2^r p)$-weighted zero-sum sequence.*

Proof. If $W^{(p)}$ has at least two units, by Lemma 3.1 it is a $U(p)$-weighted zero-sum sequence. If $W^{(p)}$ has no unit, then all its terms are zero and so it is a $U(p)$-weighted zero-sum sequence. If $W^{(2)}$ has a non-zero even number of units, by Lemma 6.1 it is a $U(2^r)$-weighted zero-sum sequence. Thus, by Observation 5.1 we see that W is a $U(2^r p)$-weighted zero-sum sequence. □

Lemma 6.3. *Let $n = 2^r p$ where p is an odd prime and $r \geq 1$. Let S be a sequence in $\mathbb{Z}_n$ of length at least $n+2$. If S has at least three odd terms, then S has a $U(n)$-weighted zero-sum subsequence of length n.*

Proof. Let $S = (x_1, \ldots, x_k)$ be a sequence in $\mathbb{Z}_n$ of length at least $n+2$. Suppose S has at least three odd terms. We consider the following two possibilities.

`Case 6.3.1.` Suppose $S^{(p)}$ has at least two units.

As the length of S is at least $n+2$, we can get a subsequence T of S having length n which has a non-zero even number of odd terms and at least two terms which are coprime to p. By Lemma 6.2 we see that T is a $U(n)$-weighted zero-sum subsequence of S having length n.

`Case 6.3.2.` Suppose $S^{(p)}$ has at most one unit.

As the length of S is at least $n+2$, we can get a subsequence T of S having length n which has a non-zero even number of odd terms and no term which is coprime to p. By Lemma 6.2 we see that T is a $U(n)$-weighted zero-sum subsequence of S having length n. □

Lemma 6.4. *Let $n = 2^r p$ where p is an odd prime. Suppose S is a sequence in $\mathbb{Z}_n$ which does not have any $U(n)$-weighted zero-sum subsequence of length n. If S has length at least $n+2$ and has exactly two odd terms, then exactly one term of S is coprime to p and it is a unit.*

Proof. Let $S = (x_1, \ldots, x_k)$ be a sequence in $\mathbb{Z}_n$ of length at least $n+2$ which does not have any $U(n)$-weighted zero-sum subsequence of length n. Suppose S has exactly two odd terms. If $S^{(p)}$ has at least two units, we can find a subsequence T having length n which has both the odd terms and at least two terms which are coprime to p. Now by Lemma 6.2 we get a contradiction. If $S^{(p)}$ has no unit, we can find a subsequence T having length n which has both the odd terms and all of whose terms are divisible by p. So again by Lemma 6.2 we get a contradiction.

Thus, we see that $S^{(p)}$ has exactly one unit. Suppose both the odd terms of S are divisible by p. We can find a subsequence T having length n which has both the odd terms and whose every term is divisible by p. By Lemma 6.2 we get a contradiction. So both the odd terms cannot be divisible by p. Hence, exactly one of the two odd terms is coprime to p and so it is a unit. □

Theorem 6.5. *Let $n = 2^r p$ where p is an odd prime and $r \geq 2$. Suppose a sequence S in $\mathbb{Z}_n$ is an E-extremal sequence for $U(n)$. Then one of the following two cases can occur:*

- *For each $i \in [0, r-2]$ an odd multiple of 2^i occurs exactly once in S.*
- *There is a unique $j \in [0, r-2]$ such that an odd multiple of 2^j occurs exactly twice in S, and for each $i \in [0, r-2] \setminus \{j\}$ an odd multiple of 2^i occurs exactly once in S.*

Proof. Let S be an E-extremal sequence for $U(n)$ of length k. As we have that $E_{U(n)}(n) = n + \Omega(n)$, it follows that $k = n + r$. Let I be the set of all $i \in [0, r-2]$ such that an odd multiple of 2^i occurs at least twice in S.

`Case 6.5.1.` I is empty.

For each $i \in [0, r-2]$ an odd multiple of 2^i occurs at most once in S. Suppose there exists $j \in [0, r-2]$ such that an odd multiple of 2^j does not occur in S. Then there are at most $r-2$ terms of S which can be expressed as an odd multiple of a power of 2 where the power of 2 is at most $r-2$. As any element of $\mathbb{Z}_n$ is an odd multiple of some power of two, we get a subsequence T of S having length at least $n+2$ such that all the terms of T are divisible by 2^{r-1}.

Let $m = 2p$ and $a = \Omega(m)$. Suppose T' is the sequence in $\mathbb{Z}_m$ whose terms are obtained by dividing the terms of T by 2^{r-1} and taking the image under $f_{n,m}$. Then T' has length $n+2 = n+a$ and $n = 2^r p \geq 4 = 2^a$. By Theorem 1.3 (ii) of [3] we see that T' has a $U(m)$-weighted zero-sum subsequence of length n. Thus, by Lemma 3.2 we get the contradiction that T has $U(n)$-weighted zero-sum subsequence of length n. Hence, for each $i \in [0, r-2]$ an odd multiple of 2^i occurs exactly once in S.

`Case 6.5.2.` I is non-empty.

Let j be the minimum of the set I. Then at most j terms of S will not be divisible by 2^j. It follows that at least $n+2$ terms of S are divisible by 2^j, since $j \le r-2$ and $k-(r-2) = n+r-(r-2) = n+2$. Let S' be the subsequence of S consisting of all the terms of S which are divisible by 2^j. As S does not have a $U(n)$-weighted zero-sum subsequence of length n, it follows that S' also does not have such a subsequence.

By considering the sequence S'' which is obtained by dividing the terms of S' by 2^j and from Lemma 6.3, it follows that at most two terms of S' are an odd multiple of 2^j as S' has length at least $n+2$. Thus, by the choice of j we see that exactly two terms of S are an odd multiple of 2^j. By applying Lemma 6.4 to the sequence S'', we get that exactly one of these two terms of S is coprime to p and all the other terms of S are divisible by p. (†)

We will first show that for each $i \in [0, r-2] \setminus \{j\}$, an odd multiple of 2^i occurs at most once in S. From our choice of j, this is clear if $i < j$. So we may assume that $j \le r-3$. Suppose an odd multiple of 2^i for some $i \in [j+1, r-2]$ occurs at least twice in S. Let us assume that i_0 is the smallest such i. Let T denote the subsequence of S consisting of all the terms which are divisible by 2^{i_0}. As $i_0 > j$, it follows that T has length at least $k-(i_0+1)$. As $i_0 \le r-2$, we see that $k-(i_0+1) \ge n+r-(r-1) = n+1$ and so T has length at least $n+1$.

Consider the sequence which is obtained by dividing the terms of T by 2^{i_0}. By our choice of i_0 and from Lemma 6.1, we can find a subsequence U of T having length n such that $U^{(2)}$ is a $U(2^r)$-weighted zero-sum subsequence of $T^{(2)}$. As $i_0 > j$ and from (†), we see that all terms of T are divisible by p and so $U^{(p)}$ is the zero sequence. Thus, by Observation 5.1, we get the contradiction that U is a $U(n)$-weighted zero-sum subsequence of length n of S. Hence, for each $i \in [j+1, r-2]$ an odd multiple of 2^i occurs at most once in S and so for each $i \in [0, r-2] \setminus \{j\}$ an odd multiple of 2^i occurs at most once in S.

We will now show that for each $i \in [0, r-2] \setminus \{j\}$ an odd multiple of 2^i occurs exactly once in S. Suppose this does not happen. Then at most $r-1$ terms of S are not divisible by 2^{r-1}. So we will have at least $k-(r-1) = n+1$ terms in $S^{(2)}$ which are either zero or 2^{r-1}. Hence we can find a subsequence V of S having length n such that $V^{(2)}$ has an even number of terms which are equal to 2^{r-1}. Thus, we see that the sum of the terms of $V^{(2)}$ is an even multiple of 2^{r-1}.

So $V^{(2)}$ is a zero-sum sequence in $\mathbb{Z}_{2^r}$. As $j < r-1$ and from (†), we see that all the terms of V are divisible by p and so $V^{(p)}$ is the zero sequence. Thus, by Observation 5.1 we get the contradiction that V is a $U(n)$-weighted zero-sum subsequence of S having length n. Hence, for each $i \in [0, r-2] \setminus \{j\}$ an odd multiple of 2^i occurs exactly once in S. □

Corollary 6.6. *Let $n = 2^r p$ where p is an odd prime and $r \ge 2$. Suppose S is an E-extremal sequence for $U(n)$. Then the number of odd terms in S is either one or two.*

Proof. Let S be an E-extremal sequence for $U(n)$. As $E_{U(n)}(n) = n + \Omega(n)$, it follows that S has length $k = n + r$ which is at least $n + 2$ as $r \geq 2$. So by Lemma 6.3 we see that S has at most two odd terms. Also, Theorem 6.5 shows that S has at least one odd term. □

Theorem 6.7. *Let $n = 2^r p$ where p is an odd prime and $r \geq 2$. Suppose S is a sequence in $\mathbb{Z}_n$ which has exactly two odd terms. Then S is an E-extremal sequence for $U(n)$ if and only if S is a permutation of a sequence which is of the type* (2).

Proof. From Lemma 4.2 we see that if a sequence S is a permutation of a sequence which is of the type (2), then S is an E-extremal sequence for $U(n)$.

Let S be an E-extremal sequence for $U(n)$ where n is as in the statement of the theorem. Then S has length $k = n + r$. If S has exactly two odd terms, by Lemma 6.4 exactly one of the odd terms is a unit and all the other terms of S are divisible by p. By Theorem 6.5 for each $i \in [1, r-2]$, an odd multiple of 2^i occurs exactly once in S.

Thus, there are exactly $k - r = n$ terms of S which are divisible by 2^{r-1}. Let T denote the subsequence of S consisting of these n terms. As the only term of S which is coprime to p is odd, all the terms of T are divisible by p and hence by $2^{r-1}p$. So all the non-zero terms of T must be equal to $2^{r-1}p = n/2$. If T has an even number of these terms, then T is a zero-sum sequence. This is a contradiction as T has length n. So T has an odd number of non-zero terms.

Hence, we see that S is a permutation of the sequence

$$(\, u,\, p\,a_0,\, 2p\,a_1,\, 2^2p\,a_2,\, \ldots,\, 2^{r-2}p\,a_{r-2},\, \overbrace{n/2,\, \ldots,\, n/2}^{m \text{ times}},\, 0,\, \ldots,\, 0\,)$$

where m is odd, u is a unit and all the a_i's are odd. This is of the type (2). □

Theorem 6.8. *Let $n = 2^r p$ where p is an odd prime and $r \geq 2$. Suppose S is a sequence in $\mathbb{Z}_n$ which has exactly one odd term. Then S is an E-extremal sequence for $U(n)$ if and only if S is a permutation of a sequence which is of the type* (1), (2) *or* (3).

Proof. From Lemmas 2.3, 4.2 and 4.4 we see that for any even n, if a sequence in $\mathbb{Z}_n$ is a permutation of a sequence which is of the type (1), (2) or (3), then it is an E-extremal sequence for $U(n)$.

Let S be an E-extremal sequence for $U(n)$ where n is as in the statement of the theorem. Then S has length $k = n + r$. By Theorem 6.5 we see that the following two cases can occur.

`Case 6.8.1.` An odd multiple of 2^i occurs exactly once in S for each $i \in [0, r-2]$.

Let T denote the subsequence of S consisting of all the terms which are divisible by 2^{r-1}. Then T has length $k - (r-1) = n + 1$. Let T' denote the sequence in

$\mathbb{Z}_{2p}$ whose terms are obtained by dividing the terms of T by 2^{r-1} and taking their images under $f_{n,2p}$. As T does not have any $U(n)$-weighted zero-sum subsequence of length n, by Lemma 3.2 it follows that T' does not have any $U(2p)$-weighted zero-sum subsequence of length n. Thus, from Theorem 5.3 we see that the following three cases can occur.

- T' is a permutation of a sequence which is of the type $(a, 2b, \overbrace{0, \ldots, 0}^{n-1 \text{ times}})$ where a is odd and b is coprime to p. Then T is a permutation of a sequence of the type $(2^{r-1}a,\, 2^r b,\, \overbrace{0,\, \ldots,\, 0}^{n-1 \text{ times}})$ where a is odd and b is coprime to p. So S is a permutation of a sequence of the type
$$(a_0,\, 2a_1,\, \ldots,\, 2^{r-2}a_{r-2},\, 2^{r-1}a_{r-1},\, 2^r b,\, \overbrace{0,\, \ldots,\, 0}^{n-1 \text{ times}})$$
where the a_i's are odd and b is coprime to p. Thus, we see that S is a permutation of a sequence which is of the type (1).

- T' is a permutation of a sequence which is of the type $(b, \overbrace{p, \ldots, p}^{m \text{ times}}, \overbrace{0, \ldots, 0}^{n-m \text{ times}})$ where b is coprime to p and m is odd. Then T is a permutation of a sequence of the type $(2^{r-1}b,\, \overbrace{2^{r-1}p\,\tilde{a}_1,\, \ldots,\, 2^{r-1}p\,\tilde{a}_m}^{m \text{ terms}},\, \overbrace{0,\, \ldots,\, 0}^{n-m \text{ times}})$, where the $\tilde{a}_i$'s are odd, b is coprime to p and m is odd. So S is a permutation of a sequence of the type
$$(a_0,\, 2a_1,\, \ldots,\, 2^{r-2}a_{r-2},\, 2^{r-1}b,\, \overbrace{n/2,\, \ldots,\, n/2}^{m \text{ times}},\, \overbrace{0,\, \ldots,\, 0}^{n-m \text{ times}})$$
where the a_i's are odd, b is coprime to p and m is odd. Thus, we see that S is a permutation of a sequence which is of the type (2).

- T' is a permutation of a sequence which is of the type $(u, c, \overbrace{p, \ldots, p}^{n-1 \text{ times}})$ where u is a unit and c is even. Then T is a permutation of a sequence of the type $(c,\, 2^{r-1}u,\, \overbrace{2^{r-1}p\,\tilde{a}_2,\, \ldots,\, 2^{r-1}p\,\tilde{a}_n}^{n-1 \text{ terms}})$ where the $\tilde{a}_i$'s are odd, u is a unit and c is divisible by 2^r. So S is a permutation of a sequence of the type
$$(a_0,\, 2a_1,\, \ldots,\, 2^{r-2}a_{r-2},\, c,\, 2^{r-1}u,\, \overbrace{n/2,\, \ldots,\, n/2}^{n-1 \text{ times}})$$
where the a_i's are odd, u is a unit and c is divisible by 2^r. Thus, we see that S is a permutation of a sequence which is of the type (3).

Case 6.8.2. There is a unique $j \in [1, r-2]$ such that an odd multiple of 2^j occurs exactly twice in S, and for each $i \in [0, r-2] \setminus \{j\}$ an odd multiple of 2^i occurs exactly once in S.

Let T denote the subsequence of S consisting of the $k - r = n$ terms which are divisible by 2^{r-1}. We claim that all the terms of T are divisible by p. Let U be the subsequence consisting of all the terms of S which are divisible by 2^j. Then U has length $k - j = n + r - j$ which is at least $n + 2$ as $j \leq r - 2$. Let U' be the sequence which is obtained by dividing the terms of U by 2^j. Lemma 6.4 shows that exactly one of the terms of U' is coprime to p and it is a unit.

Thus, exactly one term of U is coprime to p and it is an odd multiple of 2^j. This proves our claim as $j < r - 1$. So each term of T is divisible by $2^{r-1}p$. Thus, all the non-zero terms of T are equal to $2^{r-1}p = n/2$. If T has an even number of non-zero terms, then T will be a zero-sum subsequence of S having length n. This gives us a contradiction. So the number of non-zero terms in T is odd. Thus, we see that S is a permutation of the sequence

$$(a_0,\, 2a_1,\, 4a_2,\, \ldots,\, 2^{j-1}a_{j-1},\, 2^j u,\, 2^j p\, a_j,\, 2^{j+1} p\, a_{j+1},\, \ldots,\, 2^{r-2} p\, a_{r-2},\, \frac{n}{2}^{[m]},\, 0^{[n-m]})$$

where the a_i's are odd, u is a unit, m is odd, $\frac{n}{2}^{[m]}$ denotes m consecutive terms which are $\frac{n}{2}$ and $0^{[n-m]}$ denotes $n - m$ consecutive terms which are 0. Thus, we see that S is a permutation of a sequence which is of the type (2). □

7. Concluding Remarks

From Theorem 3.3 we see that when n is odd, there is only one type of E-extremal sequence for $U(n)$ (and hence also D-extremal sequence). In this paper we have seen that for any even n, there are at least three types of E-extremal sequences for $U(n)$. The only intersection between these types are sequences of the type (1) which have a term equal to $n/2$ as they are also of the type (2).

The constant $C_A(n)$ has been defined and the C-extremal sequences for $U(n)$ have been characterized in [6] when n is odd. When n is a power of 2, such sequences have been characterized in [7]. It will be interesting to characterize the C-extremal sequences for $U(n)$ when n is an even number which is not a power of 2.

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