EXCEPTIONAL SETS FOR SPINOR-REGULAR TERNARY QUADRATIC FORMS

A.G. Earnest
School of Mathematical and Statistical Sciences, Southern Illinois University, Carbondale, Illinois
aearnest@siu.edu

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Abstract

A positive definite ternary integral quadratic form is called regular if it represents all the positive integers that are represented by its genus, and spinor-regular if it satisfies the weaker condition of representing all the positive integers represented by its spinor genus. There are known to be exactly 29 equivalence classes of primitive ternary forms that are spinor-regular, but not regular. In this paper, an analysis of the positive integers that are represented everywhere locally, but not globally, by forms in each of these 29 equivalence classes is obtained via the general theory of spinor exceptions for genera of ternary quadratic forms.

1. Introduction

It is well-known that there is no local-global principle for the representation of integers by integral quadratic forms. Consequently, it is of interest to identify the set consisting of those integers that are represented everywhere locally, but not globally, by such a form. In this paper, we will use the theory of spinor exceptions to determine these sets for a group of 29 positive definite ternary integral quadratic forms.

Throughout this paper, all forms under consideration will be positive definite ternary integral quadratic forms, so the term *ternary form* when used here will always refer to a quadratic form with those properties. That is, by ternary form we will mean a homogeneous polynomial of the type

$$F(x,y,z) = ax^2 + by^2 + cz^2 + dyz + exz + fxy, \tag{1}$$

for which the coefficients are in $\mathbb{Z}$ and the symmetric 3×3 matrix M_F consisting of the second partial derivatives of F is positive definite. Such a form is *primitive*

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if the greatest common divisor of its coefficients is 1. A positive integer n is said to be *represented by* F if the equation

$$F(x, y, z) = n \tag{2}$$

has a solution x, y, z in the ring $\mathbb{Z}$ of rational integers, and to be *represented everywhere locally by* F if, for every prime p, Equation (2) has a solution x, y, z in the ring $\mathbb{Z}_p$ of p-adic integers. The *exceptional set for* F consists of those positive integers (if any) that are represented everywhere locally by F, but not represented by F. In terminology introduced by Dickson [5], the ternary forms for which the exceptional set is empty are called *regular*.

It was first observed by Jones and Pall [12] in their study of the companion forms in the genera of regular diagonal ternary forms that for certain forms, the exceptional set for the form consists precisely of the integers that lie in one or more square classes (that is, sets of the type $\{at^2 : t \in \mathbb{Z}\}$) and whose prime divisors satisfy certain congruence conditions. For example, they observed that the form

$$F(x, y, z) = 4x^2 + 9y^2 + 9z^2 + 2yz + 4xz + 4xy \tag{3}$$

represents all those integers that it represents everywhere locally except for the odd squares m^2 for which every prime factor p of m satisfies $p \equiv 1 \pmod 4$. In all, Jones and Pall listed seven forms with similar properties. A full explanation for this type of behavior later emerged through spinor genus theory in the work of Schulze-Pillot [16], which we will now describe.

Generally, a genus of integral quadratic forms consists of a finite union of spinor genera, each of which in turn consists of a finite union of equivalence classes. Here, two ternary forms are *equivalent* if one can be transformed into the other via an invertible integral linear transformation of the variables. An integer is said to be *represented by a genus* (or *represented by a spinor genus*) if it is represented by at least one form in that genus (or spinor genus, respectively). To say that an integer is represented everywhere locally by a form F is equivalent to saying that it is represented by the genus of F. So the exceptional set for a form F consists of those integers that are represented by the genus of F, but not by F itself.

An integer n that is represented by a genus of ternary forms is represented either by all of the spinor genera in that genus, or by exactly half of them. In the latter instance, n is said to be a *spinor exception for the genus*. Schulze-Pillot proved that for each of the seven forms identified by Jones and Pall the exceptional set for the form consists precisely of those integers that are spinor exceptions for their genus.

Generalizing the terminology of Dickson, a ternary form is said to be *spinor-regular* if it represents all of the integers that are represented by its spinor genus[1] [2].

[1]Forms with this property have previously been called spinor regular; it is hoped that the hyphenated version adopted here adds clarity to the meaning and helps to emphasize that a spinor-regular form may not be regular.

So every regular ternary form is spinor-regular, while the form given by Equation (3) is spinor-regular, but not regular. In light of a result of the author and Haensch [7], it is known that there are exactly 29 equivalence classes of primitive ternary forms that are spinor-regular, but not regular. In 27 of those cases, the spinor genus and equivalence class coincide. For those 27 cases, explicit formulas for the numbers of representations of all positive integers by a form in each such equivalence class were recently obtained by Aygin, Doyle, Münkel, Pehlivan and Williams [1].

In this note, the general theory of spinor exceptions will be used to determine the exceptional sets for representative forms from each of the equivalence classes of spinor-regular primitive ternary forms for which the exceptional set is nonempty. The present paper can thus be viewed as an extension of the work of Schulze-Pillot, in which the theory of spinor exceptions is applied to complete the determination of the exceptional sets for each of these spinor-regular ternary forms.

We will follow the notation describing the spinor-regular ternary forms in [1]. In that paper, the forms are labelled as A1–A13, B1–B12 and C1–C4, where the forms are grouped according to the prime factors of their discriminant. In the present paper, the main results for the three groups appear in Propositions 1, 2 and 3, respectively.

The remainder of the paper is organized as follows. Section 2 contains notation and conventions that will be in effect throughout the paper. Some pertinent facts from the general theory of spinor exceptions will be reviewed in Section 3. The statements given there are special cases of general results from [16], stated here only in the generality needed to analyze the genera of the spinor-regular ternary forms. On the topic of spinor exceptions, the interested reader may also wish to consult the excellent survey [17]. Results for the forms A1–A13, B1–B12 and C1–C4 are given in Sections 4, 5 and 6, respectively. In Section 7, the spinor-regular ternary forms for which the spinor genus and equivalence class do not coincide, namely B4 and B11, are analyzed in more detail. Since the results on the positive integers not represented by these forms are obtained only for even integers in [1], we give the full statements here as Propositions 4 and 5. The final section of the paper contains some closing remarks relating the results found in this paper to those obtained by previous authors.

2. Ternary Quadratic Forms and Lattices

A ternary form will be described by the sextuple of integers appearing as its coefficients, so that (a, b, c, d, e, f) denotes the ternary form given by Equation (1). The form is referred to as *classic* if d, e and f are even, and *non-classic* otherwise. The *discriminant of* F, which will be denoted by Δ_F, is defined to be $\frac{1}{2}\det M_F$.

For referencing the literature on spinor exceptions, it will be convenient to freely switch between the language of quadratic forms and lattices. For quadratic lattices, we will follow the terminology and notation of O'Meara's book [15]. We will use the term *ternary lattice* to mean a finitely-generated $\mathbb{Z}$-module L in a positive definite quadratic space (V, Q) of dimension 3 over the rational field $\mathbb{Q}$ whose vectors span V, and which is integral in the sense that $Q(v) \in \mathbb{Z}$ for all $v \in L$. For such a ternary lattice K, $\text{gen}K$ and $\text{spn}K$ will denote the genus and spinor genus of K, respectively. For an integer n, we will use the notation $n \to N$ to indicate that n is represented by N, where N can be the lattice K, its p-adic completion K_p with respect to some prime p, its genus $\text{gen}K$, or its spinor genus $\text{spn}K$. The notation $n \not\to N$ will be used to indicate that n is not represented by N. The *discriminant of* K, denoted by dK, is the determinant of the Gram matrix of K with respect to any basis.

To the ternary form F given by Equation (1), we associate the ternary lattice L_F having Gram matrix $\frac{1}{2}M_F$ with respect to some basis. Then $\Delta_F = 4dL_F$. Note that $n \to L_F$ if and only if n is represented by the original form F.

3. Determination of Spinor Exceptions for a Genus

Throughout this section, L will denote an arbitrary ternary lattice, Δ will be the positive integer $4dL$, and n will be a positive integer represented by $\text{gen}L$.

3.1. General Criteria for Spinor Exceptions

For a prime p, the p-adic numbers and p-adic integers will be denoted by $\mathbb{Q}_p$ and $\mathbb{Z}_p$, respectively, the nonzero elements of $\mathbb{Q}_p$ by $\dot{\mathbb{Q}}_p$, and the group of units of $\mathbb{Z}_p$ by $\mathbb{Z}_p^\times$. The order of an element λ of $\dot{\mathbb{Q}}_p$ will be denoted by $\text{ord}_p(\lambda)$; so $\lambda = p^{\text{ord}_p(\lambda)}\lambda_0$, with $\lambda_0 \in \mathbb{Z}_p^\times$. For a set S, the notation S^2 will be used for $\{s^2 : s \in S\}$.

In general, the determination of whether n is a spinor exception for the genus of L involves three basic ingredients; namely, the groups $\theta(O^+(L_p))$ of spinor norms of rotations on L_p, the relative spinor norm groups $\theta(L_p, n)$, and the groups $N_p(-n\Delta)$ of local norms at p of $\mathbb{Q}(\sqrt{-\text{sqf}(n\Delta)})$, where $\text{sqf}(\gamma)$ denotes the squarefree part of the positive integer γ. Here

$$\theta(O^+(L_p)) = \{c \in \dot{\mathbb{Q}}_p : c \in \theta(\sigma) \text{ for some } \sigma \in O^+(L_p)\},$$

where $\theta : O^+(V_p) \to \dot{\mathbb{Q}}_p/\dot{\mathbb{Q}}_p^2$ is the spinor norm mapping, and

$$N_p(-n\Delta) = \{\gamma \in \dot{\mathbb{Q}}_p : (\gamma, -n\Delta)_p = +1\},$$

where $(\cdot,\cdot)_p$ denotes the Hilbert symbol at p. To define the relative spinor norm group, fix $v \in L_p$ such that $Q(v) = n$ (such vectors exist by the assumption that

$n \to \text{gen}L$). Then $\theta(L_p, n)$ is defined to be the subgroup of $\dot{\mathbb{Q}}_p$ generated by

$$\{c \in \dot{\mathbb{Q}}_p : c \in \theta(\sigma) \text{ for some } \sigma \in O^+(V_p) \text{ such that } \sigma(v) \in L_p\}.$$

By Witt's Theorem, it can be seen that this definition is independent of the particular choice of v. In order to see a general relationship between $\theta(L_p, n)$ and $\mathbb{N}_p(-n\Delta)$, note that the one-dimensional subspace $\mathbb{Q}_p v$ orthogonally splits V_p; say $V_p \cong \mathbb{Q}_p v \perp W$, for some binary quadratic space W over $\mathbb{Q}_p$. Any element $\sigma \in O^+(W)$ can be extended to an element $\hat{\sigma} \in O^+(V_p)$ such that $\hat{\sigma}(v) = v$. Then $\theta(\hat{\sigma}) = \theta(\sigma)$ and it follows that $\theta(O^+(W)) \subseteq \theta(L_p, n)$. As $\theta(O^+(W))$ is unaffected by scaling the underlying form, one may assume without loss of generality that $1 \to W$. It can then be shown that

$$\theta(O^+(W)) = Q(W) \setminus \{0\} = N_p(-n\Delta),$$

and it follows that

$$N_p(-n\Delta) \subseteq \theta(L_p, n).$$

From this, the following result is immediate.

Lemma 1. *If $-n\Delta \in \dot{\mathbb{Q}}_p^2$, then $\theta(L_p, n) = N_p(-n\Delta) = \dot{\mathbb{Q}}_p$.*

It will also be useful to note a relationship between the groups $N_p(-n\Delta)$ and the elements of even order in $\dot{\mathbb{Q}}_p$.

Lemma 2. *If $\mathbb{Z}_p^\times \dot{\mathbb{Q}}_p^2 \subseteq N_p(-n\Delta)$, then $\text{ord}_p(n\Delta)$ is even. Moreover, the reverse implication is true when p is odd.*

Proof. Suppose first that $\text{ord}_p(-n\Delta)$ is odd. Then $(\varepsilon_p, -n\Delta)_p = (\varepsilon_p, p)_p = -1$ by [15, 63:11a], where ε_p denotes a unit of $\mathbb{Z}_p$ of quadratic defect $4\mathbb{Z}_p$. This establishes the first implication. If p is odd and $\text{ord}_p(n\Delta)$ is even, then $-n\Delta \in \mathbb{Z}_p^\times \dot{\mathbb{Q}}_p^2$. Hence $(\gamma, -n\Delta)_p = +1$ for any $\gamma \in \mathbb{Z}_p^\times \dot{\mathbb{Q}}_p^2$ by [15, 63:12]. $\square$

The following theorem of Schulze-Pillot [16, Satz 2] gives complete criteria, in terms of the groups described above, for n to be a spinor exception for $\text{gen}L$.

Theorem 1. *The integer n is a spinor exception for genL if and only if*

$$\theta(O^+(L_p)) \subseteq N_p(-n\Delta) \quad \textit{and} \quad \theta(L_p, n) = N_p(-n\Delta)$$

for all primes p.

The local conditions appearing in the statement of Theorem 1 will now be discussed in more detail in the following two subsections.

3.2. Primes Not Dividing 2Δ

Throughout this subsection, p will denote a prime not dividing 2Δ. Consequently, p is odd, $\mathrm{ord}_p(\Delta) = 0$ and L_p is unimodular. So $\theta(O^+(L_p)) = \mathbb{Z}_p^\times \dot{\mathbb{Q}}_p^2$ by [15, 92:5].

Lemma 3. *$\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$ holds if and only if $\mathrm{ord}_p(n)$ is even.*

Proof. This follows from Lemma 2 since $\mathrm{ord}_p(n\Delta) = \mathrm{ord}_p(n)$. □

Lemma 4. *Assume that $\mathrm{ord}_p(n)$ is even.*

(i) If $\mathrm{ord}_p(n) = 0$, then $\theta(L_p, n) = N_p(-n\Delta)$.

(ii) If $\mathrm{ord}_p(n) > 0$, then $\theta(L_p, n) = N_p(-n\Delta)$ holds if and only if $-n\Delta \in \dot{\mathbb{Q}}_p^2$.

Proof. If $-n\Delta \in \dot{\mathbb{Q}}_p^2$, then $\theta(L_p, n) = N_p(-n\Delta)$ by Lemma 1. So suppose that $-n\Delta \notin \dot{\mathbb{Q}}_p^2$. Here $\mathrm{ord}_p(n)$ even implies that $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$ by Lemma 3. Since $p \nmid \Delta$ and $\mathrm{ord}_p(n)$ is even, p is unramified in $\mathbb{Q}(\sqrt{-\mathrm{sqf}(n\Delta)})$. So [16, Satz 3(a)] applies with $r = s = 0$. Hence, $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$ holds if and only if $\mathrm{ord}_p(n) = 0$. □

For a positive integer t, let M_t denote the multiplicative semigroup generated by 1 and the set of all primes p such that $-t \in \dot{\mathbb{Q}}_p^2$. For any $s \in \dot{\mathbb{Q}}$, note that $M_t = M_{s^2 t}$; in particular, $M_t = M_{\mathrm{sqf}(t)}$.

Corollary 1. *Assume that $\mathrm{ord}_p(n)$ is even for all $p \nmid 2\Delta$. Write $n = kw^2$, where all prime divisors of the positive integer k divide 2Δ and w is a positive integer with g.c.d.$(w, 2\Delta) = 1$. Then $\theta(L_p, n) = N_p(-n\Delta)$ holds for all $p \nmid 2\Delta$ if and only if $w \in M_{n\Delta}$.*

3.3. Primes Dividing 2Δ

Throughout this subsection, p will denote a prime divisor of 2Δ. For $a_1, \ldots, a_n \in \mathbb{Z}_p$, $\langle a_1, \ldots, a_n \rangle$ denotes the quadratic $\mathbb{Z}_p$-lattice with orthogonal basis $\{v_1, \ldots, v_n\}$ such that $Q(v_i) = a_i$ for $i = 1, \ldots, n$.

Lemma 5. *Let p be an odd prime such that $-n\Delta \notin \dot{\mathbb{Q}}_p^2$ and $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$.*

(i) If $L_p \cong \langle 1, p, p^2 \rangle$ or $L_p \cong \langle 1, p^2, p^3 \rangle$, then $\theta(L_p, n) = N_p(-n\Delta)$ holds if and only if $\mathrm{ord}_p(n) \leq 1$.

(ii) If $L_p \cong \langle 1, p, p^3 \rangle$, then $\theta(L_p, n) = N_p(-n\Delta)$ holds if and only if $\mathrm{ord}_p(n) \leq 2$.

Proof. By the assumptions, we are in the situation of Satz 3(b) of [16]. For L_p isometric to either $\langle 1, p, p^2 \rangle$ or $\langle 1, p, p^3 \rangle$, the stated result follows from subcase (ii) of Satz 3(b) with $r = 1, s = 2$ or 3. For $L_p \cong \langle 1, p^2, p^3 \rangle$, it follows from subcase (i) with $r = 2, s = 3$. □

For the prime 2, there are numerous possibilities for the nature of a Jordan splitting of L_2. So in this case, we will state only the general form of the needed result, and provide specific references to the relevant subcases of the original result in [16, Satz 4] as they arise.

Lemma 6. *Assume that* $-n\Delta \notin \dot{\mathbb{Q}}_2^2$ *and* $\theta(O^+(L_2)) \subseteq N_2(-n\Delta)$. *Then there exists a nonnegative integer* λ *such that* $\theta(L_2, n) = N_2(-n\Delta)$ *holds if and only if* $\mathrm{ord}_2(n) \leq \lambda$.

4. Discriminants Divisible Only by 2

The spinor-regular ternary forms with discriminants divisible only by 2 are labelled as A1–A13 in [1]. Table 1 contains a list of representatives for all of the equivalence classes of forms in the genus of each of these forms, along with their separation into spinor genera and their discriminant. These representatives and their separation into spinor genera were obtained by the use of Magma [4]. For this purpose, the Gram matrix of the form was entered, and representatives for the equivalence classes in the genus and spinor genus of the form were produced using the GenusRepresentatives and SpinorRepresentatives commands, respectively. In each case, the spinor-regular form is a representative of the only equivalence class in its spinor genus, which is labelled Spinor Genus I, and the remaining equivalence classes constitute a second spinor genus labelled Spinor Genus II. These spinor genera will subsequently be referred to simply as SGI and SGII. In the table, the regular ternary forms that occur are marked with an asterisk (see [11] for the complete list of all candidates for regular ternary forms).

For each of the forms A1–A13, the data needed to determine the candidates for spinor exceptions for their genus is given in Table 2. The second column of the table gives the splitting of L_2 for the ternary lattice L corresponding to the form. The splittings for L_2 given in the table can be obtained by routine calculation using one of the representative forms in the genus. In all cases, the 2-adic splitting is of the type $\langle b_1, b_2 2^r, b_3 2^s\rangle$, where $b_i \in \mathbb{Z}_2^\times$ and $r \leq s$ are nonnegative integers. The third column contains the local spinor norm group $\theta(O^+(L_2))$ for each lattice L. Here and in the remainder of the paper, $\{a_1, \ldots, a_k\}\dot{F}^2$ is used to denote $a_1\dot{F}^2 \cup \cdots \cup a_k\dot{F}^2$. For the lattices for which the splitting of L_2 has $0 < r < s$ (that is, A4, A6, A7, A8, A9, A11 and A12), the corresponding spinor norm groups $\theta(O^+(L_2))$ can be determined by Propositions 1.4, 1.6, 1.7 and 1.8 and Theorem 2.7 of [8]. For the remainder of the cases, L_2 is split by a multiple of $\langle 1, 1\rangle$. Here $\theta(O^+(\langle 1, 1\rangle) = \{\gamma \in \dot{Q}_2 : (\gamma, -1)_2 = +1\} = \{1, 2, 5, 10\}\dot{\mathbb{Q}}_2^2$ by Proposition B of [10]. Then $\theta(O^+(L_2)) = \{1, 2, 5, 10\}\dot{\mathbb{Q}}_2^2$ follows from Theorem 3.14(iv) of [8]. In the course of the proof of Proposition 1, it will be necessary to know the specific value of the integer λ appearing in Lemma 6 for each of the forms in the table. From each 2-adic splitting, this value

#	Spinor Genus I	Spinor Genus II	Δ
A1	(2,2,5,2,2,0)	(1,1,16,0,0,0)*	2^6
A2	(1,4,9,4,0,0)	(1,1,32,0,0,0) (2,2,9,-2,2,0)	2^7
A3	(2,5,8,4,0,2)	(1,1,64,0,0,0) (2,2,17,2,-2,0) (1,4,17,-4,0,0)	2^8
A4	(4,4,5,0,4,0)	(1,4,16,0,0,0)*	2^8
A5	(4,9,9,2,4,4)	(1,16,16,0,0,0)*	2^{10}
A6	(4,5,13,2,0,0)	(1,16,20,-16,0,0) (4,5,17,2,-4,-4)	2^{10}
A7	(5,8,8,0,4,4)	(1,4,64,0,0,0) (4,4,17,0,4,0)	2^{10}
A8	(4,8,17,0,4,0)	(1,8,64,0,0,0)*	2^{11}
A9	(9,9,16,8,8,2)	(1,16,64,0,0,0) (4,16,17,0,-4,0)	2^{12}
A10	(4,9,32,0,0,4)	(1,32,32,0,0,0) (4,17,17,2,4,4)	2^{12}
A11	(5,13,16,0,0,2)	(4,16,21,16,4,0) (4,5,64,0,0,-4)	2^{12}
A12	(9,17,32,-8,8,6)	(1,16,256,0,0,0) (16,16,17,-8,0,0) (4,16,65,0,4,0)	2^{14}
A13	(9,16,36,16,4,8)	(1,64,64,0,0,0) (4,33,33,2,4,4) (4,17,64,0,0,-4)	2^{14}

Table 1: Genera containing spinor-regular ternary forms with 2-power discriminant

can be obtained from one of the subcases of [16, Satz 4]. For each form, the specific subcase that applies is identified in the fourth column of the table, and the value of λ appears in the last column.

#	L_2	$\theta(O^+(L_2))$	subcase	λ
A1	$\langle 1,1,2^4\rangle$	$\{1,2,5,10\}\dot{\mathbb{Q}}_2^2$	(b)(iii)	1
A2	$\langle 1,1,2^5\rangle$	$\{1,2,5,10\}\dot{\mathbb{Q}}_2^2$	(b)(iii)	2
A3	$\langle 1,1,2^6\rangle$	$\{1,2,5,10\}\dot{\mathbb{Q}}_2^2$	(b)(iii)	3
A4	$\langle 1,2^2,2^4\rangle$	$\{1,5\}\dot{\mathbb{Q}}_2^2$	(b)(iii)	1
A5	$\langle 1,2^4,2^4\rangle$	$\{1,2,5,10\}\dot{\mathbb{Q}}_2^2$	(b)(i)	1
A6	$\langle 5,2^2,5\cdot 2^6\rangle$	$\{1,5\}\dot{\mathbb{Q}}_2^2$	(b)(ii)	1
A7	$\langle 1,2^2,2^6\rangle$	$\{1,5\}\dot{\mathbb{Q}}_2^2$	(b)(iii)	3
A8	$\langle 1,2^3,2^6\rangle$	$\{1,2,3,6\}\dot{\mathbb{Q}}_2^2$	(c)(iii)	1
A9	$\langle 1,2^4,2^6\rangle$	$\{1,5\}\dot{\mathbb{Q}}_2^2$	(b)(iii)	3
A10	$\langle 1,2^5,2^5\rangle$	$\{1,2,5,10\}\dot{\mathbb{Q}}_2^2$	(b)(iv)	1
A11	$\langle 5,2^4,5\cdot 2^6\rangle$	$\{1,5\}\dot{\mathbb{Q}}_2^2$	(b)(ii)	3
A12	$\langle 1,2^4,2^8\rangle$	$\{1,5\}\dot{\mathbb{Q}}_2^2$	(b)(iii)	5
A13	$\langle 1,2^6,2^6\rangle$	$\{1,2,5,10\}\dot{\mathbb{Q}}_2^2$	(b)(i)	3

Table 2: Data for A1–A13

Proposition 1. *Let f be one of the forms A1–A13, and let n be a positive integer. Then n is represented everywhere locally, but not globally, by f if and only if n lies in:*

(i) M_1^2 *for A1, A4, A5, A6, A10;*

(ii) $2M_1^2$ *for A2;*

(iii) $M_1^2, 4M_1^2$ *for A3, A7, A9, A13;*

(iv) M_2^2 *for A8;*

(v) $4M_1^2$ *for A11;*

(vi) $M_1^2, 4M_1^2, 16M_1^2$ *for A12.*

Proof. First consider the lattice L corresponding to the form A1. Suppose first that n is represented everywhere locally, but not globally, by f. So $n \to \mathrm{gen}L$, but $n \not\to L$. It follows that $n \not\to \mathrm{spn}L$, since L is spinor-regular. Thus, n is a spinor exception for $\mathrm{gen}L$. So by Theorem 1 and Lemma 3 we see that $\mathrm{ord}_p(n)$ is even for all odd primes p; thus,

$$n = 2^t w^2,$$

for some nonnegative integer t and some odd positive integer w. As $\Delta = 2^6$, we then have $\mathrm{sqf}(n\Delta) = 1$ or 2, depending upon whether t is even or odd, respectively. Since $\theta(O^+(L_2)) \subseteq N_2(-n\Delta)$ and $5 \in \theta(O^+(L_2)) \setminus N_2(-2)$, it must be that t is even and $M_{n\Delta} = M_1$. By Corollary 1 it follows that $w \in M_1$. Since $-1 \notin \dot{\mathbb{Q}}_2^2$, it follows from Lemma 6 that

$$t = \mathrm{ord}_2(n) \leq \lambda = 1.$$

Since t is even, this implies that $t = 0$. Hence, $n \in M_1^2$ as claimed.

For the reverse implication, assume that $n \in M_1^2$. So $n = w^2$, $w \in M_1$ and $\mathrm{sqf}(n\Delta) = 1$. The form $x^2 + y^2 + 16z^2$ lies in SGII for the genus of A1. So $1 \to SGII$ and hence $n \to SGII$ since n is a square. In particular, $n \to \mathrm{gen}L$. Note that w is odd since $2 \notin M_1$, as $-1 \notin \dot{\mathbb{Q}}_2^2$. For p odd, $\mathrm{ord}_p(n)$ even implies that $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$ by Lemma 3, and $\theta(L_p, n) = N_p(-n\Delta)$ by Lemma 4. Also $\theta(O^+(L_2)) \subseteq N_2(-1)$. Since $\mathrm{ord}_2(n) = 0$, it follows from Lemma 6 that $\theta(L_2, n) = N_2(-n\Delta)$ since $\lambda = 1$. So the criteria of Theorem 1 are met and n is a spinor exception for $\mathrm{gen}L$. But then it must be that $n \not\to SGI$ and so $n \not\to L$. This completes the proof for A1.

The proofs for the forms A4, A5, A6 and A10 are identical. For A3, A7, A9, A11 and A13, the proof remains the same except that $\lambda = 3$ for these cases. Thus the conditions for $\theta(L_2, n) = N_2(-n\Delta)$ in Lemma 6 hold also with $t = 2$, and integers of the type $4M_1^2$ are also spinor exceptions for these genera. Note that for A11,

$1 \not\to \mathrm{gen}L$, so the integers in M_1^2 need not be listed among the spinor exceptions. In this one instance, $4 \to SGII$, and so all integers of the type $4M_1^2$ are represented by SGII, and hence not by L. For the form A12, the proof is again unchanged except that in this case $\lambda = 5$; thus the conditions for $\theta(L_2, n) = N_2(-n\Delta)$ in Lemma 6 hold also with $t = 4$, and the integers of the type $M_1^2, 4M_1^2$ and $16M_4^2$ are spinor exceptions for the genus.

This leaves two remaining cases, A2 and A8. The only difference for the proof of A2 is that $\mathrm{ord}_2(\Delta)$ is odd. Since the other aspects of the proof remain the same, including specifically that $\mathrm{sqf}(n\Delta) = 1$, we see in this case that $n = 2^t w^2$, where t and w are odd integers. As $\lambda = 1$, it must be that $t = 1$ and so $n = 2w^2$. Here $2 \to SGII$, so we conclude that no elements of the type $2M_1^2$ are represented by L.

Finally, in the case of A8, we have $\theta(O^+(L_2)) = \{1, 2, 3, 6\}\dot{\mathbb{Q}}_2^2$. From this we see that $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$ holds for all p if and only if $\mathrm{sqf}(n\Delta) = 2$. Hence $\mathrm{sqf}(n) = 1$ for any spinor exception n for this genus, since $\mathrm{ord}_2(\Delta) = 11$ is odd. As $\lambda = 1$, the spinor exceptions for the genus must be of the form w^2, for some odd integer w. In this case, the prime divisors p of w must satisfy the condition that $-2 \in \dot{\mathbb{Q}}_p^2$. Hence, $w \in M_2$, as asserted. □

Remark 1. The set M_1 appearing in the statement of the proposition is generated by 1 and the primes congruent to 1 modulo 4, and the set M_2 is generated by 1 and the primes congruent to 1 or 3 modulo 8.

5. Discriminants Divisible by 2 and 3

The spinor-regular ternary forms with discriminants divisible by both 2 and 3 are labelled as B1–B12 in [1]. Table 3 contains a list of representatives for all of the equivalence classes of forms in the genus of each of these forms, along with their separation into spinor genara and their discriminant. In each case, as for the previous grouping, the genus splits into two spinor genera, which are labelled as before. The spinor-regular forms B4 and B11 are the first forms listed in SGI for their genus.

The data necessary to determine the spinor exceptions for these genera is summarized in Table 4. We first record the local splittings at the primes 2 and 3 for the corresponding lattices. The splittings for L_3 are given in the second column of Table 4. The 2-adic splittings appear in the third column. Some additional explanation is in order here. This grouping contains both classic and non-classic forms. For the non-classic forms B1–B4, the splitting given in the table is for the lattice L_2' obtained by scaling L_2 by 2. For the remaining forms B5–B12, which are classic, the splitting of $L_2' = L_2$ is given. Since the spinor norm group $\theta(O^+(L_2))$ is unaffected by scaling, this makes it possible to directly apply the results of [8] to determine $\theta(O^+(L_2))$ from L_2' in all cases. For a nonnegative integer r, $2^r\mathbb{H}$ and

#	Spinor Genus I	Spinor Genus II	Δ
B1	(3,3,4,0,0,3)	(1,1,36,0,0,1)*	$2^2 3^3$
B2	(3,4,4,4,3,3)	(1,3,10,-3,1,0)*	$2^2 3^3$
B3	(1,7,12,0,0,1)	(3,3,13,-3,3,-3) (1,1,108,0,0,1)	$2^2 3^4$
B4	(3,7,7,5,3,3) (3,3,16,0,0,-3)	(1,1,144,0,0,1) (1,3,37,3,1,0)*	$2^4 3^3$
B5	(4,4,9,0,0,4)	(1,12,12,12,0,0)*	$2^4 3^3$
B6	(3,4,9,0,0,0)	(1,3,36,0,0,0)*	$2^4 3^3$
B7	(4,9,12,0,0,0)	(1,12,36,0,0,0)*	$2^6 3^3$
B8	(4,9,28,0,4,0)	(1,36,36,-36,0,0) (9,13,13,-10,-6,-6)	$2^4 3^5$
B9	(9,16,16,16,0,0)	(1,48,48,-48,0,0)*	$2^8 3^3$
B10	(13,13,16,-8,8,10)	(4,13,37,-2,4,-4)*	$2^8 3^3$
B11	(9,16,48,0,0,0) (16,25,25,-14,16,-16)	(1,48,144,0,0,0)* (4,49,49,-46,4,4)	$2^{10} 3^3$
B12	(9,16,112,16,0,0)	(1,144,144,144,0,0) (9,49,49,-46,6,6)	$2^8 3^5$

Table 3: Genera containing spinor-regular ternary forms with discriminant divisible by 2 and 3

$2^r\mathbb{A}$ denote the binary $\mathbb{Z}_2$-lattices with Gram matrices

$$\begin{pmatrix} 0 & 2^r \\ 2^r & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2^{r+1} & 2^r \\ 2^r & 2^{r+1} \end{pmatrix},$$

respectively. For the forms in this grouping, the value of λ in Lemma 6 can be determined by one of the subcases of [16, Satz 4(a)]. For each form, the specific subcase that applies is identified in the fourth column of the table, and the value of λ appears in the last column.

For the proof of Proposition 2, it will be necessary to determine the local spinor norm groups $\theta(O^+(L_p))$ for $p = 2$ and $p = 3$. For all cases occurring in Table 4, it follows from [13, Satz 3] that

$$\theta(O^+(L_3)) = \{1,3\}\dot{\mathbb{Q}}_3^2.$$

The determination of $\theta(O^+(L_2))$ requires some further explanation. First recall that a quadratic $\mathbb{Z}_2$-lattice K is said to have *even order* (*odd order*, respectively) if $\mathrm{ord}_2(Q(v))$ is even (odd, respectively) for all maximal vectors $v \in K$ such that the symmetry τ_v lies in $O(K)$ [8]. For a nonnegative integer r, a unary lattice $\langle 2^r\mu\rangle$, $\mu \in \mathbb{Z}_2^\times$, or a binary lattice $\langle 2^r, 3\cdot 2^r\rangle$, has even order (odd order, respectively) if r is even (odd, respectively), and $2^r\mathbb{A}$ and $2^r\mathbb{H}$ have even order (odd order, respectively) if r is odd (even, respectively), by [8, Proposition 3.2]. Using this information, it can be seen that in all cases occurring in Table 4, L_2' has a binary Jordan component either of odd order or of even order, and either all of the Jordan components of L_2' have even order, or all of them have odd order. It then follows from 1.2 of [9] that in all of these cases

$$\theta(O^+(L_2)) = \mathbb{Z}_2^\times \dot{\mathbb{Q}}_2^2.$$

#	L_3	L_2'	subcase	λ
B1	$\langle 1,3,3^2\rangle$	$\mathbb{A} \perp \langle 2^3\rangle$	(ii)(β)	1
B2	$\langle 1,3,3^2\rangle$	$\mathbb{H} \perp \langle 5\cdot 2^3\rangle$	(ii)(α)	1
B3	$\langle 1,3,3^3\rangle$	$\mathbb{A} \perp \langle 3\cdot 2^3\rangle$	(ii)(β)	1
B4	$\langle 1,3,3^2\rangle$	$\mathbb{A} \perp \langle 2^5\rangle$	(ii)(β)	3
B5	$\langle 1,3,3^2\rangle$	$\langle 1\rangle \perp 2\mathbb{A}$	(ii)(γ)	1
B6	$\langle 1,3,3^2\rangle$	$\langle 1,3,2^2\rangle$	(i)(β)	1
B7	$\langle 1,3,3^2\rangle$	$\langle 1,2^2,3\cdot 2^2\rangle$	(i)(α)	1
B8	$\langle 1,3^2,3^3\rangle$	$\langle 1\rangle \perp 2\mathbb{A}$	(ii)(γ)	1
B9	$\langle 1,3,3^2\rangle$	$\langle 1\rangle \perp 2^3\mathbb{A}$	(ii)(γ)	3
B10	$\langle 1,3,3^2\rangle$	$\langle 5\rangle \perp 2^3\mathbb{H}$	(ii)(γ)	3
B11	$\langle 1,3,3^2\rangle$	$\langle 1,2^4,3\cdot 2^4\rangle$	(i)(α)	3
B12	$\langle 1,3^2,3^3\rangle$	$\langle 1\rangle \perp 2^3\mathbb{A}$	(ii)(γ)	3

Table 4: Data for B1–B12

Proposition 2. *Let f be one of the forms B1–B12, and let n be a positive integer. Then n is represented everywhere locally, but not globally, by f if and only if n lies in:*

(i) M_3^2 *for B1, B2, B5, B6, B7, B8;*

(ii) $3M_3^2$ *for B3;*

(iii) $M_3^2, 4M_3^2$ *for B4, B9, B11, B12;*

(iv) $4M_3^2$ *for B10.*

Proof. First consider the lattice L corresponding to the form B1. Assume first that $n \to \mathrm{gen}L$ but $n \not\to L$. Then $n \not\to \mathrm{spn}L$, since L is spinor-regular. So n is a spinor exception for $\mathrm{gen}L$. From Theorem 1, Lemmas 2 and 3, and the computation of $\theta(O^+(L_2))$, it follows that $\mathrm{ord}_p(n)$ is even for all $p \neq 3$. So $\mathrm{sqf}(n\Delta) = 1$ or 3. Since $\theta(O^+(L_3)) \not\subseteq N_3(-1)$, it must be that $\mathrm{sqf}(n\Delta) = 3$. So $n = 2^t 3^s w^2$ with s, t even and $w \in M_3$, by Corollary 1. By Lemma 5 and Lemma 6, $\mathrm{ord}_p(n) \le 1$ for $p = 2, 3$. Hence, $s = t = 0$ and the conclusion follows.

For the reverse implication, assume that $n = w^2$ with $w \in M_3$. The form $(1, 1, 36, 0, 0, 1)$ lies in SGII and represents 1, and so $n \to SGII$; in particular, $n \to \text{gen}L$. Since $\Delta = 2^4 3^3$, we have $\text{sqf}(n\Delta) = 3$; thus, $N_p(-n\Delta) = N_p(-3)$ and $M_{n\Delta} = M_3$. Since $\text{ord}_p(n\Delta)$ is even for $p \neq 2, 3$, it follows from Lemma 3 that $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$, and, by Corollary 1, $\theta(L_p, n) = N_p(-n\Delta)$ for all $p \neq 2, 3$. For the primes 2 and 3, direct computations show that

$$\theta(O^+(L_2)) = \mathbb{Z}_2^\times \dot{\mathbb{Q}}_2^2 = N_2(-3) = N_2(-n\Delta)$$

and

$$\theta(O^+(L_3)) = \{1, 3\}\dot{\mathbb{Q}}_3^2 = N_3(-3) = N_3(-n\Delta).$$

It then follows by Lemma 6 for $p = 2$ and [16, Satz 3(b)] for $p = 3$ that $\theta(L_p, n) = N_p(-n\Delta)$. So the criteria of Theorem 1 are met and n is a spinor exception for $\text{gen}L$. So $n \not\to SGI$ and, in particular, $n \not\to L$. This completes the proof for B1.

The proofs for the forms B2, B5, B6, B7 and B8 proceed in exactly the same way. For B4, B9, B11 and B12, the proof is analogous except that $\lambda = 3$ for these cases. Thus the conditions for $\theta(L_2, n) = N_2(-n\Delta)$ in Lemma 6 hold also with $t = 2$, and integers of the type $4M_3^2$ are also spinor exceptions for these genera.

This leaves two remaining cases, B3 and B10. The only difference for the proof of B3 is that $\text{ord}_3(\Delta)$ is even. Since the other aspects of the proof remain the same, including specifically that $\text{sqf}(n\Delta) = 1$, we see in this case that if n is a spinor exception for $\text{gen}L$, then $n = 2^t 3^s w^2$, where t is even, s is odd, and $w \in M_3$. As $\lambda = 1$, it must be that $t = 0$, and it follows from Lemma 5 that $s = 1$; so $n = 3w^2$, $w \in M_3$. In this case, $3 \to SGII$, so we conclude that no elements of the type $3M_3^2$ are represented by L.

For the case B10, the proof of the forward implication proceeds as before up to the point where we conclude that $n = 2^t w^2$ with t even and $w \in M_3$. In this case, $L_2 \cong \langle 5 \rangle \perp 2^3\mathbb{A}$. So the assumption that $n \to L_2$ implies that $t \neq 0$. Hence, $n \in 4M_3^2$, as claimed. For the reverse implication, observe that $4 \to SGII$, and so $4M_3^2 \to SGII$. The remainder of the argument then proceeds as before. $\square$

Remark 2. The set M_3 appearing in the statement of the proposition is generated by 1 and the primes congruent to 1 modulo 3.

6. Discriminants Divisible by 2 and 7

The spinor-regular ternary forms with discriminants divisible by 2 and 7 are labelled as C1–C4 in [1]. Table 5 contains a list of representatives for all of the equivalence classes of forms in the genus of each of these forms, along with their separation into spinor genara and their discriminant. In each case, as for the previous two groupings, the genus splits into two spinor genera, which are labelled as before.

#	Spinor Genus I	Spinor Genus II	Δ	L'_2
C1	(2,7,8,7,1,0)	(1,7,14,7,0,0) (1,2,49,0,0,1)	$2^2 7^3$	$\mathbb{H} \perp \langle 2 \rangle$
C2	(7,8,9,6,7,0)	(4,7,15,-7,4,0) (1,7,51,-7,-1,0)	$2^4 7^3$	$\mathbb{A} \perp \langle 5 \cdot 2^3 \rangle$
C3	(8,9,25,2,4,8)	(1,28,56,-28,0,0) (4,8,49,0,0,4)	$2^6 7^3$	$\langle 1 \rangle \perp 2\mathbb{H}$
C4	(29,32,36,32,12,24)	(4,29,197,-2,-4,4) (16,32,53,-8,16,-16)	$2^{10} 7^3$	$\langle 5 \rangle \perp 2^3\mathbb{A}$

Table 5: Genera containing spinor-regular ternary forms with discriminant divisible by 2 and 7

For each lattice L corresponding to one of the forms C1–C4, L_7 is isometric to $\langle 1, 7, 7^2 \rangle$, and it follows from [13, Satz 3] that

$$\theta(O^+(L_7)) = \{1, 7\}\dot{\mathbb{Q}}_7^2 = N_7(-7).$$

The 2-adic splittings for these lattices are given in the last column of Table 5. The forms C1 and C2 are non-classic and the splittings listed for them are for the lattice L'_2 obtained from L_2 by scaling by 2; for C3 and C4, which are classic, the splittings listed are for $L'_2 = L_2$. In all cases, it again follows from 1.2 of [9] that

$$\theta(O^+(L_2)) = \mathbb{Z}_2^\times \dot{\mathbb{Q}}_2^2.$$

Proposition 3. *Let f be one of the forms C1–C4, and let n be a positive integer. Then n is represented everywhere locally, but not globally, by f if and only if n lies in:*

(i) M_7^2 *for C1, C2, C3;*

(ii) $4M_7^2$ *for C4.*

Proof. Let L be a lattice corresponding to one of the forms C1-C3. Assume first that $n \to \text{gen}L$ but $n \not\to L$. As before, n is a spinor exception for $\text{gen}L$. From Theorem 1, Lemma 3 and the computation of $\theta(O^+(L_2))$, it follows that $\text{ord}_p(n\Delta)$ is even for all $p \neq 7$. So $\text{sqf}(n\Delta) = 1$ or 7. Since $\theta(O^+(L_7)) \not\subseteq N_7(-1)$, it must be that $\text{sqf}(n\Delta) = 7$. So $n = 2^s 7^t w^2$ with s, t even and $w \in M_7$, by Corollary 1. By Lemma 5, $\text{ord}_7(n) \leq 1$ and so $t = 0$. Since $2 \in M_7$, it follows that $n \in M_7^2$ as claimed.

For the reverse implication, assume that $n = w^2$ with $w \in M_7$. It can be seen from the representatives of SGII that $1 \to SGII$, and so $n \to SGII$; in particular, $n \to \text{gen}L$. Since $\text{ord}_2(\Delta)$ is even and $\text{ord}_7(\Delta)$ is odd, we have $\text{sqf}(n\Delta) = 7$; thus,

$N_p(-n\Delta) = N_p(-7)$ for all p, and $M_{n\Delta} = M_7$. Since $\mathrm{ord}_p(n\Delta)$ is even for $p \neq 2,7$, it follows from Lemma 3 that $\theta(O^+(L_p)) \subseteq N_p(-n\Delta)$, and from Corollary 1 that $\theta(L_p, n) = N_p(-n\Delta)$ for all $p \neq 2,7$. Also, $\theta(O^+(L_7)) = N_7(-7) = N_7(-n\Delta)$ and $\mathrm{ord}_7(n) = 0$ gives $\theta(L_7, n) = N_7(-n\Delta)$, by [16, Satz 3(b)]. Since $-7 \in \dot{\mathbb{Q}}_2^2$, we have $\theta(O^+(L_2)) = \mathbb{Z}_2^\times \dot{\mathbb{Q}}_2^2 \subseteq N_2(-7) = N_2(-n\Delta)$, and, by Lemma 1, $\theta(L_7, n) = N_2(-n\Delta)$. So the criteria of Theorem 1 are met and n is a spinor exception for gen$L$. So $n \not\to SGI$ and, in particular, $n \not\to L$. This completes the proof for C1-C3.

For the case C4, the proof of the forward implication proceeds as above to the point where we conclude that $n = 2^s w^2$ with s even and $w \in M_7$. In this case, $L_2 \cong \langle 5 \rangle \perp 2^3\mathbb{A}$. So the assumption that $n \to L_2$ implies that $s \neq 0$. Hence, $n \in 4M_7^2$, as claimed. For the reverse implication, observe that $4 \to SGII$, and so $4M_7^2 \to SGII$. The remainder of the argument then proceeds as before. □

Remark 3. The set M_7 appearing in the statement of the proposition is generated by 1 and the primes congruent to 1, 2 or 4 modulo 7.

7. Forms for Which the Equivalence Class and Spinor Genus Do Not Coincide

For the two spinor-regular ternary forms for which the equivalence class and spinor genus do not coincide, the method of [1] yields only the even integers that fail to be represented. For completeness, we will state here the full results for these two forms, which are B4 and B11 in the list. Before stating the results, we will establish three lemmas needed to analyze the local obstructions to representation by these forms.

Lemma 7. *Let L be a ternary lattice such that $L_2 \cong M \perp \langle 16 \rangle$, where M is a lattice corresponding to $x^2 + xy + y^2$, and let n be a positive integer. Then $n \not\to L_2$ if and only if $n = 2 + 4\ell$ or $8 + 16\ell$, for some nonnegative integer ℓ.*

Proof. Since $\mathbb{Z}_2^\times \to M$, it follows that $\mathbb{Z}_2^\times, 4\mathbb{Z}_2^\times \to L_2$. Also, for any $x, y \in \mathbb{Z}_2$, $x^2 + xy + y^2$ lies in either $\mathbb{Z}_2^\times$ or $4\mathbb{Z}_2$. So all odd integers and integers of the type $4n_0$ with n_0 odd are represented by L_2, and no integer of the type $2 + 4\ell$ can be represented by L_2.

Consider $n = 4n_0$ with n_0 even. If there exist $x, y \in \mathbb{Z}_2$ such that

$$n = x^2 + xy + y^2 + 16z^2, \tag{4}$$

then $x, y \in 2\mathbb{Z}_2$ and the right-hand side of Equation (4) is in $4\mathbb{Z}_2$. Hence, no element of the type $8 + 16\ell$ is represented by L_2.

Finally, let $n = 16n_0$, $n_0 \in \mathbb{Z}$. Since $\mathbb{Z}_2^\times \to M$, either n_0 or $n_0 - 1$ is represented by M. So there exist $x_0, y_0, z \in \mathbb{Z}_2$ such that

$$n_0 = x_0^2 + x_0 y_0 + y_0^2 + z^2.$$

It follows that Equation (4) is satisfied with $x = 4x_0, y = 4y_0$. This completes the proof. $\square$

Lemma 8. *Let L be a ternary lattice such that $L_2 \cong \langle 1, 16, 48\rangle$, and let n be a positive integer. Then $n \not\to L_2$ if and only if $n = 5 + 8\ell$, $2 + 4\ell$, $3 + 4\ell$, $8 + 16\ell$, or $12 + 16\ell$, for some nonnegative integer ℓ.*

Proof. Here $n \to L_2$ if and only if there exist $x, y, z \in \mathbb{Z}_2$ such that

$$n = x^2 + 16y^2 + 48z^2. \tag{5}$$

If $2 \nmid n$, then Equation (5) is solvable if and only if $n \equiv 1 \pmod 8$, thus ruling out integers of the type $5 + 8\ell$ and $3 + 4\ell$. If $2 \mid n$, then $x = 2x_0$ for some $x_0 \in \mathbb{Z}_2$, so

$$n = 4x_0^2 + 16y^2 + 48z^2. \tag{6}$$

So it must be that $4 \mid n$, ruling out integers of the type $2 + 4\ell$. Write $n = 4n_0$, with $n_0 \in \mathbb{Z}$. Dividing Equation (6) through by 4 then gives

$$n_0 = x_0^2 + 4y^2 + 12z^2. \tag{7}$$

If $2 \nmid n_0$, then Equation (7) is solvable if and only if $n_0 \equiv 1 \pmod 4$. This rules out integers of the type $4(3 + 4\ell) = 12 + 16\ell$. If $2 \mid n_0$ then $x_0 \in 2\mathbb{Z}_2$, and so the right-hand side of Equation (7) is in $4\mathbb{Z}_2$. This rules out integers of the type $8+16\ell$. Finally, the lattice $\langle 1, 1, 3\rangle$ is isotropic over $\mathbb{Z}_2$ and is therefore $\mathbb{Z}_2$-universal by [6, Proposition 4.1]. Hence, all positive integers divisible by 16 are represented by L_2. This completes the proof. $\square$

Lemma 9. *Let L be a ternary lattice such that $L_3 \cong \langle 1, 3, 9\rangle$, and let n be a positive integer. Then $n \not\to L_3$ if and only if $n = 2 + 3\ell$ or $n = 9^k(6 + 9\ell)$, for some nonnegative integers k, ℓ.*

Proof. Let V denote the underlying quadratic space. By a computation of Hasse symbols, it follows that V_3 is anisotropic; hence $\alpha \not\to V_3$ for any $\alpha \in -dV = -3\dot{\mathbb{Q}}_3^2$ (see, e.g., [6, Lemma 2.2]). As $9^k(6 + 9\ell) = 3^{2k} \cdot 3(2 + 3\ell) \in -3\dot{\mathbb{Q}}_3^2$, no such integer can be represented by L_3. Integers of the type $n = 2 + 3\ell$ are ruled out for representation by L_3 by the Local Square Theorem [15, 63:1].

It remains to show that all other integers are represented by L_3. Write $n = 3^t n_0$, with t a nonnegative integer and $n_0 \equiv 1, 2 \pmod 3$. If $n_0 \equiv 1 \pmod 3$, then there exists $\lambda \in \mathbb{Z}_3^\times$ such that $n_0 = \lambda^2$. So if $t = 0$ and $n_0 \equiv 1 \pmod 3$ then $n = \lambda^2 \to L_3$. If $t = 2k + 1$ is odd and $n_0 \equiv 1 \pmod 3$, then $n = 3(3^k\lambda)^2 \to L_3$. By [15, 92:1b], $\mathbb{Z}_3^\times \to \langle 1, 1\rangle$. So $3^2\mathbb{Z}_3^\times \to \langle 1, 3^2\rangle$. It then follows that $n \to L_3$ whenever t is even and $t \geq 2$. This exhausts all cases and completes the proof. $\square$

Proposition 4. *The form $3x^2+7y^2+7z^2+5yz+3xz+3xy$ represents a positive integer n if and only if n is not of the type*

$$2+3\ell, 9^k(6+9\ell), 4^a(2+4\ell) \text{ or } 4^aM_3^2,$$

where k,ℓ are nonnegative integers and $a\in\{0,1\}$.

Proof. This is the form B4. Let L be a lattice corresponding to this form and let n be a positive integer. By [15, 92:1b], $n\to L_p$ for all primes $p\neq 2,3$. For $p=2$ and $p=3$, the conditions for representation by L_p are given in Lemmas 7 and 9, respectively. Consequently, $n\to \text{gen}L$ if and only if n is not of one of the types $2+3\ell$, $9^k(6+9\ell)$ or $4^a(2+4\ell)$ for some nonnegative integers k,ℓ and $a\in\{0,1\}$. If n is not of one of these types, then $n\to L$ if and only if n does not lie in M_3^2 or $4M_3^2$, by Proposition 2(iii). □

Proposition 5. *The form $9x^2+16y^2+48z^2$ represents a positive integer n if and only if n is not of the type*

$$5+8\ell, 4^a(2+4\ell), 4^a(3+4\ell) \text{ or } 4^aM_3^2,$$

where ℓ is a nonnegative integer and $a\in\{0,1\}$.

Proof. This is the form B11. Let L be a lattice corresponding to this form and let n be a positive integer. By [15, 92:1b], $n\to L_p$ for all primes $p\neq 2,3$. For $p=2$ and $p=3$, the conditions for representation by L_p are given in Lemmas 8 and 9, respectively. Consequently, $n\to \text{gen}L$ if and only if n is not of one of the types $5+8\ell$, $4^a(2+4\ell)$ or $4^a(3+4\ell)$ for some nonnegative integer ℓ and $a\in\{0,1\}$. If n is not of one of these types, then $n\to L$ if and only if n does not lie in M_3^2 or $4M_3^2$, by Proposition 2(iii). □

8. Concluding Remarks

For each of the 27 spinor-regular ternary forms for which the equivalence class and spinor genus coincide, Aygin et al [1] summarize in Table A.17 the list of all positive integers not represented by the form. As detailed in that paper, the results for several of these forms appeared earlier in work of Lomadze [14] and Berkovich [3]. Note that in all cases the integers in the arithmetic progressions listed in Table A.17 are those that are excluded from representation because they are not represented everywhere locally by the form. The remaining excluded integers listed in Table A.17 occur in one or more square classes. These agree with the integers appearing in Propositions 1, 2 and 3 of this paper, except for the forms B10 and C4. In each of those cases, it can be seen from the 2-adic structure of the corresponding lattice (as given in Table 4 or Table 5, respectively) that if n is any integer represented 2-adically by the form, then $n \equiv 5 \pmod 8$. Consequently, the odd squares are excluded from the lists for these forms appearing in Propositions 2 and 3. Specifically, for the form B10, M_3^2 appears in Table A.17 but not in Proposition 2(iv), and for the form C4, the excluded set M_7^2 in Table A.17 is replaced by $4M_7^2$ in Proposition 3(ii).

In light of the fact that a number of the forms studied here were first identified by Jones and Pall [12] as genus mates of regular diagonal ternary forms, it may also be of interest to add a few additional comments regarding regular forms. Note that the integers represented everywhere locally, but not globally, by the forms studied here are squares, except in the cases of the forms A2 and B3, where they are of the types $2\lambda^2$ and $3\lambda^2$, respectively. In all cases, the opposite spinor genus SGII represents 1, except for the cases A11, B10 and C4, where SGII represents 4. Note also that SGII represents 2 in case A2, and SGII represents 3 in case B3. Consequently, in all cases, SGII can be seen to be a regular spinor genus, in the sense that the forms within that spinor genus collectively represent all integers represented by the genus. So when SGII consists of a single equivalence class, every form in that class is a regular form. This occurs for the cases A1, A4, A5, A8, B1, B2, B5, B6, B7, B9 and B10. Among this list, the cases when SGII contains a diagonal form (namely, A1, A4, A5, A8, B6 and B7) are those investigated by Jones and Pall [12] and Schulze-Pillot [16], along with the case B11, where one of the two classes in SGII is a regular diagonal form.

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