ON THE EQUATION $\sigma^*(n) = 1 + mn$

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Abstract

Let M be a positive integer with $M \geq 3$, and let $\sigma^*(n)$ denote the unitary analogue of the sum of divisors function $\sigma(n)$. We strengthen considerably the lower estimations of the solutions n of the equation $\sigma^*(n) = 1 + Mn$.

1. Introduction

We say that d is a *unitary* divisor of n if $d|n$ and $\left(d, \frac{n}{d}\right) = 1$. Let $\sigma^*(n) = \sum_{d|n,\ (d, \frac{n}{d})=1} d$ be the sum of all unitary divisors of n. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ where p_i are distinct primes and $\alpha_i > 0$ for all i, then it is easy to see that

$$\sigma^*(n) = \prod_{i=1}^{r} (1 + p_i^{\alpha_i})$$

and that $\sigma^*(n)$ is a multiplicative function.

The *Dedekind psi function* $\psi(n)$ is a multiplicative function given by

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

Let $M \geq 1$ be a positive integer. Subbarao [6] studied the Diophantine equations

$$\psi(n) = 1 + Mn \tag{1}$$

and

$$\sigma^*(n) = 1 + Mn. \tag{2}$$

He formulated the following conjecture.

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Conjecture 1. The congruence $\sigma^*(n) \equiv 1 \pmod{n}$ is possible if and only if n is a prime power. In particular, for arbitrary distinct primes $p_1, \ldots, p_r$, $r \geq 1$,

$$(p_1 + 1)\ldots(p_r + 1) \equiv 1 \pmod{p_1 \ldots p_r} \text{ if and only if } r = 1.$$

Equivalently, $\psi(n) \equiv 1 \pmod{n}$ if and only if n is a prime.

No counterexample to Conjecture 1 is known although there are some partial results. For $M = 1, 2, 3, \ldots$ define

$$T_M = \{n : \psi(n) = 1 + Mn\},\ \mathcal{T} = \bigcup_{M \geq 2} T_M,$$
$$T_M^* = \{n : \sigma^*(n) = 1 + Mn\},\ \mathcal{T}^* = \bigcup_{M \geq 2} T_M^*.$$

Note that $T_M \subset T_M^*$. Subbarao's conjecture is that $\mathcal{T}$ and $\mathcal{T}^*$ are empty.

In the sequel we assume that $M > 1$. Subbarao himself obtained the following results:

i) If $n \in T_M^*$, then n is not a powerful number, M is odd ≥ 3 and $\omega(n) \geq 16$, $n > 10^{20}$, where $\omega(n)$ denotes the number of distinct prime factors of n;

ii) If $n \in T_M^*$ with $\omega(n) = r$, then

$$n < (r-1)^{2^r - 1}; \tag{3}$$

iii) If $n \in T_M$ and $3|n$, then we have

$$\omega(n) > 2557,\ n > 5.9 \cdot 10^{10766}, \tag{4}$$

(stated without proof);

iv) If $n \in T_M$ and $3 \nmid n$, then we have

$$\omega(n) \geq 123. \tag{5}$$

Our first theorem improves the upper bound in Inequality (3).

Theorem 1. *If $n \in T_M^*$, then*

$$n \leq 2^{2^r - 1} - 2^{2^{r-1} - 1}, \tag{6}$$

where r denotes the number of distinct prime divisors of n.

Using Grytczuk-Wójtowicz's techniques from the paper [3] we obtain the lower bounds for $\omega(n)$ and n for $n \in T_M^*$ with $M \geq 3$, which are also valid for $n \in T_M$ with $M > 3$; thereby improving Inequalities (4)-(5).

The main result of this paper is the following theorem.

Theorem 2. *Let $M \geq 3$ and $n \in T_M^*$.*

(a) If $3|n$, then $\omega(n) = r > \frac{1}{2} \cdot 1578^{AM^2/9}$, where $A = 0.998\ldots$.

(b) If $3 \nmid n$, then $\omega(n) \geq 51^{M/3} - 1$.

The corollary below is an immediate consequence of Theorem 2 and Robin's inequality (see [4, Théorème 6]) that for every positive integer n we have

$$n > \left(\frac{r \log r}{3}\right)^r, \tag{7}$$

where $r = \omega(n)$. Note that for $M \geq 3$ we have the following rough bounds: $\frac{1}{2} \cdot 1578^{AM^2/9} > 2^{AM^2-1}$ and $51^{M/3} - 1 > 3^M$.

Corollary 1. *Let $M \geq 3$ and $n \in T_M^*$.*

(a) If $3|n$, then $n > (c(AM^2-1)2^{AM^2-1})^{2^{AM^2-1}}$, where $c = 0.231\ldots = \log 2/3$.

(b) If $3 \nmid n$, then $n > (dM3^M)^{3^M}$, where $d = 0.366\ldots = \log 3/3$.

Thus, if $3|n$ and $M \geq 5$, we have that $\omega(n) > 3.7 \cdot 10^8$ and $n > 3.9 \cdot 10^{10^8}$ (compare with Inequality (4)). If $3 \nmid n$ and $M \geq 5$, then $\omega(n) > 700$ and $n > 10^{643}$ (compare with Inequality (5)).

Using Inequality (6) we obtain the following analogue of [3, Theorem 2].

Theorem 3. *Let $\mathcal{P} = \{P_1, P_2 \ldots\}$, where $P_i < P_{i+1}$ for all $i \geq 1$ denote the set of all odd prime numbers. For every integer $k \geq 2$ there exists an infinite subset $\mathcal{P}(k)$ of the set $\mathcal{P}$ such that*

(a) for every pairwise distinct primes $p_1, p_2, \ldots, p_k \in \mathcal{P}(k)$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ the number $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ does not fulfill Equation (2);

(b) $\mathcal{P}(k)$ is maximal with respect to inclusion.

(Note that, since $\omega(n) \geq 16$ in general, we have $\mathcal{P}(k) = \mathcal{P}$ for $k \leq 15$.)

2. Preliminaries

In this section we shall collect some preliminary results.

Lemma 1. *Let $n \in T_M^*$, $M > 1$. If $p|n$ and $q^\beta + 1 \equiv 0 \pmod p$, then q^β cannot be a unitary divisor of n. In particular, if $p|n$, $q + 1 \equiv 0 \pmod p$, and q^β is a unitary divisor of n, then β cannot be odd.*

Proof. Given $p|n$ and $q^\beta + 1 \equiv 0 \pmod p$, if $q^\beta || n$, then also $\sigma^*(q^\beta) = q^\beta + 1 | \sigma^*(n)$, so that $p|\sigma^*(n)$. Thus, $p|(n, \sigma^*(n))$, leading to a contradiction. Hence, the lemma is proven. $\square$

The next lemma due to Goto [2] plays an important role in the proof of the Theorem 1.

Lemma 2. *Let r, a, b be positive integers. If integers $x_1, \ldots, x_r$ satisfy $1 \le x_1 < x_2 < \ldots < x_r$ and*

$$1 \le \prod_{i=1}^{r-1}\left(1+\frac{1}{x_i}\right) < \frac{a}{b} \le \prod_{i=1}^{r}\left(1+\frac{1}{x_i}\right), \tag{8}$$

then it follows that

$$\prod_{i=1}^{r} x_i \le (b+1)^{2^r-1} - (b+1)^{2^{r-1}-1} \tag{9}$$

with equality if and only if $x_i = m_i$ for $1 \le i \le r$, where

$$m_i = \begin{cases} (b+1)^{2^{i-1}} & i = 1, \ldots, r-1, \\ (b+1)^{2^{i-1}} - 1 & i = r. \end{cases}$$

3. Proofs

Proof of Theorem 1. Let $n \in T_M^*$ with $\omega(n) = r$. Write $n = N_1 \cdots N_r$ where $N_1, \ldots, N_r$ denote prime powers satisfying $N_i < N_j$, $(N_i, N_j) = 1$ for $i < j$. Then

$$\frac{\sigma^*(n)-1}{n} < \frac{\sigma^*(n)}{n} = \prod_{i=1}^{r}\left(1+\frac{1}{N_i}\right).$$

Moreover,

$$1 < \prod_{i=1}^{r-1}\left(1+\frac{1}{N_i}\right) < \prod_{i=1}^{r}\left(1+\frac{1}{N_i}\right) - \frac{1}{N_1\cdots N_r} = \frac{\sigma^*(n)-1}{n}.$$

Thus,

$$1 < \prod_{i=1}^{r-1}\left(1+\frac{1}{N_i}\right) < \frac{\sigma^*(n)-1}{n} < \prod_{i=1}^{r}\left(1+\frac{1}{N_i}\right).$$

Hence, the Inequality (8) is satisfied for $x_i = N_i$, $a = \frac{\sigma^*(n)-1}{n}$, $b = 1$. From Inequality (9) we get $n = N_1 \cdots N_r \le 2^{2^r-1} - 2^{2^{r-1}-1}$, and the theorem follows. □

Proof of Theorem 2. From Equation (2) we get that if $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \in T_M^*$, then

$$M = \frac{\sigma^*(n)-1}{n} < \frac{\sigma^*(n)}{n} = \prod_{i=1}^{r}\left(1+\frac{1}{p_i^{\alpha_i}}\right) < \prod_{i=1}^{r}\left(1+\frac{1}{p_i}\right)$$

and hence

$$\log M < \sum_{i=1}^{r} \log\left(1 + \frac{1}{p_i}\right). \tag{10}$$

Part (a). From Lemma 1 it follows that $p_i \equiv 1 \pmod 6$ or α_i is even for $i \geq 2$. We define the set $\mathcal{A} := \{3\} \cup \{p \in \mathcal{P} :\ p \equiv 1 \pmod 6\} \cup \{p^2 :\ p \equiv 5 \pmod 6\} = \{a_1, a_2, \ldots\}$, where $a_j < a_{j+1}$ for $j = 1, 2, \ldots$.

Put

$$\alpha(k) = \sum_{i=1}^{k} \log\left(1 + \frac{1}{a_i}\right),$$

where $k \geq 16$. Thus,

$$\log M < \alpha(r). \tag{11}$$

Let k_0 be the least integer k with $\log 3 \leq \alpha(k)$. Since $\alpha(805) = 1.098538\ldots < \log 3 = 1.098612\ldots < 1.098613\ldots = \alpha(806)$, we obtain that $k_0 = 806$. Now from Inequality (11) it follows that for $n \in T_M^*$ with $M \geq 3$ we have $\omega(n) = r \geq 806$.

Let $\mathcal{Q}' = \{Q'_1, Q'_2, \ldots\}$ denote the set of all primes $\equiv 1 \pmod 6$ with $Q'_i < Q'_{i+1}$ and $\mathcal{Q}^* = \{Q^*_1, Q^*_2, \ldots\}$ denote the set of all primes $\equiv 5 \pmod 6$ with $Q^*_i < Q^*_{i+1}$ for $i = 1, 2, \ldots$. Since $i \geq 806$, then $a_i \equiv 1 \pmod 6$ and $a_i \geq 13441 = Q'_{790}$ or $a_i = p^2$ with $p \equiv 5 \pmod 6$ a prime and $p \geq 131 = Q^*_{16}$. Hence,

$$\begin{aligned}\log M &< \alpha(805) + \sum_{i=806}^{r} \log\left(1 + \frac{1}{a_i}\right) < \log 3 + \sum_{i=806}^{r} \frac{1}{a_i} \\ &\qquad\qquad < \log 3 + \sum_{j=790}^{r} \frac{1}{Q'_i} + \sum_{i=16}^{\infty} \frac{1}{{Q^*_j}^2}.\end{aligned}$$

Using the estimate $Q'_n, Q^*_n > 2n \log(2n)$ for $Q'_n, Q^*_n > 198$, which follows from [1, Corollary 1.6], we get

$$\begin{aligned}\log M &< \log 3 + \sum_{j=16}^{23} \frac{1}{{Q^*_j}^2} + \sum_{j=790}^{r} \frac{1}{2j \log(2j)} + \sum_{j=24}^{\infty} \frac{1}{4j^2 \log^2(2j)} \\ &< \log 3 + 0.00081 + \int_{789}^{r} \frac{dx}{2x \log(2x)}\end{aligned}$$

Hence, $\log M < \log 3 + 0.00081 + \frac{1}{2} \log\log(2r) - \frac{1}{2} \log\log 1578$. Therefore $\omega(n) = r > \frac{1}{2} \cdot 1578^{AM^2/9}$ where $A = 0.998\ldots$.

Part (b). Let $\mathcal{Q} = \{Q_1, Q_2, \ldots\}$ denote the set of all primes with $Q_i < Q_{i+1}$ for $i = 1, 2, \ldots$. We have in Inequality (10) that $p_1 \geq 5 = Q_3$. Put

$$\beta(k) = \sum_{i=1}^{k} \log\left(1 + \frac{1}{Q_{i+2}}\right),$$

where $k \geq 16$. Now from Inequality (10) we obtain

$$\log M < \beta(r). \tag{11'}$$

Since $\beta(49) = 1.09651\ldots < \log 3 < 1.10069\ldots = \beta(50)$, from Inequality (11′) it follows that for $n \in T_M^*$ with $M \geq 3$ we have $\omega(n) = r \geq 50$. Thus,

$$\begin{aligned}\log M <&\beta(49) + \sum_{i=50}^{r} \log\left(1 + \frac{1}{Q_{i+2}}\right) < \log 3 + \sum_{i=50}^{r} \frac{1}{Q_{i+2}} \\ &< \log 3 + \int_{49}^{r} \frac{dx}{(x+2)\log(x+2)},\end{aligned}$$

as

$$Q_m > m \log m \text{ for } m \geq 1.$$

(see [5]). Hence, $\log M < \log 3 + \log\log(r+2) - \log\log 51$ i.e., $\omega(n) = r \geq 51^{M/3} - 1$, as claimed. □

Proof of Theorem 3. The theorem immediately follows from Theorem 1 since we can take $\mathcal{P}(k)$ to be the set of primes larger than $2^{2^k-1} - 2^{2^{k-1}-1}$. Indeed, n cannot fulfil $\sigma^*(n) = 1 + Mn$ if any prime factor of n is larger than $2^{2^k-1} - 2^{2^{k-1}-1}$.

The existence of a maximal (with respect to inclusion) set $\mathcal{P}(k)$ follows from Kuratowski-Zorn's lemma. □

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