INTEGRAL REPRESENTATIONS OF CATALAN NUMBERS AND SUMS INVOLVING CENTRAL BINOMIAL COEFFICIENTS

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Abstract

In the paper, the authors collect several integral representations of the Catalan numbers and central binomial coefficients, supply alternative proofs of two integral representations of the Catalan numbers, and apply these integral representations to alternatively prove several combinatorial identities for finite and infinite sums in which central binomial coefficients are involved.

1. Preliminaries and Motivations

In this paper, we use the following notation:

$$\mathbb{Z} = \{0, \pm1, \pm2, \dots\}, \qquad \mathbb{N} = \{1, 2, \dots\},$$
$$\mathbb{N}_0 = \{0, 1, 2, \dots\}, \qquad \mathbb{N}_- = \{-1, -2, \dots\}.$$

The classical Euler's *gamma function* $\Gamma(z)$ can be defined [49, Chapter 3] by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

The logarithmic derivative $[\ln\Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$, denoted by $\psi(z)$, is called the *psi function* or the *digamma function*. It is known [49, Chapter 3] that,

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- the gamma function $\Gamma(x)$ is positive on the intervals $(0, \infty)$ and $(-2k, 1-2k)$ for $k \in \mathbb{N}$ (see [49, p. 44, Figure 3.1]);
- the gamma function $\Gamma(x)$ is negative on the intervals $(1-2k, 2-2k)$ for $k \in \mathbb{N}$ (see [49, p. 44, Figure 3.1]);
- the gamma function $\Gamma(z)$ is single-valued and analytic over the punctured complex plane $\mathbb{C} \setminus \{1-k, k \in \mathbb{N}\}$ (see [1, p. 255, Entry 6.1.3]);
- the gamma function $\Gamma(z)$ has simple poles in the left half-plane at the points $1-k$ and the residue at $1-k$ is $\frac{(-1)^{k-1}}{(k-1)!}$ for $k \in \mathbb{N}$ (see [49, p. 44]);
- the reciprocal $\frac{1}{\Gamma(z)}$ is an entire function possessing simple zeros at the points $1-k$ for $k \in \mathbb{N}$ (see [1, p. 255, Entry 6.1.3]);
- the gamma function $\Gamma(z)$ satisfies the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \quad \text{(see [1, p. 256, Entry 6.1.18]).} \tag{1}$$

The *Catalan numbers* C_n for $n \in \mathbb{N}_0$ form a positive integer sequence. This sequence was first used in 1730 by the Mongolian mathematician Ming Antu, described in the 18th century by Leonhard Euler, and named after the Belgian mathematician Eugéne Charles Catalan; see the papers [8, 9, 17, 18, 29]. This sequence is one of the more fascinating sequences in combinatorial number theory with over fifty significant combinatorial interpretations (see [7, 47]). We now mention several properties of the Catalan numbers C_n as follows.

- Two analytic expressions of the Catalan numbers C_n for $n \in \mathbb{N}_0$ are

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{4^n \Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \quad n \in \mathbb{N}_0 \quad \text{(see [29, 44]).} \tag{2}$$

- The Catalan numbers C_n for $n \in \mathbb{N}_0$ can be analytically generated by

$$\frac{2}{1+\sqrt{1-4t}} = \sum_{n=0}^{\infty} C_n t^n \quad \text{(see [6, 32]).} \tag{3}$$

- For $n \in \mathbb{N}_0$, the Catalan numbers C_n can be represented by the integral

$$C_n = \frac{1}{2\pi} \int_0^4 \sqrt{\frac{4-t}{t}}\, t^n \, dt \quad \text{(see [21, p. 2, Eq. (10)]).} \tag{4}$$

  The integral representation (4) has been applied in [43]. In the paper [48], a simple proof and some applications of the integral representation (4) were discussed once again.

- The Catalan numbers C_n for $n \in \mathbb{N}_0$ can be represented by

$$C_n = \frac{2}{\pi}\int_0^\infty \frac{t^2}{(t^2+1/4)^{n+2}}dt = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{(t+1/4)^{n+2}}dt \tag{5}$$

(see [24, Theorem 3], [32, Theorem 5.2], and [38, Theorem 1.3]).

In [3, Corollary 3.2], [10, Section 4.2], [21, p. 2, Eq. (10)], and [23, Theorem 3.1], among other things, the integral representations

$$\begin{aligned}\binom{2n}{n} &= \frac{1}{\pi}\int_0^4 \sqrt{\frac{t}{4-t}}\, t^{n-1}dt \\ &= \frac{2}{\pi}\int_0^{\pi/2} (2\sin t)^{2n}dt \\ &= \frac{1}{\pi}\int_0^\infty \frac{1}{(t^2+1/4)^{n+1}}dt\end{aligned} \tag{6}$$

of central binomial coefficients $\binom{2n}{n}$ for $n \in \mathbb{N}_0$ were established and applied. See also [28, p. 57], [29, Section 2.4, Theorem 7], and [43, Lemma 2.5 and Theorem 5.5].

In the preprint [37, Remark 1] and its formally published version [40, Eq. (9)], the mathematician Feng Qi, the first author of the papers from [23] to [43], and his coauthors generalized the Catalan numbers C_n to the so-called *Catalan–Qi function*

$$C(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z \frac{\Gamma(z+a)}{\Gamma(z+b)}, \quad \Re(a), \Re(b) > 0, \quad \Re(z) \ge 0.$$

See also the paper [36] and many closely related references therein. It is clear that

$$C(b,a;z) = \frac{1}{C(a,b;z)}, \quad C\left(\frac{1}{2}, 2; n\right) = C_n, \quad C(a,b;n) = \left(\frac{b}{a}\right)^n \frac{(a)_n}{(b)_n}$$

for all $n \in \mathbb{N}_0$, where

$$(z)_n = \prod_{\ell=0}^{n-1}(z+\ell) = \begin{cases} z(z+1)\cdots(z+n-1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}$$

is called the rising factorial or the Pochhammer symbol of any complex number $z \in \mathbb{C}$. For $b > a > 0$ and $x \ge 0$, the Catalan–Qi function $C(a,b;x)$ has two integral representations

$$C(a,b;x) = \left(\frac{a}{b}\right)^{b-1} \frac{1}{B(a,b-a)} \int_0^{b/a} \left(\frac{b}{a} - t\right)^{b-a-1} t^{x+a-1}dt$$

and

$$C(a,b;x) = \left(\frac{a}{b}\right)^{a} \frac{1}{B(a,b-a)} \int_0^{\infty} \frac{t^{b-a-1}}{(t+a/b)^{x+b}}dt,$$

where the classical beta function $B(z,w)$ can be defined by

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt=\int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}}dt=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

for $\Re(z),\Re(w)>0$ (see [29, Theorem 12] and [35, Theorem 4]).

The *extended binomial coefficient* $\binom{z}{w}$ for $z,w\in\mathbb{C}$ is defined [50] by

$$\binom{z}{w}=\begin{cases}\dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z\notin\mathbb{N}_-, \quad w,z-w\notin\mathbb{N}_-;\\ 0, & z\notin\mathbb{N}_-, \quad w\in\mathbb{N}_- \text{ or } z-w\in\mathbb{N}_-;\\ \dfrac{\langle z\rangle_w}{w!}, & z\in\mathbb{N}_-, \quad w\in\mathbb{N}_0;\\ \dfrac{\langle z\rangle_{z-w}}{(z-w)!}, & z,w\in\mathbb{N}_-, \quad z-w\in\mathbb{N}_0;\\ 0, & z,w\in\mathbb{N}_-, \quad z-w\in\mathbb{N}_-;\\ \infty, & z\in\mathbb{N}_-, \quad w\notin\mathbb{Z},\end{cases}$$

where

$$\langle\alpha\rangle_n=\prod_{k=0}^{n-1}(\alpha-k)=\begin{cases}\alpha(\alpha-1)\cdots(\alpha-n+1), & n\in\mathbb{N}\\ 1, & n=0\end{cases}$$

is called the *falling factorial*. In the paper [50], integral representations in (6) were extended as

$$\begin{aligned}\binom{2z}{z}&=\frac{1}{\pi}\int_0^4\sqrt{\frac{x}{4-x}}\,x^{z-1}dx\\ &=\frac{2}{\pi}\int_0^{\pi/2}(2\sin x)^{2z}dx\\ &=\frac{1}{\pi}\int_0^\infty\frac{1}{(x^2+1/4)^{z+1}}dx\end{aligned}\tag{7}$$

for $z\in\mathbb{C}$ such that $\Re(z)>-\frac{1}{2}$. Meanwhile, integral representations in (7) were also applied in [50] to establish inequalities and monotonicity of functions concerning extended central binomial coefficients $\binom{2x}{x}$ for $x\in\mathbb{R}$.

In the papers [2, 10, 11, 13, 14, 16, 19, 23, 25, 26, 27, 30, 31, 33, 39, 42, 46, 52], there have been many other new conclusions about the Catalan numbers C_n and central binomial coefficients $\binom{2n}{n}$.

Recall from Chapter XIII in [20], Chapter 1 in [45], and Chapter IV in [51] that, if a function $f(x)$ on an interval I has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x)\ge 0$ for $x\in I$ and $n\in\mathbb{N}_0$, then we call $f(x)$ a *completely monotonic function* on I. Theorem 12b in [51, p. 161] characterized that a function $f(x)$ is completely monotonic on $(0,\infty)$ if and only if

$$f(x)=\int_0^\infty \mathrm{e}^{-xt}\,d\sigma(t), \quad x\in(0,\infty), \tag{8}$$

where $\sigma(s)$ is non-decreasing and the integral in (8) converges for $x \in (0,\infty)$. The integral representation (8) means that a function $f(x)$ is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform of a non-decreasing measure $\sigma(s)$ on $(0,\infty)$.

In [4, Remark 8], it was pointed out that the function

$$S : s \mapsto \frac{(s+1)^2}{s^2(2s+1)}\left[1-\frac{\Gamma^2(s+1)}{\Gamma(2s+1)}\right] \tag{9}$$

decreases on $(0,\infty)$ from $\frac{\pi^2}{6}$ to 0. In [4, Corollary 5], the inequality

$$\frac{\Gamma^2(s+2)}{\Gamma(2s+1)} \ge 1+2s-s^2, \quad s \ge -\frac{3}{2}, \tag{10}$$

which is strict except at $s = 0, 1$, was established. In the proof of [4, Corollary 5], Dulac and Simon pointed out that,

- for $s \in \left[-\frac{3}{2}, -1\right] \cup \left(1+\sqrt{2}, \infty\right)$ the strict inequality (10) is trivial,
- by virtue of the duplication formula (1), the strict inequality (10) for $s \in \left(-1, -\frac{1}{2}\right]$ can be reformulated as

$$\frac{\sqrt{\pi}\,(s+1)\Gamma(s+2)}{4^s\Gamma(s+1/2)} \ge 1+2s-s^2. \tag{11}$$

In [4, Remark 9], the inequalities (10) and (11) were discussed and compared.

By virtue of the duplication formula (1), we can reformulate the function $S(s)$ defined by (9) as

$$\begin{aligned} S(s) &= \frac{(s+1)^2}{s^2(2s+1)}\left[1-\frac{1}{s+1}\frac{\sqrt{\pi}\,\Gamma(s+2)}{4^s\Gamma(s+1/2)}\right] \\ &= \frac{(s+1)^2}{s^2(2s+1)}\left[1-\frac{1}{s+1}C\left(2,\frac{1}{2};s\right)\right]. \end{aligned}$$

The decreasing monotonicity of the function $S(s)$ implies that

$$(s+1)\left[1-\frac{\pi^2}{6}\frac{s^2(2s+1)}{(s+1)^2}\right] < \frac{\sqrt{\pi}\,\Gamma(s+2)}{4^s\Gamma(s+1/2)} < s+1, \quad s>0 \tag{12}$$

or, equivalently,

$$(s+1)\left[1-\frac{\pi^2}{6}\frac{s^2(2s+1)}{(s+1)^2}\right] < C\left(2,\frac{1}{2};s\right) < s+1, \quad s>0.$$

By virtue of the duplication formula (1), the inequality (10) can be rewritten as (11) for $s \ge -\frac{3}{2}$. When $s > -1$, the inequality (11) can be rearranged as

$$\frac{\sqrt{\pi}\,\Gamma(s+2)}{4^s\Gamma(s+1/2)} \ge \frac{1+2s-s^2}{s+1}. \tag{13}$$

It is easy to verify that, when $s > 0$, the lower bound $(s+1)\left[1-\frac{\pi^2}{6}\frac{s^2(2s+1)}{(s+1)^2}\right]$ in (12) is better than the lower bound $\frac{1+2s-s^2}{s+1}$ in (13).

We also notice that, when

$$s > \frac{36+60\pi^2+\pi^4-(\pi^2-6)A^{1/3}+A^{2/3}}{6\pi^2 A^{1/3}} = 0.88273\ldots,$$

where

$$A = 216+540\pi^2+234\pi^4-\pi^6-18\pi^3\sqrt{72+132\pi^2-2\pi^4},$$

the lower bound $(s+1)\left[1-\frac{\pi^2}{6}\frac{s^2(2s+1)}{(s+1)^2}\right]$ in (12) becomes negative.

In October 2022, Dr. Feng Qi, the first author of the papers between [23] and [43], proposed the following problem and conjecture:

- can one find a positive lower bound of the inequality (12) for all $s > 0$?
- we conjecture that the function $S(s)$ is completely monotonic on $(0, \infty)$.

In this paper, we will provide alternative proofs and applications of three integral representations in (5) and (6).

2. Alternative Proofs of Two Integral Representations

In this section, we supply alternative and direct proofs of those two integral representations in (5).

Theorem 1 ([38, Theorem 1.3])**.** *The Catalan numbers C_n for $n \in \mathbb{N}_0$ can be represented by*

$$C_n = \frac{2}{\pi}\int_0^\infty \frac{t^2}{(t^2+1/4)^{n+2}}dt. \tag{14}$$

First proof. Let

$$\Theta_n = \frac{2}{\pi}\int_0^\infty \frac{t^2}{(t^2+1/4)^{n+2}}dt$$

for $n \in \mathbb{N}_0$. Then

$$\begin{aligned}\sum_{n=0}^\infty \Theta_n x^n &= \frac{2}{\pi}\sum_{n=0}^\infty\left[\int_0^\infty \frac{t^2}{(t^2+1/4)^{n+2}}dt\right]x^n \\ &= \frac{2}{\pi}\int_0^\infty \frac{t^2}{(t^2+1/4)^2}\left[\sum_{n=0}^\infty \frac{x^n}{(t^2+1/4)^n}\right]dt \\ &= \frac{2}{\pi}\int_0^\infty \frac{t^2}{(t^2+1/4)^2}\frac{4t^2+1}{(4t^2+1)-4x}dt\end{aligned}$$

$$\begin{aligned}
&= \frac{32}{\pi}\int_0^\infty \frac{t^2}{(4t^2+1)(4t^2+1-4x)}dt\\
&= \frac{2}{\pi x}\int_0^\infty \left[\frac{1}{4t^2+1} - \frac{1-4x}{4t^2+(1-4x)}\right]dt\\
&= \frac{2}{\pi x}\left(\frac{\pi}{4} - \frac{\pi\sqrt{1-4x}}{4}\right)\\
&= \frac{1-\sqrt{1-4x}}{2x}\\
&= \frac{2}{1+\sqrt{1-4x}}
\end{aligned}$$

for $x < \frac{1}{4}$. Comparing this with Equation (3) implies that $C_n = \Theta_n$ for $n \in \mathbb{N}_0$. The proof of the integral representation (14) is complete. □

Second proof. By virtue of the last integral representation in (6), we obtain

$$\begin{aligned}
\frac{1}{n+1}\binom{2n}{n} &= \frac{1}{\pi(n+1)}\int_0^\infty \frac{1}{(t^2+1/4)^{n+1}}dt\\
&= \frac{1}{\pi(n+1)}\left[\frac{t}{(t^2+1/4)^{n+1}}\bigg|_{t=0}^{t=\infty} - \int_0^\infty t\frac{d}{dt}\left(\frac{1}{(t^2+1/4)^{n+1}}\right)dt\right]\\
&= \frac{2}{\pi}\int_0^\infty \frac{t^2}{(t^2+1/4)^{n+2}}dt.
\end{aligned}$$

The proof of the integral representation (14) is complete. □

Theorem 2 ([38, Theorem 1.3])**.** *The Catalan numbers C_n for $n \in \mathbb{N}_0$ can be represented by*

$$C_n = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{(t+1/4)^{n+2}}dt. \tag{15}$$

First proof. Let

$$\Lambda_n = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{(t+1/4)^{n+2}}dt$$

for $n \in \mathbb{N}_0$. Then

$$\begin{aligned}
\sum_{n=0}^\infty \Lambda_n x^n &= \frac{1}{\pi}\sum_{n=0}^\infty\left[\int_0^\infty \frac{\sqrt{t}}{(t+1/4)^{n+2}}dt\right]x^n\\
&= \frac{1}{\pi}\int_0^\infty\left[\frac{\sqrt{t}}{(t+1/4)^2}\sum_{n=0}^\infty \frac{x^n}{(t+1/4)^n}\right]dt\\
&= \frac{16}{\pi}\int_0^\infty \frac{\sqrt{t}}{(4t+1)(4t-4x+1)}dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{32}{\pi}\int_0^\infty \frac{s^2}{(4s^2+1)(4s^2-4x+1)}ds\\
&= \frac{2}{\pi x}\int_0^\infty \left[\frac{4x-1}{4s^2-(4x-1)}+\frac{1}{4s^2+1}\right]ds\\
&= \frac{2}{\pi x}\left[\frac{\pi}{4}-\int_0^\infty \frac{1-4x}{4s^2+(1-4x)}ds\right]\\
&= \frac{2}{\pi x}\left(\frac{\pi}{4}-\frac{\pi\sqrt{1-4x}}{4}\right)\\
&= \frac{1-\sqrt{1-4x}}{2x}\\
&= \frac{2}{1+\sqrt{1-4x}}
\end{aligned}
$$

for $x < \frac{1}{4}$. Comparing this with Equation (3) reveals that $C_n = \Lambda_n$ for $n \in \mathbb{N}_0$. The proof of the integral representation (15) is complete. □

Second proof. Replacing t^2 by s under the integration on the right-hand side of (14) gives

$$
\begin{aligned}
C_n &= \frac{2}{\pi}\int_0^\infty \frac{s}{(s+1/4)^{n+2}}d\left(\sqrt{s}\,\right)\\
&= \frac{2}{\pi}\int_0^\infty \frac{s}{(s+1/4)^{n+2}}\frac{1}{2\sqrt{s}}ds\\
&= \frac{1}{\pi}\int_0^\infty \frac{\sqrt{s}}{(s+1/4)^{n+2}}ds.
\end{aligned}
$$

The proof of the integral representation (15) is complete. □

3. Several Applications of Two Integral Representations

In this section, we apply integral representations (14) and (15) to alternatively prove several known combinatorial identities involving central binomial coefficients $\binom{2n}{n}$ or the Catalan numbers C_n.

Theorem 3 ([41, Lemma 3])**.** *For $n \in \mathbb{N}_0$, we have*

$$
\sum_{k=0}^{n}\frac{1}{k+1}\frac{1}{2^{2k}}\binom{2k}{k} = 2\left[1-\frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1}\right] \tag{16}
$$

and

$$
\sum_{k=0}^{\infty}\frac{1}{k+1}\frac{1}{2^{2k}}\binom{2k}{k} = 2. \tag{17}
$$

Proof. Considering the explicit expression (2) and using the integral representation (15), the left-hand side of (16) can be rearranged as

$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{k+1}\frac{1}{2^{2k}}\binom{2k}{k} &= \sum_{k=0}^{n}\frac{C_k}{2^{2k}}\\ &= \frac{1}{\pi}\sum_{k=0}^{n}\frac{1}{2^{2k}}\int_0^{\infty}\frac{\sqrt{t}}{(t+1/4)^{k+2}}dt\\ &= \frac{1}{\pi}\int_0^{\infty}\left[\frac{\sqrt{t}}{(t+1/4)^2}\sum_{k=0}^{n}\frac{1}{(4t+1)^k}\right]dt\\ &= \frac{1}{\pi}\int_0^{\infty}\frac{\sqrt{t}}{(t+1/4)^2}\frac{1}{4t}\left[4t+1-\frac{1}{(4t+1)^n}\right]dt\\ &= \frac{1}{\pi}\int_0^{\infty}\left[\frac{1}{\sqrt{t}\,(t+1/4)}-\frac{1}{4^{n+1}\sqrt{t}\,(t+1/4)^{n+2}}\right]dt.\end{aligned}$$

Making use of the integral formula

$$\int_0^{\infty}\frac{x^{a-1}dx}{(1+bx)^{\nu}} = \frac{B(a,\nu-a)}{b^a}, \quad |\arg(b)|<\pi, \quad \Re(\nu)>\Re(a)>0,$$

which is listed in [5, p. 318, Entry 3.194.3], and taking $a=\frac{1}{2}$, $b=4$, and $\nu=k\in\mathbb{N}$, we obtain

$$\int_0^{\infty}\frac{1}{\sqrt{x}\,(x+1/4)^k}dx = 2^{2k-1}B\left(\frac{1}{2},k-\frac{1}{2}\right), \quad k\in\mathbb{N}.$$

Consequently, it follows that

$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{k+1}\frac{1}{2^{2k}}\binom{2k}{k} &= \frac{2}{\pi}\left[B\left(\frac{1}{2},\frac{1}{2}\right)-B\left(\frac{1}{2},n+\frac{3}{2}\right)\right]\\ &= \frac{2}{\pi}\left[\frac{\Gamma^2(1/2)}{\Gamma(1)}-\frac{\Gamma(1/2)\Gamma(n+3/2)}{\Gamma(n+2)}\right]\\ &= 2\left[1-\frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1}\right].\end{aligned}$$

It is straightforward that

$$\frac{\Gamma(n+3/2)}{\Gamma(n+2)} = \Gamma\left(\frac{1}{2}\right)\prod_{k=0}^{n}\frac{2k+1}{2n+2}\to 0, \quad n\to\infty.$$

Therefore, Equation (17) is valid. □

Theorem 4 ([29, Theorem 23]). *For $n\in\mathbb{N}_0$, we have*

$$\sum_{k=0}^{n}\frac{1}{2^{2k}}\binom{2k}{k} = \frac{2n+1}{2^{2n}}\binom{2n}{n}. \tag{18}$$

Proof. The left-hand side of Equation (18) can be rewritten as

$$\sum_{k=0}^{n}\frac{1}{2^{2k}}\binom{2k}{k}=\sum_{k=0}^{n}\frac{k+1}{2^{2k}}C_k.$$

Utilizing the integral representation (14) gives

$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{2^{2k}}\binom{2k}{k}&=\frac{2}{\pi}\sum_{k=0}^{n}\frac{k+1}{2^{2k}}\int_0^\infty\frac{t^2}{(t^2+1/4)^{k+2}}dt\\&=\frac{32}{\pi}\int_0^\infty\left[\frac{t^2}{(4t^2+1)^2}\sum_{k=0}^{n}\frac{k+1}{(4t^2+1)^k}\right]dt\\&=\frac{2}{\pi}\int_0^1\left[\frac{\sqrt{1-s}}{\sqrt{s}}\sum_{k=0}^{n}(k+1)s^k\right]ds.\end{aligned}$$

Employing the formula

$$\sum_{k=0}^{n-1}(a+kr)q^k=\frac{a-[a+(n-1)r]q^n}{1-q}+\frac{rq(1-q^{n-1})}{(1-q)^2}$$

for $q\ne1$ and $n\in\mathbb{N}$, which is listed in [5, p. 1, Entry 0.113], we arrive at

$$\sum_{k=0}^{n}(k+1)s^k=\frac{1-(n+1)s^{n+1}}{1-s}+\frac{s(1-s^n)}{(1-s)^2}$$

for $n\in\mathbb{N}_0$. Accordingly, we find

$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{2^{2k}}\binom{2k}{k}&=\frac{2}{\pi}\int_0^1\left[\frac{1-(n+1)s^{n+1}}{\sqrt{s(1-s)}}+\frac{\sqrt{s}\,(1-s^n)}{(1-s)^{3/2}}\right]ds\\&=\frac{2}{\pi}\left[B\left(\frac12,\frac12\right)-(n+1)B\left(\frac12,n+\frac32\right)+\int_0^1\frac{\sqrt{s}\,(1-s^n)}{(1-s)^{3/2}}ds\right]\end{aligned}$$

and, by integration by parts,

$$\begin{aligned}\int_0^1\frac{\sqrt{s}\,(1-s^n)}{(1-s)^{3/2}}ds&=\int_0^1\sqrt{s}\,(1-s^n)d\left[\frac{2}{(1-s)^{1/2}}\right]\\&=\frac{2\sqrt{s}\,(1-s^n)}{(1-s)^{1/2}}\bigg|_{s=0}^{s=1}-\int_0^1\frac{2[\sqrt{s}\,(1-s^n)]'}{(1-s)^{1/2}}ds\\&=-2\int_0^1\frac{s^{-1/2}/2-(n+1/2)s^{n-1/2}}{(1-s)^{1/2}}ds\\&=-2\left[\frac12B\left(\frac12,\frac12\right)-\left(n+\frac12\right)B\left(\frac12,n+\frac12\right)\right].\end{aligned}$$

In conclusion, we obtain

$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{2^{2k}}\binom{2k}{k}&=\frac{2}{\pi}\left[2\left(n+\frac{1}{2}\right)B\left(\frac{1}{2},n+\frac{1}{2}\right)-(n+1)B\left(\frac{1}{2},n+\frac{3}{2}\right)\right]\\&=\frac{2n+1}{2^{2n}}\binom{2n}{n}.\end{aligned}$$

The proof of Equation (18) is complete. ☐

Theorem 5 ([12, Theorem 1] and [34, Theorem 2])**.** *The sum*

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\frac{1}{2^{2k}}\binom{2k}{k}=2\ln\bigl[2\bigl(\sqrt{2}-1\bigr)\bigr] \tag{19}$$

is valid.

Proof. Reformulating the left-hand side of Equation (19) and making use of the integral representation (15) yield

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\frac{1}{2^{2k}}\binom{2k}{k}&=\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\frac{1}{2^{2k}}(k+1)C_k\\&=\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\frac{k+1}{2^{2k}}\int_0^\infty\frac{\sqrt{t}}{(t+1/4)^{k+2}}dt\\&=\frac{1}{\pi}\int_0^\infty\left[\frac{\sqrt{t}}{(t+1/4)^2}\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)\left(-\frac{1}{4t+1}\right)^k\right]dt\\&=\frac{1}{\pi}\int_0^\infty\frac{\sqrt{t}}{(t+1/4)^2}\left[-\frac{1}{2(2t+1)}-\ln\frac{2(2t+1)}{4t+1}\right]dt\\&=-\frac{1}{\pi}\left[8\int_0^\infty\frac{\sqrt{t}}{(2t+1)(4t+1)^2}dt\right.\\&\qquad\left.+16\int_0^\infty\frac{\sqrt{t}}{(4t+1)^2}\ln\left(1+\frac{1}{4t+1}\right)dt\right],\end{aligned}$$

where

$$\begin{aligned}\int_0^\infty\frac{\sqrt{t}}{(2t+1)(4t+1)^2}dt&=2\int_0^\infty\frac{s^2}{(2s^2+1)(4s^2+1)^2}ds\\&=2\int_0^\infty\left[\frac{1}{4s^2+1}-\frac{1}{2(4s^2+1)^2}-\frac{1}{2(2s^2+1)}\right]ds\\&=2\left[\frac{1}{4}B\left(\frac{1}{2},\frac{1}{2}\right)-\frac{1}{8}B\left(\frac{3}{2},\frac{1}{2}\right)-\frac{1}{4\sqrt{2}}B\left(\frac{1}{2},\frac{1}{2}\right)\right]\\&=\frac{1}{8}\bigl(3-2\sqrt{2}\bigr)\pi\end{aligned}$$

by virtue of the formula

$$\int_0^\infty \frac{x^{\mu-1}dx}{(p+qx^\nu)^{n+1}} = \frac{1}{\nu p^{n+1}}\left(\frac{p}{q}\right)^{\mu/\nu} B\left(1+n-\frac{\mu}{\nu}, \frac{\mu}{\nu}\right) \tag{20}$$

for $0 < \frac{\mu}{\nu} < n+1$, $n \in \mathbb{N}_0$, $p \ne 0$, and $q \ne 0$, which is given in [5, p. 325, Entry 4], and

$$\begin{aligned}\int_0^\infty \frac{\sqrt{t}}{(4t+1)^2} \ln\left(1+\frac{1}{4t+1}\right) dt &= \frac{1}{8}\int_0^1 \sqrt{\frac{1-s}{s}}\, \ln(1+s)ds \\ &= \frac{\pi}{8}\left[\ln\frac{1+\sqrt{2}}{2} + \frac{1-\sqrt{2}}{2\left(1+\sqrt{2}\,\right)}\right]\end{aligned}$$

by virtue of the formula

$$\int_0^a \sqrt{a-x}\,\frac{\ln(d+cx)}{\sqrt{x}}dx = \pi a \ln\frac{d+\sqrt{d(d+ac)}}{2} + \frac{\pi a}{2}\frac{\sqrt{d}-\sqrt{d+ac}}{\sqrt{d}+\sqrt{d+ac}}$$

for $a, b, c > 0$, which is found in [22, p. 503, Entry 34]. Consequently, we find that

$$\begin{aligned}\sum_{k=1}^\infty \frac{(-1)^k}{k}\frac{1}{2^{2k}}\binom{2k}{k} &= -\frac{1}{\pi}\left(\left(3-2\sqrt{2}\,\right)\pi + 2\pi\left[\ln\frac{1+\sqrt{2}}{2} + \frac{1-\sqrt{2}}{2\left(1+\sqrt{2}\,\right)}\right]\right) \\ &= 2\ln\left[2\left(\sqrt{2}-1\right)\right].\end{aligned}$$

The proof of Theorem 5 is complete. □

Theorem 6 ([15, Theorem 1.2])**.** *The identity*

$$\sum_{k=1}^\infty \frac{1}{k}\frac{1}{2^{2k}}\binom{2k}{k} = 2\ln 2 \tag{21}$$

is valid.

Proof. Rearranging the left-hand side of Equation (19) and using the integral representation (14) reveal that

$$\begin{aligned}\sum_{k=1}^\infty \frac{1}{k}\frac{1}{2^{2k}}\binom{2k}{k} &= \sum_{k=1}^\infty \frac{k+1}{k}\frac{1}{2^{2k}}C_k \\ &= \frac{2}{\pi}\sum_{k=1}^\infty \frac{k+1}{k}\frac{1}{2^{2k}}\int_0^\infty \frac{t^2}{(t^2+1/4)^{k+2}}dt \\ &= \frac{32}{\pi}\int_0^\infty \left[\frac{t^2}{(4t^2+1)^2}\sum_{k=1}^\infty \frac{k+1}{k}\frac{1}{(4t^2+1)^k}\right]dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{32}{\pi}\int_0^\infty \left[\frac{t^2}{(4t^2+1)^2}\left(\frac{1}{4t^2} - \ln\frac{4t^2}{4t^2+1}\right)\right]dt\\
&= \frac{32}{\pi}\int_0^\infty \left[\frac{1}{4(4t^2+1)^2} - \frac{t^2}{(4t^2+1)^2}\ln\frac{4t^2}{4t^2+1}\right]dt\\
&= \frac{32}{\pi}\left[\frac{1}{16}B\left(\frac{3}{2},\frac{1}{2}\right) - \int_0^\infty \frac{t^2}{(4t^2+1)^2}\ln\frac{4t^2}{4t^2+1}dt\right],
\end{aligned}
$$

where we used the formula (20) for $\mu = 1$, $\nu = 2$, $p = 1$, $q = 4$, and $n = 1$, and

$$
\begin{aligned}
\int_0^\infty \frac{t^2}{(4t^2+1)^2}\ln\frac{4t^2}{4t^2+1}dt &= \int_0^\infty \frac{t^2}{(4t^2+1)^2}\ln\left(1 - \frac{1}{4t^2+1}\right)dt\\
&= \frac{1}{16}\int_0^1 \sqrt{\frac{1-s}{s}}\,\ln(1-s)ds\\
&= \frac{1}{16}B\left(\frac{1}{2},\frac{3}{2}\right)\left[\psi\left(\frac{3}{2}\right) - \psi(2)\right]\\
&= -\frac{2\ln 2 - 1}{32}\pi,
\end{aligned}
$$

where we used the special values

$$
\psi(n+1) = -\gamma + \sum_{k=1}^{n}\frac{1}{k}, \quad \psi\left(n+\frac{1}{2}\right) = -\gamma - 2\ln 2 + 2\sum_{k=1}^{n}\frac{1}{2k-1}
$$

for $n \in \mathbb{N}$ and $\gamma = 0.57721566\ldots$, which are given in [1, p. 258, Items 6.3.2 and 6.3.4], and the formula

$$
\int_0^a x^{\alpha-1}(a-x)^{\beta-1}\ln(a-x)dx = a^{\alpha+\beta-1}B(\alpha,\beta)[\ln a + \psi(\beta) - \psi(\alpha+\beta)]
$$

for $a, \Re(\beta) > 0$ and $\Re(\alpha) > 0$ (or $\Re(\alpha) > -1$ for $a = 1$), which is included in [5, p. 502, Entry 25]. Consequently, we conclude

$$
\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{2^{2k}}\binom{2k}{k} = \frac{32}{\pi}\left[\frac{1}{16}B\left(\frac{3}{2},\frac{1}{2}\right) + \frac{2\ln 2 - 1}{32}\pi\right] = 2\ln 2.
$$

The proof of Equation (21) is complete. □

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